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Average degree exponent sum energy of graphs

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Abstract

In this paper we introduce new energy of graph that is average degree exponent sum energy. We obtain characteristic polynomial of the average degree exponent sum of standard graphs and also obtained few graphs by some graph operations.

Keywords: Average degree exponent sum matrix, Average degree exponent sum polynomial and energy.
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1 Introduction

All the graphs considered here are simple, finite and undirected. Basic terminologies and notations can be found in [15]. Let $A = (a_{ij})$ be an adjacency matrix of order n of the graph G. The characteristic polynomial of G is denoted by $Ch(G, \lambda) = (\lambda I - G)$, where λ is an eigenvalue of the graph G. Hence, by [13], the energy of G is defined as $E(G) = \sum_{i=1}^{n} \mid \lambda_i \mid$.
The concept of energy of graph arose from Huckel theory in which the total π-electron energy of a conjugated carbon molecule was computed, which coincides with the energy of a graph. Let $V(G)$ be the vertex set and $E(G)$ be an edge set of G. The degree of a vertex G is denoted by $d_u(G)$. The average degree exponent sum matrix of a graph G is denoted by $AD(G) = (s_{ij})$ and whose elements are defined as

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$$s_{ij} = \begin{cases} \dfrac{d_i^{d_j} + d_j^{d_i}}{2} & \text{if } v_i \sim v_j \\ 0 & \text{if otherwise} \end{cases}.$$

2 Some basic properties of largest average degree square sum eigenvalue

Let us define number p as

$$p = \sum_{i<j} (\frac{d_i^{d_j} + d_j^{d_i}}{2})^2$$

Proposition 2.1. *The first three coefficient of the polynomial $Ch(AD(G, \lambda))$ are as follows*

(i) $a_0 = 1$

(ii) $a_1 = 0$

(iii) $a_2 = -p$

Proof. (i) By the definition of characteristic poynomial we get, $a_0 = 1$

(ii) The sum of determinants of all 1×1 principal submatrices of $AD(G)$ is equal to the trace of $AD(G)$ so,

$$a_1 = tr(AD(G)) = 0$$

(iii) We have ,

$$(-1)^2 a_2 = \sum_{1 \le i < j \le n} (a_{ii} a_{jj} - a_{ji} a_{ij})$$
$$= -p$$

□

Proposition 2.2. *If λ_1, λ_2 ,..., λ_n are the average degree exponent sum eigenvalues of AD(G) then,*

$$\sum_{i=1}^{n} \lambda_i^2 = 2p$$

Proof.

$$\sum_{i=1}^{n} \lambda_i^2 = tr([AD(G)]^2) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} a_{ji}$$

$$= 2 \sum_{i<j} (a_{ij})^2 + \sum_{i=1}^{n} (a_{ii})^2$$

$$= 2\sum_{i<j}(a_{ij})^2$$

$$= 2p$$

□

Theorem 2.3 ([18]). *Let a_i and b_i be nonnegative real numbers, then*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \frac{n^2}{4}(M_1 M_2 - m_1 m_2)^2 \tag{1}$$

where, $M_1 = max(a_i)$, $M_2 = max(b_i), m_1 = min(a_i)$, $m_2 = min(b_i)$ also $i = 1, 2, ..., n$

Theorem 2.4 ([5]). *Let a_i and b_i be nonnegative real numbers, then*

$$\left| n\sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \right| \leq \alpha(n)(A-a)(B-b) \tag{2}$$

where a, b, A and B are real constants such that $a \leq a_i \leq A$ and $b \leq b_i \leq B$ for each i, $1 \leq i \leq n$. Further, $\alpha(n) = n\lfloor \frac{n}{2} \rfloor(1 - \frac{1}{n}\lfloor \frac{n}{2} \rfloor)$.

Theorem 2.5 ([11]). *Let a_i and b_i be nonnegative real numbers, then*

$$\sum_{i=1}^{n} b_i^2 + C_1 C_2 \sum_{i=1}^{n} a_i^2 \leq (C_1 + C_2) \sum_{i=1}^{n} a_i b_i \tag{3}$$

where C_1 and C_2 are real constants such that $C_1 a_i \leq b_i \leq C_2 a_i$ for each i, $1 \leq i \leq n$.

Theorem 2.6. *Let G be a r-regular graph of order n. Then G has only one positive average degree exponent sum eigenvalue $\lambda = r^r(n-1)$.*

Proof. Let G be a connected r-regular graph of order n and $\{v_1, v_2, ..., v_n\}$ be the vertex set of G. Let $d_i = r$ be the degree of v_i, $i = 1, 2, ...n$. Then the characteristic polynomial of $AD(G)$

$$Ch[AD(G), \lambda] = (\lambda - r^r(n-1))(\lambda + r^r)^{n-1} \tag{4}$$

Therefore, the eigenvalues are $r^r(n-1)$ and $-r^r$ which repeats $(n-1)$ times.

□

Theorem 2.7. *Let G be a graph of order n and λ_1 be the largest average degree exponent sum eigenvalue. Then*

$$\lambda_1 \leq \sqrt{\frac{2p(n-1)}{n}}$$

Proof. By the Cauchy-Schwartz inequality [[2]] we have

$$(\sum_{i=1}^{n} a_i^2 b_i^2)^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2$$

where a_i and b_i are nonnegative real numbers.
now, by substituting $a_i = 1$ and $b_i = \lambda_i$,we have

$$(\sum_{i=2}^{n} \lambda_i^2)^2 \leq (n-1) \sum_{i=2}^{n} \lambda_i^2$$

By using propositions 2.1 and 2.2 in the above inequality

$$(-\lambda_1)^2 \leq (n-1)(2p - \lambda_1^2)$$

Hence,

$$\lambda_1 \leq \sqrt{\frac{2p(n-1)}{n}}$$

Remark 2.8. *If G is a regular graph, then*

$$\lambda_1 = \sqrt{\frac{2p(n-1)}{n}}$$

□

Remark 2.9. *Let G be a r-regular graph of order n, then $AD(G) = r^r J - r^r I$. Where J is the matrix of order n whose all entries are equal to one and I is an identity matrix of order n.*
The characteristic polynomial is given by

$$Ch[AD(G), \lambda] = (\lambda - r^r(n-1))(\lambda + r^r)^{n-1}$$

Hence ,

$$E[AD(G)] = 2r^2(n-1) \tag{5}$$

The complement $\overline{G}$ [15] of a graph G is a graph with vertex set $V(G)$ and two vertices of $\overline{G}$ are adjacent if and only if they are nonadjacent in G.

