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A NOTE ON THE MODIFIED TYPE 2 DEGENERATE POLY-CHANGHEE-GENOCCHI NUMBERS AND POLYNOMIALS

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ABSTRACT. In this paper, we introduce the modified type 2 degenerate poly-Changhee-Genocchi numbers and polynomials, and derive several explicit expressions and some identities for those numbers and polynomials. In particular, we provide interesting identities related to the Chnaghee-Genocchi polynomials and numbers of the second kind.

1. INTRODUCTION

As is well known, the Genocchi polynomials are defined by the generating function to be

$$\frac{2t}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!},\quad \text{(see [21, 24])}. \tag{1}$$

When $x=0$, $G_n=G_n(0)$ are called the Genocchi numbers. From (1), we note that

$$G_0(x)=0,\ \ E_n(x)=\frac{G_{n+1}(x)}{n+1},(n\geq 0),\quad \text{(see [21, 24])}, \tag{2}$$

where $E_n(x)$ are the ordinary Euler polynomials which are given by the generating function to be

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},\quad \text{(see [2, 25])}. \tag{3}$$

The Changhee polynomials have been defined by the generating function to be

$$\frac{2}{t+2}(1+t)^x=\sum_{n=0}^{\infty}Ch_n(x)\frac{t^n}{n!},\quad \text{(see [12, 18, 20, 22, 23, 26])}. \tag{4}$$

When $x=0$ $Ch_n=Ch_n(0)$ are called the Changhee numbers. From (4), it is easy to see that

$$Ch_n(x)=\sum_{k=0}^{n}E_k(x)S_1(n,k),\ (n\geq 0),\quad \text{(see [1, 12, 18, 27])}, \tag{5}$$

where $S_1(n,k)$ are called the Stirling numbers of the first kind as follows

$$(x)_n=\sum_{l=0}^{n}S_1(n,l)x^l,\quad (n\geq 0),\qquad \text{(see [1, 2, 3, 9, 10, 11, 17])}. \tag{6}$$

The Bernoulli numbers of the second kind are defined by the generating function to be

$$\frac{t}{\log(1+t)}=\sum_{n=0}^{\infty}b_n\frac{t^n}{n!},\quad \text{(see [5, 8, 15])}. \tag{7}$$

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From (7), we note that

$$\left(\frac{t}{\log(1+t)}\right)^{r}(1+t)^{x-1}=\sum_{n=0}^{\infty}B_{n}^{(n-r+1)}(x)\frac{t^{n}}{n!},\quad \text{(see [5, 8, 15])}. \tag{8}$$

where $B_n^{(r)}(x)$ are the higher-order Bernoulli polynomials defined by the generating function to be

$$\left(\frac{t}{e^{t}-1}\right)^{r}e^{xt}=\sum_{n=0}^{\infty}B_{n}^{(r)}(x)\frac{t^{n}}{n!},\quad \text{(see [5, 8, 15])}. \tag{9}$$

When $x=1$ and $r=1$, we get

$$b_{n}=B_{n}^{(n)}(1),\quad \text{(see [5, 8, 15])}. \tag{10}$$

The Changhee-Genocchi polynomials are defined by the generating function to be

$$\frac{2\log(1+t)}{2+t}(1+t)^{x}=\sum_{n=0}^{\infty}CG_{n}(x)\frac{t^{n}}{n!},\quad \text{(see [16, 21])}. \tag{11}$$

The degenerate Changhee-Genocchi polynomials are defined by the generating function to be

$$\frac{2\lambda\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)}{2\lambda+\log(1+\lambda t)}\left(1+\lambda^{-1}\log(1+\lambda t)\right)^{x}=\sum_{n=0}^{\infty}CG_{n,\lambda}(x)\frac{t^{n}}{n!},\quad \text{(see [11, 21])}. \tag{12}$$

From (11) and (12), we note that

$$\lim_{\lambda\to 0}CG_{n,\lambda}(x)=CG_{n}(x),\ (n\geq 0),\quad \text{(see [16, 21])}. \tag{13}$$