Remark 2.10. *If G is a r-regular graph , its complement $\overline{G}$ is $(n-1-r)$ regular graph then, we have,*

$$Ch[AD(\overline{G}), \lambda] = (\lambda - (n-1)(n-1-r)^{n-1-r})(\lambda + (n-1-r)^{n-1-r})^{n-1}$$

Thus ,

$$E[AD(\overline{G})] = 2(n-1-r)^{n-1-r}(n-1) \tag{6}$$

Theorem 2.11. *Let G be a graph of order n and size m. Then*

$$E[AD(G)] \geq \sqrt{2np - \frac{n^2}{4}(\mid \lambda_1 \mid - \mid \lambda_2 \mid)^2}$$

Proof. Let λ_1, λ_2, ..., λ_n be the average degree exponent sum eigenvalues of G. Substituting $a_i = 1$ and $b_i =\mid \lambda_i \mid$ in the equation (1) We get

$$\sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} \mid \lambda_i \mid^2 - \left(\sum_{i=1}^{n} \mid \lambda_i \mid^2 \right)^2 \leq \frac{n^2}{4} (\mid \lambda_1 \mid - \mid \lambda_n \mid)^2$$

$$2pn - (E[AD(G)])^2 \leq \frac{n^2}{4} (\mid \lambda_1 \mid - \mid \lambda_n \mid)^2$$

$$E[AD(G)] \geq \sqrt{2np - \frac{n^2}{4} (\mid \lambda_1 \mid - \mid \lambda_n \mid)^2}$$

□

Corollary 2.12. *If G is a r-regular graph of order n, then*

$$E[AD(G)] \geq nr^2 \sqrt{8(n-1) - n^2}$$

Theorem 2.13. *Let G be a graph of order n, then*

$$\sqrt{2p} \leq E[AD(G)] \leq \sqrt{2np}$$

Proof. By the Cauchy-Schwartz inequality [[2]] we have

$$\left(\sum_{i=1}^{n} a_i b_i \right)^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2$$

where a_i and b_i are non-negative real numbers.
Now, substituting $a_i = 1$ and $b_i = \lambda_i$ we have

$$\left(\sum_{i=1}^{n} \mid \lambda_i \mid \right)^2 \leq \sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} \mid \lambda_i \mid^2$$

$$(E[AD(G)])^2 \leq 2pn$$

Thus,

$$E[AD(G)] \leq \sqrt{2pn}$$

and

$$\sum_{i=1}^{n} \mid \lambda_i \mid^2 \leq \left(\sum_{i=1}^{n} \mid \lambda_i \mid \right)^2$$

$$2p \leq (E[AD(G)])^2$$

thus

$$E[AD(G)] \geq \sqrt{2p}.$$

□

Theorem 2.14. *Let G be a graph of order n and Δ be the absolute value of the determinent of $AD(G)$. Then*

$$\sqrt{2p+n(n-1)\Delta^{\frac{2}{n}}} \leq E[AD(G)] \leq \sqrt{2np}$$

Proof.

$$(E[AD(G)])^2 = \left(\sum_{i=1}^{n} \mid \lambda_i \mid\right)^2$$

$$= \sum_{i=1}^{n} \lambda_i^2 + 2\sum_{i<j} \mid \lambda_i \mid\mid \lambda_j \mid$$

$$= 2p + 2\sum_{i<j} \mid \lambda_i \mid\mid \lambda_j \mid$$

$$(E[AD(G)])^2 = 2p + \sum_{i\neq j} \mid \lambda_i \mid\mid \lambda_j \mid \tag{7}$$

We know that for nonnegative numbers , the arithmatic mean is always greater than or equal to the geometric mean, so

$$\frac{1}{n(n-1)} \sum_{i\neq j} \mid \lambda_i \mid\mid \lambda_j \mid \geq \left(\prod_{i\neq j} \mid \lambda_i \mid\mid \lambda_j \mid\right)^{\frac{1}{n(n-1)}}$$

$$= \left(\prod_{i=1} \mid \lambda_i \mid^{2(n-1)}\right)^{\frac{1}{n(n-1)}}$$

$$= \prod_{i\neq j} \mid \lambda_i \mid^{\frac{2}{n}}$$

$$= \Delta^{\frac{2}{n}}$$

Therefore,

$$\sum_{i\neq j} \mid \lambda_i \mid\mid \lambda_j \mid \geq n(n-1)\Delta^{\frac{2}{n}}$$

from equation (7) we have,

$$E[AD(G)] \geq \sqrt{2p+n(n-1)\Delta^{\frac{2}{n}}}$$

Consider a nonnegative quantity

$$Y = \sum_{i=1}^{n}\sum_{j=1}^{n} (\mid \lambda_i \mid - \mid \lambda_j \mid)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} (\mid \lambda_i \mid^2 + \mid \lambda_j \mid^2 - 2 \mid \lambda_i \mid\mid \lambda_j \mid)$$

$$Y = n\sum_{i=1}^{n} \mid \lambda_i \mid^2 + n\sum_{j=1}^{n} \mid \lambda_j \mid^2 - 2\sum_{i=1}^{n} \mid \lambda_i \mid \sum_{j=1}^{n} \mid \lambda_j \mid$$

$$Y = 4np - 2(E[AD(G)])^2$$

since

$$Y \geq 0$$
$$4np - 2(E[AD(G)])^2 \geq 0$$
$$E[AD(G)] \leq \sqrt{2np}$$

□

Corollary 2.15. *If G is a r-regular graph of order n, then*

$$E[AD(G)] \leq 2nr^2\sqrt{n-1}$$

Theorem 2.16. *Let G be a graph f order n and size m. Let $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ be a non-increasing arrangement of average degree exponent sum eigenvalues. Then*

$$E[AD(G)] \geq \sqrt{2np - \alpha(n)(\mid \lambda_1 \mid - \mid \lambda_n \mid)^2}$$

where $\alpha(n) = n\lfloor \frac{n}{2} \rfloor (1 - \frac{1}{n}\lfloor \frac{n}{2} \rfloor)$.

Proof. Let λ_1, λ_2, ..., λ_n be the average degree exponent sum eigenvalues of G. Substituting $a_i =\mid \lambda_i \mid= b_i$ and $a =\mid \lambda_n \mid= b$, $A =\mid \lambda_1 \mid= B$ in the equation (2)

$$\left| n\sum_{i=1}^{n} \mid \lambda_i \mid^2 - \left(\sum_{i=1}^{n} \mid \lambda_i \mid\right)^2 \right| \leq \alpha(n)(\mid \lambda_1 \mid - \mid \lambda_n \mid)^2$$

Since $E[AD(G)] = \sum_{i=1}^{n} \mid \lambda_i \mid$ and $\sum_{i=1}^{n} \mid \lambda_i \mid^2 = 2p$ we get the required result. □

Theorem 2.17. *Let G be a graph of order n and size m. Let $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ be a non-increasing arrangement of average degree exponent sum eigenvalues. Then*

$$E[AD(G)] \geq \frac{\mid \lambda_1 \mid\mid \lambda_n \mid n + 2p}{\mid \lambda_1 \mid + \mid \lambda_n \mid}$$

where $\mid \lambda_1 \mid$ and $\mid \lambda_2 \mid$ are maximum and minimum of the absolute value of $\lambda_i's$

Proof. Let λ_1, λ_2, ..., λ_n be the average degree exponent sum eigenvalues of G. Substituting $a_i = 1$ and $b_i =\mid \lambda_i \mid$, $C_1 =\mid \lambda_n \mid$, $C_2 =\mid \lambda_1 \mid$ in the equation (4)

$$\sum_{i=1}^{n} \mid \lambda_i \mid^2 + \mid \lambda_1 \mid\mid \lambda_n \mid \sum_{i=1}^{n} 1^2 \leq (\mid \lambda_1 \mid + \mid \lambda_n \mid)\left(\sum_{i=1}^{n} \mid \lambda_i \mid\right)$$

Since $E[AD(G)] = \sum_{i=1}^{n} \mid \lambda_i \mid$ and $\sum_{i=1}^{n} \mid \lambda_i \mid^2 = 2p$ we get the required result. □

Definition 2.18. *[15] The line graph $L(G)$ of a graph G is a graph with vertex set as the edge set of G and two vertices of $L(G)$ are adjacent whenever the corresponding edges in G are adjacent.*

The k^{th} iterated line graph [6, 7, 15] of G is defined as $L^k(G) = L(L^{k-1}(G))$, $k = 1, 2, 3..$ where $L^0(G) \cong G$ and $L^1(G) \cong L(G)$

Remark 2.19 ([6, 7]). *The line graph $L(G)$ of a r-regular graph of G of order n is an $r_1 = (2r-2)$-regular graph of order $n_1 = \frac{nr}{2}$. Thus, $L^k(G)$ is an r_k-regular graph of order n_k given by*

$$n_k = \frac{n}{2^k}\prod_{i=1}^{k-1}(2^i r - 2^{i+1} + 2) \quad and \quad r_k = 2^k r - 2^{k+1} + 2$$

Theorem 2.20. *Let G be a r-regular graph of order n and let $L^k(G)$ be the r_k-regular graph of order n_k then average degree exponent sum energ of $L^k(G)$*

$$E[AD(L^k(G))] = 2r_k^{r_k}(n-1) \quad where, \quad r_k = 2^k r - 2^{k+1} + 2$$

Proof. The average degree exponent sum characteristic polynomial of $L^k(G)$ with vertex set n_k (see equation (1) and remark 2.15) is given by

$$Ch[AD(L^k(G)),\lambda] = [\lambda - (2^k r - 2^{k+1} + 2)^{2^k r - 2^{k+1}+2}(n_k - 1)][\lambda + (2^k r - 2^{k+1} + 2)^{2^k r - 2^{k+1}+2}]^{n_k - 1}$$

Thus,

$$E[AD(L^k(G))] = 2r_k^{r_k}(n-1) \quad where, \quad r_k = 2^k r - 2^{k+1} + 2$$

□

Lemma 2.21 ([21]). *If a,b,c and d are real numbers, then the determinent of the form*

$$\begin{vmatrix} (\lambda + a)I_{n_1} - aJ_{n_1} & -cJ_{n_1\times n_2} \\ -dJ_{n_2\times n_1} & (\lambda + b)I_{n_2} - bJ_{n_2} \end{vmatrix}$$

$$= (\lambda + a)^{n_1 - 1}(\lambda + b)^{n_2 - 1}[(\lambda - (n_1 - 1)a][\lambda - (n_2 - 1)b] - n_1 n_2 cd)]$$

Definition 2.22 ([15]). *The subdivision graph $S(G)$ of a graph G is a graph with vertex set $V(G) \cup E(G)$ and is obtained by inserting a new vertex of degree 2 into each edge of G.*

Definition 2.23 ([22]). *The semitotal line graph $T_1(G)$ of a graph G is a graph with vertex set $V(G) \cup E(G)$ where two vertices of $T_1(G)$ are adjacent if and only if they corresponds to two adjacent edges of G or one is a vertex of G and another is an edge G incident with it in G.*

Definition 2.24 ([22]). *The semitotal point graph $T_2(G)$ of a graph G is a graph with vertex set $V(G) \cup E(G)$ where two vertices of $T_2(G)$ are adjacent if and only if they corresponds to two adjacent vertices of G or one is a vertex of G and another is an edge G incident with it in G.*

Definition 2.25 ([15]). *The total graph $T(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices of $T(G)$ are adjacent if and only if the corresponding elements of G are either adjacent or incident.*

Definition 2.26 ([21]). *The graph G^{+k} is a graph obtained from the graph G by attaching k pendant edges to each vertex of G. If G is a graph of order n and size m, then G^{+k} is graph of order $n+nk$ and size $m+nk$.*

Definition 2.27 ([15]). *The union of the graphs G_1 and G_2 is a graph $G_1 \cup G_2$ whose vertex set is $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and the edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.*

Definition 2.28 ([15]). *The join $G_1 + G_2$ of two graphs G_1 and G_2 is the graph obtained from G_1 and G_2 by joining every vertex of G_1 to all vertices of G_2.*

Definition 2.29 ([15]). *The product $G \times H$ of two graphs G and H is defined as follows*
Consider any two points $u = (u_1, u_2)$ and $u = (v_1, v_2)$ in $V = V_1 \times V_2$. Then u and v are adjacent in $G \times H$ whenever [$u_1 = v_1$ and u_2 adj v_2] or [$u_2 = v_2$ and u_1 adj v_1].

Definition 2.30 ([15]). *The composition $G[H]$ of two graphs G and H is defined as follows: Consider any two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$) in $V = V_1 \times V_2$. Then u and v are adjacent in $G[H]$ whenever [u_1 adj v_1] or [$u_1 = v_1$ and u_2 adj v_2].*

Definition 2.31 ([15]). *The corona $G \circ H$ of graphs G and H is a graph obtained from G and H by taking one copy of G and $\mid V(G) \mid$ copies of H and then joining by an edge each vertex of the i^{th} copy of H is named (H, i) with the i^{th} vertex of G.*

Definition 2.32 ([8]). *The jump graph $J(G)$ of a graph G is defined as a graph with vertex set as $E(G)$ where the two vertices of $J(G)$ are adjacent if and only if they correspond to two nonadjacent edges of G.*

3 Main results

Theorem 3.1. *Let G be a r- regular graph of order n and size m. Then,*

$$Ch[AD(S(G))] = (\lambda + r^r)^{n-1}(\lambda + 4)^{\frac{nr}{2}-1}[\lambda^2 - (4(\frac{nr}{2} - 1) + r^r(n-1))\lambda + \frac{1}{4}(16r^r(n-1)(\frac{nr}{2} - 1) - \frac{n^2 r}{2}(r^2 + 2^r)^2)]$$

Proof. The subdivision graph of the r-regular graph has two types of vertices. The n vertices with degree r and $\frac{nr}{2}$ vertices with degree 2. Hence

$$AD(S(G)) = \begin{bmatrix} r^r(J_n - I_n) & \frac{(r^2+2^r)}{2} J_{n \times \frac{nr}{2}} \\ \frac{(r^2+2^r)}{2} J_{\frac{nr}{2} \times n} & 4(J_{\frac{nr}{2}} - I_{\frac{nr}{2}}) \end{bmatrix}.$$

$$Ch[AD(S(G))] = \mid \lambda I - AD(S(G)) \mid$$

$$= \begin{vmatrix} (\lambda + r^r)I_n - r^r J_n & -\frac{(r^2+2^r)}{2} J_{n \times \frac{nr}{2}} \\ -\frac{(r^2+2^r)}{2} J_{\frac{nr}{2} \times n} & (\lambda + 4)I_{\frac{nr}{2}} - 4J_{\frac{nr}{2}} \end{vmatrix}.$$