The modified degenerate Changhee-Genocchi polynomials are defined by the generating function to be

$$\frac{2\lambda t}{2\lambda+\log(1+\lambda t))}\left(1+\lambda^{-1}\log(1+\lambda t)\right)^{x}=\sum_{n=0}^{\infty}CG_{n,\lambda}^{*}(x)\frac{t^{n}}{n!},\quad \text{(see [16, 21, 19])}. \tag{14}$$

Note that

$$\lim_{\lambda\to 0}CG_{n,\lambda}^{*}(x)=CG_{n}^{*}(x),\ (n\geq 0),\quad \text{(see [16, 21])}. \tag{15}$$

Here, $CG_n^*(x)$ are called the modified Changhee-Genocchi polynomials which are given by the generating function to be

$$\frac{2t}{2+t}(1+t)^{x}=\sum_{n=0}^{\infty}CG_{n}^{*}(x)\frac{t^{n}}{n!},\quad \text{(see [16, 21])}. \tag{16}$$

The degenerate Changhee polynomials are given by

$$\frac{2}{2+\lambda^{-1}\log(1+\lambda t)}(1+\lambda^{-1}\log(1+\lambda t))^{x}=\sum_{n=0}^{\infty}Ch_{n,\lambda}(x)\frac{t^{n}}{n!},\quad \text{(see [1, 13])}. \tag{17}$$

The degenerate exponential function is defined by

$$e_{\lambda}^{x}(t)=(1+\lambda t)^{\frac{x}{\lambda}},\ \ e_{\lambda}(t)=e_{\lambda}^{1}(t)=(1+\lambda t)^{\frac{1}{\lambda}},\quad \text{(see [6])}. \tag{18}$$

and the compositional inverse $\log_\lambda(t)$ of $e_\lambda(t)$ is given by $\log_\lambda(e_\lambda(t))=e_\lambda(\log_\lambda(t))=t$. Note that $\log_\lambda(t)=\frac{1}{\lambda}(t^\lambda-1)$.

Recently, Kim-Kim-Kwon-Lee [5] introduced type 2 degenerate poly-Bernoulli numbers and polynomials. In this paper, we consider the modified type 2 degenerate poly-Changhee-Genocchi numbers and polynomials, and derive several explicit expressions and some identities for those umbers and polynomials. In particular, we provide interesting identities related to the Chnaghee-Genocchi polynomials and numbers of the second kind.

2. The modified type 2 degenerate poly-Changhee-Genocchi numbers and polynomials

For $k \in \mathbb{Z}$, the polyexponential function is defined by

$$Ei_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!n^k}, \ (|x| < 1) \quad \text{(see [4, 6, 7, 28])}. \tag{19}$$

By (19), we see that $Ei_1(x) = e^x - 1$. For $k \in \mathbb{Z}$, the degenerate modified polyexponential function is given by

$$Ei_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda} x^n}{(n-1)!n^k}, \ (|x| < 1) \quad \text{(see [4, 6, 28])}. \tag{20}$$

Note that $Ei_{1,\lambda}(x) = e_\lambda(x) - 1$. From (20), we note that

$$\frac{d}{dx} Ei_{k,\lambda}(x) = \frac{1}{x} \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda} x^n}{(n-1)!n^{k-1}} = \frac{1}{x} Ei_{k-1,\lambda}(x). \tag{21}$$

For $k \geq 2$, by (21), we have

$$\begin{aligned} Ei_{k,\lambda}(x) &= \int_0^x \underbrace{\frac{1}{t} \int_0^t \frac{1}{t} \int_0^t \cdots \frac{1}{t}}_{(k-2) times} \int_0^t \frac{1}{t} Ei_{1,\lambda}(t) dt \cdots dt dt dt \\ &= \int_0^x \underbrace{\frac{1}{t} \int_0^t \frac{1}{t} \int_0^t \cdots \frac{1}{t}}_{(k-2) times} \int_0^t \frac{1}{t} (e_\lambda(t) - 1) dt \cdots dt dt dt \end{aligned}$$

$$. \tag{22}$$

Recently, Kim–Kim–Kwon–Lee[5] introduced the type 2 degenerate poly–Bernoulli polynomials which are given by the generating function to be

$$\frac{Ei_{k,\lambda}(\log_\lambda(1+t))}{e_\lambda(t) - 1} e_\lambda^x(t) = \sum_{n=0}^{\infty} B_{n,\lambda}^{(k)}(x) \frac{t^n}{n!}. \tag{23}$$