Now by using Lemma 2.17, we get the desired result. □

Theorem 3.2. *Let G be a r- regular graph of order n and size m. Then,*

$$Ch[AD(T_2(G))] = (\lambda + (2r)^{2r})^{n-1}(\lambda + 4)^{\frac{nr}{2}-1}[\lambda^2 - (4(\frac{nr}{2} - 1) + (2r)^{2r}(n-1))\lambda + 4(2r)^{2r}(n-1)(\frac{nr}{2} - 1) - \frac{n^2 r}{8}(4r^2 + 2^{2r})^2]$$

Proof. The semitotal point graph of an r-regular graph has two types of vertices. The n vertices with degree $2r$ and $\frac{nr}{2}$ vertices with degree 2. Hence

$$AD(T_2) = \begin{bmatrix} (2r)^{2r}(J_n - I_n) & \frac{(4r^2+2^{2r})}{2} J_{n \times \frac{nr}{2}} \\ \frac{(4r^2+2^{2r})}{2} J_{\frac{nr}{2} \times n} & 4(J_{\frac{nr}{2}} - I_{\frac{nr}{2}}) \end{bmatrix}.$$

$$Ch[AD(T_2)] = \mid \lambda I - AD(T_2(G)) \mid$$

$$= \begin{vmatrix} (\lambda + (2r)^{2r})I_n - (2r)^{2r} J_n & -\frac{(4r^2+2^{2r})}{2} J_{n \times \frac{nr}{2}} \\ -\frac{(4r^2+2^{2r})}{2} J_{\frac{nr}{2} \times n} & (\lambda + 4)I_{\frac{nr}{2}} - 4J_{\frac{nr}{2}} \end{vmatrix}$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.3. *Let G be a r- regular graph of order n and size m. Then,*

$$Ch[AD(T_1)] = (\lambda + r^r)^{n-1}(\lambda + (2r)^{2r})^{\frac{nr}{2}-1}[\lambda^2 - (2r)^{2r}(\frac{nr}{2} - 1) + (n-1)r^r)\lambda + r^r(n-1)(\frac{nr}{2} - 1)(2r)^{2r} - \frac{n^2 r}{8}((r^{2r} + (2r)^r)^2] \tag{8}$$

Proof. The semitotal line graph of an r-regular graph has two types of vertices. The n vertices with degree r and $\frac{nr}{2}$ vertices with degree $2r$. Hence

$$AD(T_1) = \begin{bmatrix} r^r(J_n - I_n) & \frac{(r^{2r}+(2r)^r)}{2} J_{n \times \frac{nr}{2}} \\ \frac{(r^{2r}+(2r)^r)}{2} J_{\frac{nr}{2} \times n} & (2r)^{2r}(J_{\frac{nr}{2}} - I_{\frac{nr}{2}}) \end{bmatrix}.$$

$$Ch[AD(T_1)] = \mid \lambda I - AD(T_1(G)) \mid$$

$$\begin{vmatrix} (\lambda + r^r)I_n - r^r J_n & -\frac{(r^{2r}+(2r)^r)}{2} J_{n \times \frac{nr}{2}} \\ -\frac{(r^{2r}+(2r)^r)}{2} J_{\frac{nr}{2} \times n} & (\lambda + (2r)^{2r})I_{\frac{nr}{2}} - (2r)^{2r} J_{\frac{nr}{2}} \end{vmatrix}$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.4. *Let G be a r- regular graph of order n and size m. Then,*

$$Ch[AD(T(G))] = (\lambda - (2r)^{2r}(n + \frac{nr}{2} - 1))(\lambda + (2r)^{2r})^{n + \frac{nr}{2} - 1}$$

Proof. The total graph of a r-regular graph is a regular graph of degree $2r$ with $n + \frac{nr}{2}$ vertices. Hence the result follows from equation (4)

□

Theorem 3.5. *Let G be a r- regular graph of order n and size m. Then,*

$$Ch[AD(G^{+k})] = (\lambda + (r+k)^{r+k})^{n-1}(\lambda+1)^{nk-1}[\lambda^2 - ((nk-1) + (r+k)^{r+k}(n-1))\lambda + \frac{1}{4}(4(r+k)^{r+k}(n-1)(nk-1) - n^2k(r+k+1)^2)]$$

Proof. The graph G^{+k} of a r-regular graph has two types of vertices. The n vertices with degree $r+k$ and nk vertices with degree 1. Hence

$$AD(G^{+k}) = \begin{bmatrix} (r+k)^{r+k}(J_n - I_n) & \frac{(r+k+1)}{2} J_{n\times nk} \\ \frac{(r+k+1)}{2} J_{nk\times n} & (J_{nk} - I_{nk}) \end{bmatrix}.$$

$$Ch[AD(G^{+k})] = | \lambda I - AD(G^{+k}) |$$

$$\begin{vmatrix} (\lambda + (r+k)^{r+k})I_n - (r+k)^{r+k}J_n & -\frac{(r+k+1)}{2} J_{n\times nk} \\ -\frac{(r+k)+1}{2} J_{nk\times n} & (\lambda+1)I_{nk} - J_{nk} \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.6. *Let G be a r- regular graph of order n and size m. Then,*

$$Ch[AD(G\cup H)] = Ch(AD(G))Ch(AD(H)) - (\lambda + r_1^{r_1})^{n_1-1}(\lambda + r_2^{r_2})^{n_2-1} n_1 n_2 \frac{(r_1^{r_2} + r_2^{r_1})^2}{4}$$

Proof. The graph $G \cup H$ has two types of vertices, the n_1 vertices of degree r_1 and the remaining n_2 vertices are of degree r_2. Hence

$$AD[G\cup H] = \begin{bmatrix} AD(G) & \frac{(r_1^{r_2}+r_2^{r_1})}{2} J_{n_1\times n_2} \\ \frac{(r_1^{r_2}+r_2^{r_1})}{2} J_{n_2\times n_1} & AD(H) \end{bmatrix}.$$

$$= \begin{bmatrix} r_1^{r_1}(J_{n_1} - I_{n_1}) & \frac{(r_1^{r_2}+r_2^{r_1})}{2} J_{n_1\times n_2} \\ \frac{(r_1^{r_2}+r_2^{r_1})}{2} J_{n_2\times n_1} & r_2^{r_2}(J_{n_2} - I_{n_2}) \end{bmatrix}.$$

$$Ch[AD(G\cup H)] = | \lambda I - AD(G\cup H) |$$

$$= \begin{vmatrix} (\lambda + r_1^{r_1})I_{n_1} - r_1^{r_1}J_{n_1} & -\frac{(r_1^{r_2}+r_2^{r_1})}{2} J_{n_1\times n_2} \\ -\frac{(r_1^{r_2}+r_2^{r_1})}{2} J_{n_2\times n_1} & (\lambda + r_2^{r_2})I_m - r_2^{r_2}J_{n_2} \end{vmatrix}.$$