When $x = 0$, $B_{n,\lambda}^{(k)} = B_{n,\lambda}^{(k)}(0)$ are called type 2 degenerate poly–Bernoulli numbers. In view of (14) and (23), using the degenerate modified polyexponential function, we define the modified type 2 degenerate poly–Changhee–Genocchi polynomials which are given by the generate function to be

$$\frac{Ei_{k,\lambda}(\log_\lambda(1+2t))}{2 + \lambda^{-1}\log(1+\lambda t)} (1 + \lambda^{-1}\log(1+\lambda t))^x = \sum_{n=0}^{\infty} CG_{n,\lambda}^{(k,*)}(x) \frac{t^n}{n!}. \tag{24}$$

When $x=0, CG_{n,\lambda}^{(k,*)}=CG_{n,\lambda}^{(k,*)}(0)$ are called the modified type 2 degenerate poly–Changhee–Genocchi numbers. From(24), we note that

$$\begin{aligned}&\frac{Ei_{1,\lambda}(\log_\lambda(1+2t))}{2+\lambda^{-1}\log(1+\lambda t)}(1+\lambda^{-1}\log(1+\lambda t))^x\\ &=\frac{2t}{2+\lambda^{-1}\log(1+\lambda t)}(1+\lambda^{-1}\log(1+\lambda t))^x.\end{aligned}\tag{25}$$

From (25), for $k=1$, we have that $CG_{n,\lambda}^{(1,*)}(x)=CG_{n,\lambda}^{*}(x)$. By (24), for $x=0$, we note that

$$\begin{aligned}Ei_{k,\lambda}(\log_\lambda(1+2t))&=\left(2+\lambda^{-1}\log(1+\lambda t)\right)\left(\sum_{l=0}^{\infty}CG_{l,\lambda}^{(k,*)}\frac{t^l}{l!}\right)\\ &=\left(2+\sum_{s=1}^{\infty}\frac{(-1)^{s-1}\lambda^{s-1}}{s}t^s\right)\left(\sum_{l=0}^{\infty}CG_{l,\lambda}^{(k,*)}\frac{t^l}{l!}\right)\\ &=\sum_{n=1}^{\infty}2CG_{n,\lambda}^{(k,*)}\frac{t^n}{n!}+\left(\sum_{s=1}^{\infty}(-1)^{s-1}\lambda^{s-1}(s-1)!\frac{t^s}{s!}\right)\left(\sum_{l=0}^{\infty}CG_{n,\lambda}^{(k,*)}\frac{t^l}{l!}\right)\\ &=\sum_{n=1}^{\infty}\left(2CG_{l,\lambda}^{(k,*)}+\sum_{s=1}^{n}\binom{n}{s}(s-1)!(-1)^{s-1}\lambda^{s-1}CG_{n-s,\lambda}^{(k,*)}\right)\frac{t^n}{n!}\end{aligned}\tag{26}$$

and

$$\begin{aligned}Ei_{k,\lambda}(\log_\lambda(1+2t))&=\sum_{m=1}^{\infty}\frac{(1)_{m,\lambda}(\log_\lambda(1+2t))^m}{(m-1)!m^k}\\ &=\sum_{m=1}^{\infty}\frac{(1)_{m,\lambda}}{m^{k-1}}\frac{1}{m!}(\log_\lambda(1+2t))^m\\ &=\sum_{m=1}^{\infty}\frac{(1)_{m,\lambda}}{m^{k-1}}\sum_{n=m}^{\infty}S_{1,\lambda}(n,m)\frac{(2t)^n}{n!}\\ &=\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\frac{(1)_{m,\lambda}S_{1,\lambda}(n,m)2^n}{m^{k-1}}\right)\frac{t^n}{n!}.\end{aligned}\tag{27}$$

Therefore, by (26) and (27), we obtain the following theorem.

Theorem 2.1. *For $n\in\mathbb{N}$ and $k\in\mathbb{Z}$, we have*

$$\begin{aligned}&2CG_{n,\lambda}^{(k,*)}+\sum_{s=1}^{n}\binom{n}{s}(s-1)!(-1)^{s-1}\lambda^{s-1}CG_{n-s,\lambda}^{(k,*)}\\ &=\sum_{m=1}^{n}\frac{(1)_{m,\lambda}S_{1,\lambda}(n,m)2^n}{m^{k-1}}.\end{aligned}\tag{28}$$

From Theorem 2.1 with $k=1$, we get

$$\begin{aligned}&2CG_{n,\lambda}^{(1,*)}+\sum_{s=1}^{n}\binom{n}{s}(s-1)!(-1)^{s-1}\lambda^{s-1}CG_{n-s,\lambda}^{(1,*)}\\ &=2CG_{l,\lambda}^{(*)}+\sum_{s=1}^{n}\binom{n}{s}(s-1)!(-1)^{s-1}\lambda^{s-1}CG_{n-s,\lambda}^{(*)},\end{aligned}\tag{29}$$

and

$$\sum_{m=1}^{n}(1)_{m,\lambda}S_{1,\lambda}(n,m)2^n = 2^n\left(\sum_{m=0}^{\infty}(1)_{m,\lambda}S_{1,\lambda}(n,n)2^n - S_{1,\lambda}(n,0)\right) = 2^n((1)_n - \delta_{n,0}), \tag{30}$$

where the degenerate Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{l=0}^{n} S_{1,\lambda}(n,l)(x)_{l,\lambda}.$$

From (29) and (30), we have the following corollary.

Corollary 2.2. *For $n \in \mathbb{N}$, we have*

$$2CG^{*}_{l,\lambda} + \sum_{l=1}^{n}\binom{n}{l}(l-1)!(-1)^{l-1}\lambda^{l-1}CG^{*}_{n-l,\lambda} = 2^n((1)_n - \delta_{n,0}). \tag{31}$$

From (22), we note that

$$\begin{aligned} Ei_{k,\lambda}(\log_\lambda(1+2t)) &= (e_\lambda(2t)-1)\sum_{l=0}^{\infty}B^{(k)}_{l,\lambda}\frac{(2t)^l}{l!} \\ &= \left(\sum_{m=0}^{\infty}\frac{(1)_{m,\lambda}}{m!}(2t)^m - 1\right)\left(\sum_{l=0}^{\infty}B^{(k)}_{l,\lambda}2^l\frac{t^l}{l!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}(1)_{n-m,\lambda}2^nB^{(k)}_{m,\lambda} - 2^nB^{(k)}_{n,\lambda}\right)\frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty}2^n\left(B^{(k)}_{n,\lambda}(1) - B^{(k)}_{n,\lambda}\right)\frac{t^n}{n}. \end{aligned} \tag{32}$$

By (32), (24) and (17), we observe that

$$\begin{aligned} &\sum_{n=0}^{\infty}CG^{(k,*)}_{n,\lambda}(x)\frac{t^n}{n!} \\ &= \frac{Ei_{k,\lambda}(\log_\lambda(1+2t))}{2+\lambda^{-1}\log(1+\lambda t)}(1+\lambda^{-1}\log(1+\lambda t))^x \\ &= \frac{1}{2}Ei_{k,\lambda}(\log_\lambda(1+2t))\frac{2}{2+\lambda^{-1}\log(1+\lambda t)}(1+\lambda^{-1}\log(1+\lambda t))^x \\ &= \left(\sum_{m=1}^{\infty}2^{m-1}\left(B^{(k)}_{m,\lambda}(1) - B^{(k)}_{m,\lambda}\right)\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}Ch_{l,\lambda}(x)\frac{t^l}{l!}\right) \\ &= \sum_{n=1}^{\infty}\left(\sum_{l=0}^{n-1}\binom{n}{l}2^{n-l-1}\left(B^{(k)}_{n-l,\lambda}(1) - B^{(k)}_{n-l,\lambda}\right)Ch_{l,\lambda}(x)\right)\frac{t^n}{n!}. \end{aligned} \tag{33}$$

From (33), we obtain the following theorem.