Now by using Lemma 2.21, we get

$$Ch[AD(G\cup H)] = (\lambda + r_1^{r_1})^{n_1-1}(\lambda + r_2^{r_2})^{n_2-1}[(\lambda - (n_1-1)r_1^{r_1})(\lambda - (n_2-1)r_2^{r_2}) - \frac{n_1n_2(r_1^{r_2} + r_2^{r_1})^2}{4}]$$

Since G and H are regular graphs of order n_1 and n_2 and degree r_1 and r_2 respectively, by equation (4) we have

$$Ch[AD(G)] = (\lambda - r_1^{r_1}(n_1-1))(\lambda + r_1^{r_1})^{n_1-1}$$

and

$$Ch[AD(H)] = (\lambda - r_2^{r_2}(n_2-1))(\lambda + r_2^{r_2})^{n_2-1}$$

Hence the result follows. □

Theorem 3.7. *Let G be a r- regular graph of order n and size m. Then,*

$$Ch[AD(G+H)] = (\lambda + R_1^{R_1})^{n_1-1}(\lambda + R_2^{R_2})^{n_2-1}[\lambda^2 - (R_2^{R_2}(n_2-1) + R_1^{R_1}(n_1-1))\lambda + \frac{1}{4}(R_1^{R_1}R_2^{R_2}(n_1-1)(n_2-1) - n_1n_2(R_1^{R_2} + R_2^{R_1})^2)]$$

Proof. If G is a r_1 - regular graph of order n_1 and H is a r_2-regular graph of order n_2 then $G+H$ has two types of vertices, the n_1 vertices with degree $R_1 = r_1 + n_2$ and n_2 vertices with degree $R_2 = r_2 + n_1$. Hence

$$AD(G+H) = \begin{bmatrix} R_1^{R_1}(J_{n_1} - I_{n_1}) & \frac{(R_1^{R_2}+R_2^{R_1})}{2} J_{n_1\times n_2} \\ \frac{(R_1^{R_2}+R_2^{R_1})}{2} J_{n_2\times n_1} & R_2^{R_2}(J_{n_2} - I_{n_2}) \end{bmatrix}.$$

$$Ch[AD(G+H)] = \mid \lambda I - AD(G+H) \mid$$

$$= \begin{vmatrix} (\lambda + R_1^{R_1})I_{n_1} - R_1^{R_1}J_{n_1} & -\frac{(R_1^{R_2}+R_2^{R_1})}{2} J_{n_1\times n_2} \\ -\frac{(R_1^{R_2}+R_2^{R_1})}{2} J_{n_2\times n_1} & (\lambda + R_2^{R_2})I_{n_2} - R_2^{R_2}J_{n_2} \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.8. *Let G be a r_1- regular graph of order n_1 and H be r_2-regular graph of order n_2. Then,*

$$Ch[AD(G\times H)] = (\lambda - (r_1+r_2)^{r_1+r_2}(n_1n_2-1))(\lambda + (r_1+r_2)^{r_1+r_2})^{n_1n_2-1}$$

Proof. Since G is a r_1- regular graph of order n_1 and H is r_2-regular graph of order n_2, we have $G\times H$ as an (r_1+r_2)- regular graph with n_1n_2 vertices. Hence the result follows fron equation (4). □

Theorem 3.9. *Let G be a r_1- regular graph of order n_1 and H be a r_2- regular graph of order n_2. Then,*

$$Ch[AD(G[H])] = (\lambda + (n_2r_1+r_2)^{n_2r_1+r_2})^{n_1n_2-1}(\lambda - (n_2r_1+r_2)^{n_2r_1+r_2}(n_1n_2-1))$$

Proof. Since G is a r_1- regular graph of order n_1 and H is r_2-regular graph of order n_2, we have $G[H]$ as a $(n_2r_1+r_2)$- regular graph with n_1n_2 vertices. Hence the result follows fron equation (4). □

Theorem 3.10. *Let G be a r- regular graph of order n and size m. Then,*

$$Ch[AD(G\circ H)] = (\lambda + R_1^{R_1})^{n_1-1}(\lambda + R_2^{R_2})^{n_1n_2-1}[\lambda^2 - (R_2^{R_2}(n_1n_2-1) + R_1^{R_1}(n_1-1))\lambda + \frac{1}{4}(R_1^{R_1}R_2^{R_2}(n_1-1)(n_1n_2-1) - n_1^2n_2(R_1^{R_2} + R_2^{R_1})^2)]$$

Proof. Since G is a r_1 - regular graph of order n_1 and H is a r_2-regular graph of order n_2 then $G\circ H$ has two types of vertices, the n_1 vertices with degree $R_1 = r_1 + n_2$ and remaining n_1n_2 vertices with degree $R_2 = r_2 + 1$. Hence

$$AD(G\circ H) = \begin{bmatrix} R_1^{R_1}(J_{n_1} - I_{n_1}) & \frac{(R_1^{R_2}+R_2^{R_1})}{2} J_{n_1\times n_1n_2} \\ \frac{(R_1^{R_2}+R_2^{R_1})}{2} J_{n_1n_2\times n_1} & R_2^{R_2}(J_{n_1n_2} - I_{n_1n_2}) \end{bmatrix}.$$

$$Ch[AD(G\circ H)] =\mid \lambda I - AD(G\circ H)\mid$$

$$= \begin{vmatrix} (\lambda + R_1^{R_1})I_{n_1} - R_1^{R_1}J_{n_1} & -\frac{(R_1^{R_2}+R_2^{R_1})}{2}J_{n_1\times n_1n_2} \\ -\frac{(R_1^{R_2}+R_2^{R_1})}{2}J_{n_1n_2\times n_1} & (\lambda + R_2^{R_2})I_{n_1n_2} - R_2^{R_2}J_{n_1n_2} \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.11. *If W_n is a wheel graph, then*

$$Ch[AD(W_n)] = (\lambda+27)^{n-2}[\lambda^2 - 27(n-2)\lambda - \frac{(n-1)(3^{n-1}+(n-1)^3)^2}{4}]$$

Proof. The graph W_n of order n has two types of verices namely, $(n-1)$ rim vertices are of degree 3 and central vertex has degree $(n-1)$. Hence,

$$AD(W_n) = \begin{bmatrix} 27(J_{n-1} - I_{n-1}) & \frac{(3^{n-1}+(n-1)^3)}{2}J_{(n-1)\times 1} \\ \frac{(3^{n-1}+(n-1)^3)}{2}J_{1\times(n-1)} & (n-1)^{n-1}(J_1 - I_1) \end{bmatrix}.$$

$$Ch[AD(W_n)] =\mid \lambda I - AD(W_n)\mid$$

$$= \begin{vmatrix} (\lambda+27)I_{n-1} - 27J_{n-1} & -\frac{(3^{n-1}+(n-1)^3)}{2}J_{(n-1)\times 1} \\ -\frac{(3^{n-1}+(n-1)^3)}{2}J_{1\times(n-1)} & (\lambda+(n-1)^{n-1})I_1 - (n-1)^{n-1}J_1 \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.12. *If F_t^3 be an friendship graph, then*

$$Ch[AD(F_t^3)] = (\lambda+4)^{2t-1}[\lambda^2 - 4(2t-1)\lambda - \frac{2t(2^{2t}+(2t)^2)^2}{4}]$$

Proof. The graph F_t^3 of order $2t+1$ has two types of verices namely, $2t$ vertices of degree 2 and 1 vertex of degree $2t$. Hence,

$$AD(F_t^3) = \begin{bmatrix} 4(J_{2t} - I_{2t}) & \frac{(2^{2t}+(2t)^2)}{2}J_{2t\times 1} \\ \frac{(2^{2t}+(2t)^2)}{2}J_{1\times 2t} & (2t)^{2t}(J_1 - I_1) \end{bmatrix}.$$

$$Ch[F_t^3] =\mid \lambda I - AD(F_t^3)\mid$$

$$= \begin{vmatrix} (\lambda+4)I_{2t} - 4J_{2t} & -\frac{(2^{2t}+(2t)^2)}{2}J_{2t\times 1} \\ -\frac{(2^{2t}+(2t)^2)}{2}J_{1\times 2t} & (\lambda+(2t)^{2t})I_1 - (2t)^{2t}J_1 \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.13. *If $H_n - c$ is a helm without central vertex, then*

$$Ch[AD(H_n - c)] = (\lambda+27)^{n-2}(\lambda+1)^{n-2}[\lambda^2 - 28(n-2)\lambda + 27(n-2)^2 - 4(n-1)^2$$

Proof. The graph $H_n - c$ with order $2(n-1)$ having two types of vertices namely, $(n-1)$ vertices has degree 3 and remaining $(n-1)$ vertices has degree 1. Hence,

$$AD(H_n - c) = \begin{bmatrix} 27(J_{n-1} - I_{n-1}) & 2J_{(n-1)\times(n-1)} \\ 2J_{(n-1)\times(n-1)} & (J_{n-1} - I_{n-1}) \end{bmatrix}.$$

$$Ch[AD(H_n - c)] = \mid \lambda I - AD(H_n - c) \mid$$

$$= \begin{vmatrix} (\lambda+27)I_{n-1} - 27J_{n-1} & -2J_{(n-1)\times(n-1)} \\ -2J_{(n-1)\times(n-1)} & (\lambda+1)I_{n-1} - J_{n-1} \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.14. *If $H_n^{'} - c$ is a closed helm without central vertex, then,*

$$Ch[AD(H_n^{'} - c)] = (\lambda - 27(2n-3))(\lambda + 27)^{2n-3}$$

Proof. The closed helm without central vertex $H_n^{'} - c$ is 3 - regular graph with $2(n-1)$ vertices. Hence the result follows fron equation (4). □

Theorem 3.15. *If $SF_n - c$ is a sun flower graph without central vertex, then*

$$Ch[AD(SF_n - c)] = (\lambda+27)^{n-2}(\lambda+4)^{n-2}[\lambda^2 - 31(n-2)\lambda + 108(n-2)^2 - \frac{(n-1)^2 289}{4}]$$

Proof. The sun flower graph $SF_n - c$ without central vertex is a graph of order $2(n-1)$, which has two types of vertices namely, $(n-1)$ vertices has degree 3 and the remaining $(n-1)$ vertices has degree 2. Hence,

$$AD(SF_n - c) = \begin{bmatrix} 27(J_{n-1} - I_{n-1}) & \frac{17}{2}J_{(n-1)\times(n-1)} \\ \frac{17}{2}J_{(n-1)\times(n-1)} & 4(J_{n-1} - I_{n-1}) \end{bmatrix}.$$

$$Ch[AD(SF_n - c)] = \mid \lambda I - AD(SF_n - c) \mid$$

$$= \begin{vmatrix} (\lambda+27)I_{n-1} - 27J_{n-1} & -\frac{17}{2}J_{(n-1)\times(n-1)} \\ -\frac{17}{2}J_{(n-1)\times(n-1)} & (\lambda+4)I_{n-1} - 4J_{n-1} \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.16. *If DC_n is a double cone, then,*

$$Ch[AD(C_n)] = (\lambda+256)^{n-1}(\lambda+n^n)[\lambda^2 - (n^n + 256(n-1))\lambda + 256n^n(n-1) - \frac{n(4^n+n^4)^2}{2}]$$

Proof. The double cone is a graph of of order $(n+2)$ has two types of vertices namely, n vertices having degree 4 and the remaining 2 vertices having degree n. Hence,

$$AD(DC_n) = \begin{bmatrix} 256(J_n - I_n) & \frac{(4^n+n^4)}{2}J_{n\times 2} \\ \frac{(4^n+n^4)}{2}J_{2\times n} & n^n(J_2 - I_2) \end{bmatrix}.$$

$$Ch[AD(DC_n)] = \mid \lambda I - AD(DC_n) \mid$$
$$= \begin{vmatrix} (\lambda+256)I_n - 256J_n & -\frac{(4^n+n^4)}{2}J_{n\times 2} \\ -\frac{(4^n+n^4)}{2}J_{2\times n} & (\lambda+n^n)I_2 - n^nJ_2 \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.17. *If B_b is a book graph, then*

$$Ch[AD(B_b)] = (\lambda+4)^{2b-1}(\lambda+(b+1)^{b+1})[\lambda^2 - ((b+1)^{b+1} + 4(2b-1))\lambda$$
$$+4(2b-1)(b+1)^{b+1} - b(2^{b+1} + (b+1)^2)^2]$$

Proof. The Book graph B_b of order $2b+2$ has two types of vertices namely, $2b$ vertices with degree 2 and 2 vertices are with degree $b+1$. Hence,

$$AD(B_b) = \begin{bmatrix} 4(J_{2b} - I_{2b}) & \frac{2^{b+1}+(b+1)^2}{2}J_{2b\times 2} \\ \frac{2^{b+1}+(b+1)^2}{2}J_{2\times 2b} & (b+1)^{b+1}(J_2 - I_2) \end{bmatrix}.$$

$$Ch[AD(B_b)] = \mid \lambda I - AD(B_b) \mid$$
$$= \begin{vmatrix} (\lambda+4)I_{2b} - 4J_{2b} & -\frac{2^{b+1}+(b+1)^2}{2}J_{2b\times 2} \\ -\frac{2^{b+1}+(b+1)^2}{2}J_{2\times 2b} & (\lambda+(b+1)^{b+1})I_2 - (b+1)^{b+1}J_2 \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.18. *B_t is a book with triangular pages, then*

$$Ch[AD(B_t)] = (\lambda+4)^{t-1}(\lambda+(t+1)^{t+1})[\lambda^2 - ((t+1)^{t+1} + 4(t-1))\lambda$$
$$+4(t-1)(t+1)^{t+1} - \frac{t(2^{t+1} + (t+1)^2)^2}{2}]$$

Proof. The book B_t with triangular pages of order $t+2$ has two types of vertices with t vertices having degree 2 and the remaining 2 vertices having degree $t+1$. Hence,

$$AD(B_t) = \begin{bmatrix} 4(J_t - I_t) & \frac{2^{t+1}+(t+1)^2}{2}J_{t\times 2} \\ \frac{2^{t+1}+(t+1)^2}{2}J_{2\times t} & (t+1)^{t+1}(J_2 - I_2) \end{bmatrix}.$$

$$Ch[AD(B_t)] = \mid \lambda I - AD(B_t) \mid$$
$$= \begin{vmatrix} (\lambda+4)I_t - 4J_t & -\frac{2^{t+1}+(t+1)^2}{2}J_{t\times 2} \\ -\frac{2^{b+1}+(b+1)^2}{2}J_{2\times t} & (\lambda+(t+1)^{t+1})I_2 - (t+1)^{t+1}J_2 \end{vmatrix}.$$