Theorem 2.3. *For $n \geq 1$, we have*

$$CG^{(k,*)}_{n,\lambda}(x) = \sum_{l=0}^{n-1}\binom{n}{l}2^{n-l-1}\left(B^{(k)}_{n-1,\lambda}(1) - B^{(k)}_{n-l,\lambda}\right)Ch_{l,\lambda}(x). \tag{34}$$

From (21) and (22), we note that

$$
\begin{aligned}
&\frac{d}{dx}Ei_{k,\lambda}\left(\log_\lambda(1+2x)\right)\\
&=\frac{d}{dx}\sum_{n=1}^{\infty}\frac{(1)_{n,\lambda}\left(\log_\lambda(1+2x)\right)^n}{(n-1)!n^k}\\
&=\frac{2(1+2x)^{\lambda-1}}{\log_\lambda(1+2x)}\cdot\sum_{n=1}^{\infty}\frac{(1)_{n,\lambda}\left(\log_\lambda(1+2x)\right)^n}{(n-1)!n^{k-1}}\\
&=\frac{2(1+2x)^{\lambda-1}}{\log_\lambda(1+2x)}Ei_{k-1,\lambda}\left(\log_\lambda(1+2x)\right).
\end{aligned}
\tag{35}
$$

By (35), for $k\geq 2$, we have

(36)

$$
Ei_{k,\lambda}\left(\log_\lambda(1+2x)\right)=\underbrace{\int_0^x\frac{2(1+2x)^{\lambda-1}}{\log_\lambda(1+2x)}\int_0^t\cdots\frac{2(1+2x)^{\lambda-1}}{\log_\lambda(1+2x)}}_{(k-2)times}\int_0^t\frac{2(1+2x)^{\lambda-1}}{\log_\lambda(1+2x)}2tdt\cdots dtdt.
$$

It is well known that the degenerate Bernoulli polynomials of the second kind are defined by

$$
\frac{t}{\log_\lambda(1+t)}(1+t)^x=\sum_{n=0}^{\infty}b_{n,\lambda}(x)\frac{t^n}{n!},\quad\text{(see [8]).}\tag{37}
$$

When $x=0$, $b_{n,\lambda}=b_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers of the second kind. From (24) and (36), we have

(38)

$$
\begin{aligned}
&\sum_{n=0}^{\infty}CG_{n,\lambda}^{(k,*)}\frac{x^n}{n!}=\frac{1}{2+\lambda^{-1}\log(1+\lambda x)}Ei_{k,\lambda}(\log_\lambda(1+2x))\\
&=\frac{1}{2+\lambda^{-1}\log(1+\lambda x)}\underbrace{\int_0^x\frac{2(1+2t)^{\lambda-1}}{\log_\lambda(1+2t)}\int_0^t\cdots\frac{2(1+2t)^{\lambda-1}}{\log_\lambda(1+2t)}}_{(k-2)times}\int_0^t\frac{2(1+2t)^{\lambda-1}}{\log_\lambda(1+2t)}2tdt\cdots dtdt\\
&=\frac{2x}{2+\lambda^{-1}\log(1+\lambda x)}\sum_{m=0}^{\infty}\sum_{m_1+\cdots+m_{k-1}=m}2^m\binom{m}{m_1\cdots m_{k-1}}\\
&\quad\times\frac{b_{m_1,\lambda}(\lambda-1)}{m_1+1}\cdot\frac{b_{m_2,\lambda}(\lambda-1)}{m_1+m_2+1}\cdots\frac{b_{m_{k-1},\lambda}(\lambda-1)}{m_1+\cdots+m_{k-1}+1}\cdot\frac{x^m}{m!}\\
&=\left(\sum_{l=0}^{\infty}CG_{l,\lambda}^{*}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\sum_{m_1+\cdots+m_{k-1}=m}2^m\binom{m}{m_1\cdots m_{k-1}}\right)\\
&\quad\times\frac{b_{m_1,\lambda}(\lambda-1)}{m_1+1}\cdot\frac{b_{m_2,\lambda}(\lambda-1)}{m_1+m_2+1}\cdots\frac{b_{m_{k-1},\lambda}(\lambda-1)}{m_1+\cdots+m_{k-1}+1}\frac{x^m}{m!}\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\sum_{m_1+\cdots+m_{k-1}=m}2^m\binom{m}{m_1\cdots m_{k-1}}\\
&\quad\times\frac{b_{m_1,\lambda}(\lambda-1)}{m_1+1}\cdot\frac{b_{m_2,\lambda}(\lambda-1)}{m_1+m_2+1}\cdots\frac{b_{m_{k-1},\lambda}(\lambda-1)}{m_1+\cdots m_{k-1}+1}CG_{n-m,\lambda}^{*}\frac{x^n}{n!}.
\end{aligned}
$$