□

Theorem 3.19. *If L_n is a ladder graph, then*

$$Ch[AD(L_n)] = (\lambda+27)^{2n-5}(\lambda+4)^3[\lambda^2 + -(27(2n-5)+12)\lambda + 324(2n-5) - 289(2n-4)$$

Proof. The graph L_n is a ladder graph of order $2n$ and has two types of vertices. There 2n-4 vertices has degree 3 and 4 vertices has degree 2. Hence,

$$AD(L_n) = \begin{bmatrix} 27(J_{2n-4} - I_{2n-4}) & \frac{17}{2} J_{(2n-4)\times 4} \\ \frac{17}{2} J_{4\times(2n-4)} & 4(J_4 - I_4) \end{bmatrix}.$$

$$Ch[AD(L_n)] = | \lambda I - AD(L_n) |$$
$$= \begin{vmatrix} (\lambda+27)I_{2n-4} - 27J_{2n-4} & -\frac{17}{2} J_{(2n-4)\times 4} \\ -\frac{17}{2} J_{4\times(2n-4)} & (\lambda+4)I_4 - 4J_4 \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.20. *If Pr_n is a prism graph, then*

$$Ch[AD(Pr_n)] = (\lambda + 27)^{2n-1}(\lambda - 27(2n-1))$$

Proof. The prism Pr_n is 3-regular graph with $2n$ vertices. Hence, the result follows from equation (4).

□

Theorem 3.21. *If T_n is a triangular snake, then*

$$Ch[AD(T_n)] = (\lambda+4)^n(\lambda+256)^{n-3}[\lambda^2-(256(n-3)+4n)\lambda+1024n(n-3)-256(n+1)(n-2)]$$

Proof. The triangular snake T_n has two types of vertices with n+1 vertices having degree 2 and the remaining $n-2$ vertices having degree 4. Hence,

$$AD(T_n) = \begin{bmatrix} 4(J_{n+1} - I_{n+1}) & 16J_{(n+1)\times(n-2)} \\ 16J_{(n-2)\times(n+1)} & 256(J_{n-2} - I_{n-2}) \end{bmatrix}.$$

$$Ch[AD(T_n)] = | \lambda I - AD(T_n) |$$
$$= \begin{vmatrix} (\lambda+4)I_{n+1} - 4J_{n+1} & -16J_{(n+1)\times(n-2)} \\ -16J_{(n-2)\times(n+1)} & (\lambda+256)I_{n-2} - 256J_{n-2} \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.22. *If Q_n is a quadrilateral snake, then*

$$Ch[AD(Q_n)] = (\lambda+4)^{2n-1}(\lambda+256)^{n-3}[\lambda^2 - (256(n-3)+4(2n-1))\lambda + 1024(2n-1)(n-3) - 512n(n-2)]$$

Proof. The quadrilateral snake Q_n of degree $3n-2$ has two types of vertices with 2n vertices having degree 2 and the remaining $n-2$ vertices having degree 4. Hence,

$$AD(Q_n) = \begin{bmatrix} 4(J_{2n} - I_{2n}) & 16J_{(2n)\times(n-2)} \\ 16J_{(n-2)\times(2n)} & 256(J_{n-2} - I_{n-2}) \end{bmatrix}.$$

$$Ch[AD(Q_n)] = | \lambda I - AD(Q_n) |$$
$$= \begin{vmatrix} (\lambda+4)I_{2n} - 4J_{2n} & -16J_{(2n)\times(n-2)} \\ -16J_{(n-2)\times(2n)} & (\lambda+256)I_{n-2} - 256J_{n-2} \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.23. *If G is an r-regular graph of order n, then*

$$Ch[AD(J(G))] = (\lambda + r_1^{r_1}(\frac{nr}{2} - 1))(\lambda - r_1^{r_1})^{(\frac{nr}{2}-1)} \quad where, \quad r_1 = \frac{(n-4)r}{2} + 1$$

Proof. The jump graph $J(G)$ is r-regular graph is $r_1 = (\frac{(n-4)r}{2} + 1)$-regular graph with $\frac{nr}{2}$ vertices. Hence, the result follows from equation (4). □

Theorem 3.24. *If S_n is a Star graph, then*

$$Ch[AD(S_n)] = (\lambda + 1)^{n-2}[\lambda^2 - (n-2)\lambda - \frac{(n-1)n^2}{4}]$$

Proof. The graph S_n of order n has two types of verices, namely, $n-1$ rim vertices having degree 1 and central vertex has degree $n-1$. Hence,

$$AD(S_n) = \begin{bmatrix} (J_{n-1} - I_{n-1}) & \frac{n}{2} J_{(n-1)\times 1} \\ \frac{n}{2} J_{1\times(n-1)} & (n-1)^{n-1}(J_1 - I_1) \end{bmatrix}.$$

$$Ch[AD(S_n)] = \mid \lambda I - AD(S_n) \mid$$

$$= \begin{vmatrix} (\lambda+1)I_{n-1} - J_{n-1} & -\frac{n}{2} J_{(n-1)\times 1} \\ -\frac{n}{2} J_{1\times(n-1)} & (\lambda + (n-1)^{n-1})I_1 - (n-1)^{n-1} J_1 \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.25. *If $S_{n,n}$ is a double star graph, then*

$$Ch[AD(S_{n,n})] = (\lambda+1)^{2n-3}(\lambda+n^n)[\lambda^2 - ((2n-3)+n^n)\lambda + (2n-3)n^n - (n-1)(n+1)^2]$$

Proof. The graph $S_{n,n}$ of order $2n$ has two types of vertices, with $2n-2$ vertices having degree 1 and remaining two vertices having degree n. Hence,

$$AD(S_{n,n}) = \begin{bmatrix} (J_{2n-2} - I_{2n-2}) & \frac{n+1}{2} J_{(2n-2)\times 2} \\ \frac{n+1}{2} J_{2\times(2n-2)} & n^n(J_2 - I_2) \end{bmatrix}.$$

$$Ch[AD(S_{n,n})] = \mid \lambda I - AD(S_{n,n}) \mid$$

$$= \begin{vmatrix} (\lambda+1)I_{2n-2} - J_{2n-2} & -\frac{(n+1)}{2} J_{(2n-2)\times 2} \\ -\frac{(n+1)}{2} J_{2\times(2n-2)} & (\lambda + n^n)I_2 - n^n J_2 \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.26. *If $K_{m,n}$ is a complete biparite graph, then*

$$Ch[AD(K_{m,n})] = (\lambda + n^n)^{m-1}(\lambda + m^m)^{n-1}[\lambda^2 - (m^m(n-1) + n^n(m-1))\lambda + (m-1)(n-1)m^m n^n - \frac{mn(m^n + n^m)^2}{4}]$$