Therefore, by (38), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$CG_{n,\lambda}^{(k,*)} = \sum_{m=0}^{n} \binom{n}{m} 2^m \sum_{m_1+\cdots+m_{k-1}=m} \binom{m}{m_1 \cdots m_{k-1}} \times \frac{b_{m_1,\lambda}(\lambda-1)}{m_1+1} \cdot \frac{b_{m_2,\lambda}(\lambda-1)}{m_1+m_2+1} \cdots \frac{b_{m_{k-1},\lambda}(\lambda-1)}{m_1+\cdots m_{k-1}+1} CG_{n-m}^{*}. \tag{39}$$

By replacing t by $\frac{1}{2}(e_\lambda(t)-1)$ in (24), we get

$$\begin{aligned} &\sum_{m=0}^{\infty} CG_{m,\lambda}^{(k,*)}(x) \frac{(\frac{1}{2}(e_\lambda(t)-1))^m}{m!} \\ &= \sum_{m=0}^{\infty} CG_{m,\lambda}^{(k,*)}(x) \left(\frac{1}{2^m} \sum_{n=m}^{\infty} S_{2,\lambda}(n,m) \frac{t^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} 2^{-m} CG_{m,\lambda}^{(k,*)}(x) S_{2,\lambda}(n,m) \right) \frac{t^n}{n!}, \end{aligned} \tag{40}$$

where the degenerate Stirling numbers of the second kind are given by

$$\frac{1}{m!}(e_\lambda(t)-1)^m = \sum_{l=m}^{\infty} S_{2,\lambda}(m,l) \frac{t^l}{l!}. \tag{41}$$

On the other hand,

$$\begin{aligned} &\sum_{m=0}^{\infty} CG_{n,\lambda}^{(k,*)}(x) \frac{(\frac{1}{2}(e_\lambda(t)-1))^m}{m!} \\ &= \frac{Ei_{k,\lambda}(t)}{2+\lambda^{-1}\log(1+\frac{\lambda}{2}(e_\lambda(t)-1))} \left(1+\lambda^{-1}\log\left(1+\frac{\lambda}{2}(e_\lambda(t)-1\right)\right)^x \\ &= \frac{1}{2} \cdot \frac{2\left(1+\lambda^{-1}\log\left(1+\frac{\lambda}{2}(e_\lambda(t)-1\right)\right)^x Ei_{k,\lambda}(t)}{2+\lambda^{-1}\log(1+\frac{\lambda}{2}(e_\lambda(t)-1))} \\ &= \frac{1}{2} \sum_{m=0}^{\infty} Ch_{m,\lambda}(x) \frac{\left(\frac{1}{2}(e_\lambda(t)-1)\right)^m}{m!} \sum_{l=1}^{\infty} \frac{(1)_{l,\lambda}}{l^{k-1}} \cdot \frac{t^l}{l!} \\ &= \left(\frac{1}{2} \sum_{m=0}^{\infty} Ch_{m,\lambda}(x) 2^{-m} \sum_{i=m}^{\infty} S_{2,\lambda}(i,m) \frac{t^i}{i!} \right) \left(\sum_{l=0}^{\infty} \frac{(1)_{l+1,\lambda}}{(l+1)^{k-1}} \frac{t^{l+1}}{(l+1)!} \right) \\ &= \left(\sum_{i=0}^{\infty} \left(\sum_{m=0}^{i} Ch_{m,\lambda}(x) 2^{-m-1} S_{2,\lambda}(i,m) \right) \frac{t^i}{i!} \right) \left(\sum_{l=0}^{\infty} \frac{(1)_{l+1,\lambda}}{(l+1)^{k-1}} \frac{t^{l+1}}{(l+1)!} \right) \\ &= \sum_{n=1}^{\infty} \left(\sum_{l=0}^{n-1} \sum_{m=0}^{n-l-1} \binom{n}{l+1} Ch_{m,\lambda}(x) 2^{-m-1} S_{2,\lambda}(n-l-1,m) \frac{(1)_{l+1,\lambda}}{(l+1)^{k-1}} \right) \frac{t^n}{n!}. \end{aligned} \tag{42}$$

Thereforem by (40) and (42), we obtain the following theorem.

Theorem 2.5. *For $n \geq 1$, we have*

$$\begin{aligned} &\sum_{m=0}^{n} 2^{-m} CG_{m,\lambda}^{(k,*)}(x) S_{2,\lambda}(n,m) \\ &= \sum_{l=0}^{n-1} \sum_{m=0}^{n-l-1} \binom{n}{l+1} Ch_{m,\lambda}(x) 2^{-m-1} S_{2,\lambda}(n-l-1,m) \frac{(1)_{l+1,\lambda}}{(l+1)^{k-1}}. \end{aligned} \tag{43}$$

Finally, we observe that

$$
\begin{aligned}
&\frac{Ei_{k,\lambda}(\log_\lambda(1+2t))}{2+\lambda^{-1}\log(1+\lambda t)}(1+\lambda^{-1}\log(1+\lambda t)^x \\
&=\left(\sum_{i=0}^{\infty}CG_{i,\lambda}^{(k,*)}\frac{t^i}{i!}\right)\left(\sum_{m=0}^{\infty}\frac{(x)_m}{m!}(\lambda^{-1}\log(1+\lambda t))^m\right) \\
&=\left(\sum_{i=0}^{\infty}CG_{i,\lambda}^{(k,*)}\frac{t^i}{i!}\right)\left(\sum_{m=0}^{\infty}(x)_m\frac{\lambda^{-m}}{m!}(\log(1+\lambda t))^m\right) \\
&=\left(\sum_{i=0}^{\infty}CG_{i,\lambda}^{(k,*)}\frac{t^i}{i!}\right)\left(\sum_{m=0}^{\infty}(x)_m\lambda^{-m}\sum_{j=m}^{\infty}S_1(j,m)\frac{\lambda^j t^j}{j!}\right) \\
&=\left(\sum_{i=0}^{\infty}CG_{i,\lambda}^{(k,*)}\frac{t^i}{i!}\right)\left(\sum_{j=0}^{\infty}\sum_{m=0}^{j}(x)_m\lambda^{j-m}S_1(j,m)\frac{t^j}{j!}\right) \\
&=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\sum_{m=0}^{j}\binom{n}{j}(x)_m CG_{n-j,\lambda}^{(k,*)}\lambda^{j-m}S_1(j,m)\right)\frac{t^n}{n!}
\end{aligned}
\tag{44}
$$

From (24) and (44), we obtain the following theorem.

Theorem 2.6. *For $n \geq 0$, we have*

$$
CG_{n,\lambda}^{(k,*)}(x)=\sum_{j=0}^{n}\sum_{m=0}^{j}\binom{n}{j}(x)_m CG_{n-j,\lambda}^{(k,*)}\lambda^{j-m}S_1(j,m). \tag{45}
$$

3. CONCLUSION

Lee-Kim-Jang(2020) considered the type 2 degenerate poly-Euler polynomials by using the degenerate multiple polyexponential functions. Based on these ideas and similar views, we considered the modified type 2 degenerate poly-Changhee-Genocchi polynomials in Eq. (24) by using the degenerate modified polyexponential functions. Furthermore, we obtained some explicit expressions for the degenerate type 2 multi-poly-Genocchi polynomials in Theorem 2.1, Corollary 2.2, Theorem 2.3, Theorem 2.4 and Theorem 2.5.

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