Proof. The graph $K_{m,n}$ of order $m+n$ has two types of vertices, with m vertices having degree n and n vertices having degree m. Hence,

$$AD(K_{m,n}) = \begin{bmatrix} n^n(J_m - I_m) & \frac{m^n+n^m}{2}J_{m\times n} \\ \frac{m^n+n^m}{2}J_{n\times m} & m^m(J_n - I_n) \end{bmatrix}.$$

$$Ch[AD(K_{m,n})] = \mid \lambda I - AD(K_{m,n}) \mid$$

$$= \begin{vmatrix} (\lambda + n^n)I_m - n^n J_m & -\frac{m^n+n^m}{2}J_{m\times n} \\ -\frac{m^n+n^m}{2}J_{m\times n} & (\lambda + m^m)I_n - m^m J_n \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.27. *If P_n is a path graph, then*

$$Ch[AD(P_n)] = (\lambda+4)^{n-3}(\lambda+1)[\lambda^2 - (4(n-3)+1)\lambda + 4(n-3) - \frac{9(n-2)}{2}]$$

Proof. The graph P_n of order n has two types of vertices, with $n-2$ vertices having degree 2 and remaining two end vertices having degree 1. Hence,

$$AD(P_n) = \begin{bmatrix} 4(J_{n-2} - I_{n-2}) & \frac{3}{2}J_{(n-2)\times 2} \\ \frac{3}{2}J_{2\times(n-2)} & (J_2 - I_2) \end{bmatrix}.$$

$$Ch[AD(P_n)] = \mid \lambda I - AD(P_n) \mid$$

$$= \begin{vmatrix} (\lambda+4)I_{n-2} - 4J_{n-2} & -\frac{3}{2}J_{(n-2)\times 2} \\ -\frac{3}{2}J_{2\times(n-2)} & (\lambda+1)I_2 - J_2 \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

A dumbbell is the graph obtained from two disjoint cycles by joining them by a path.

Theorem 3.28. *If $D_{n,n}$ is a dumbbell graph, then*

$$Ch[ED(D_{n,n})] = (\lambda+4)^{2n-3}(\lambda+27)[\lambda^2 - (4(2n-3)+27)\lambda + 108(2n-3) - \frac{578(n-1)}{4}]$$

Proof. The graph $D_{n,n}$ of order $2n$ has two types of vertices, with $2n-2$ vertices having degree 2 and remaining two having degree 3. Hence,

$$ED(D_{n,n}) = \begin{bmatrix} 4(J_{2n-2} - I_{2n-2}) & \frac{17}{2}J_{(2n-2)\times 2} \\ \frac{17}{2}J_{2\times(2n-2)} & 27(J_2 - I_2) \end{bmatrix}.$$

$$Ch[ED(D_{n,n})] = \mid \lambda I - ED(D_{n,n}) \mid$$

$$= \begin{vmatrix} (\lambda+4)I_{2n-2} - 4J_{2n-2} & -\frac{17}{2}J_{(2n-2)\times 2} \\ -\frac{17}{2}J_{2\times(2n-2)} & (\lambda+27)I_2 - 27J_2 \end{vmatrix}.$$

Now by using Lemma 2.19, we get the desired result. □

4 Conclusion

We conclude with the following observations.

In this paper, we have obtained the characteristic polynomial of the average degree exponent sum matrix of graphs obtained by some graphs operations. Also, bounds for both largest average degree exponent sum eigenvalue and average degree exponent sum energy of graphs are established.

References

[1] M. Aouchiche and P. Hansen, *Distance spectra of graphs:* A survey, Linear Algebra Appl., 458(2014), 301-386.

[2] S. Bernard and J. M. Child, *Higher Algebra*, Macmillan India Ltd., New Delhi, (2001).

[3] B. Basavanagoud and E. Chitra *Degree square sum energy of graphs*, Int. J. Math. And Appl.,6(2-B)(2018), 193-205.

[4] A. E. Brouwer and W. H. Haemers, *Spectra of Graphs*, Springer, Berlin, (2012).

[5] M. Biernacki, H. Pidek and C. Ryll-Nardzewsk, *Sur une ine galite entre des integrals definies.*, Maria Curie kACodowska Univ., A4(1950), 1-4.

[6] F. Buckley, *Iterated line graphs*, Congr. Numer., 33(1981), 390-394.

[7] F. Buckley, *The size of iterated line graphs*, Graph Theory Notes New York, 25(1993), 33-36.

[8] G. Chartrand, H. Hevia, E. B. Jarrett and M. Schultz, *Subgraph distances in graphs defined by edge transfers*, Discrete Math. 170 (1997), 63-79.

[9] D. Cvetkovic, M. Doob and H. Sachs, *Spectra of Graphs-Theory and Applications*, Academic Press, New York, (1980).

[10] D. Cvetkovic, P. Rowlinson and S. K. Simic, *Eigenvalue bounds for the signless Laplacian*, Publ. Inst. Math.(Beograd), 81(2007), 11-27.

[11] J. B. Diaz and F. T. Metcalf, *Stronger forms of a class of inequalities of G.* Poly-G.Szego and L. V. Kantorovich.,Bulletin of the AMS, 60(2003), 415-418.

[12] S. Gong, X. Li, G. Xu, I. Gutman, B. Furtula, *Borderenergetic graphs*, MATCH Commun. Math. Comput. Chem., 74(2015), 321–332.

[13] I. Gutman, *The energy of a graph*, Ber. Math. Statist. Sekt. Forschungsz. Graz 103 (1978), 1-22.

[14] I. Gutman, *Hyperenergetic molecular graphs*, J. Serbian Chem. Soc., 64(1999), 199–205.

[15] F. Harary, *Graph Theory*, Addison-Wesely, Reading, Mass, (1969).

[16] S. R. Jog, S. P. Hande and D. S. Revankar, *Degree sum polynomial of graph valued functions on regular graphs*, Int. J.Graph Theory, 1(2013), 108-115.

[17] B. Mohar, *The Laplacian spectrum of graphs*, in: Y. Alavi, G. Chartrand, O. R. Ollermann, A. J. Schwenk (Eds.), *Graph Theory*, Combinatorics and Applications, Wiley, New York, 2(1991), 871-898.

[18] N. Ozeki, *On the estimation of inequalities by maximum and minimum values.*, J. College Arts and Science, Chiba Univ., 5(1968), 199-203.

[19] G. Polya and G. Szego, *Problems and Theorems in analysis.*, Series, Integral calculus, Theory of Functions, Springer, Berlin, (1972).

[20] H. S. Ramane, D. S. Revankar and J. B. Patil, *Bounds for the degree sum eigenvalues and degree sum energy of a graph*, Int. J. Pure Appl. Math. Sci., 6(2013), 161-167.

[21] H. S. Ramane and S. S. Shinde, *Degree exponent polynomial of graphs obtained by some graph operations*, Electronic Notes in Discrete Math., 63(2017), 161-168.

[22] E. Sampathkumar and S. B. Chikkodimath, *Semitotal graphs of a graph-I*, J. Karnatak Univ. Sci., 18(1973), 274-280. 205