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CONSTRUCTION OF GENERALIZED LEIBNITZ TYPE NUMBERS AND THEIR PROPERTIES

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ABSTRACT. The aim of this paper is to give combinatorial sums with nonnegative real parameters a and b derived from integration of the modification of the Bernstein basis functions. For $a = 0$ and $b = 1$, these sums reduce to the combinatorial sum of the Leibnitz type numbers. We also give some properties of the Leibnitz numbers with the aid of their generating functions derived from the Volkenborn integral on the set of p-adic integers. Moreover, we give some novel identities and relations involving the Bernoulli numbers, the Stirling numbers, the Leibnitz numbers, the Daehee numbers, the Changhee numbers, inverse binomial coefficients, and combinatorial sums. Finally, by coding computation formula for the generalization of the Leibnitz numbers in Mathematica 12.0 with their implementation, we compute and present few values of these numbers with their tables. Finally, by using the applications of the Volkenborn integral to the Mahler coefficients, we derive some novel formulas involving the Leibnitz numbers.

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1. INTRODUCTION

It is known by many researchers, who work on the subject of special numbers and their applications in recent years, that this subject has became among the leading topics of mathematics and especially analytic number theory. The *Leibnitz numbers*, known by the famous German mathematician Gottfried Wilhelm Leibnitz (1646 - 1716), are considered in this paper. These numbers, which have rarely been addressed until now, are studied by using the techniques of generating functions and their Volkenborn integral representation in this paper. These numbers are also closely related to the *Leibniz Harmoic Triangle* numbers. The denominators of some of these numbers are also directly related to the *pronic numbers*. Within the scope of this paper, it has been proved that these numbers are also related to the Bernoulli numbers, the Stirling numbers, the Daehee numbers, the Changhee numbers, and the combinatorial sums and numbers.

1.1. **Definitions and Notations.** Let $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ denote the set of natural numbers, the set of real numbers, and the set of complex numbers, respectively. $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $z \in \mathbb{C}$ with $z = x + iy$, $x, y \in \mathbb{N}$, $i^2 = -1$ and also $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$.

Let $x_{(n)} = x(x-1)\ldots(x-n+1)$ with $x_{(0)} = 1$ and $n \in \mathbb{N}$.

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The Bernoulli polynomials, $B_n(x)$, are defined by the following generating function:

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}, \tag{1}$$

where $|t|<2\pi$ (*cf.* [2]-[39]).

The Stirling numbers of the first kind, $S_1(n,k)$, are defined by means of the following generating function:

$$\frac{(\log(1+t))^k}{k!}=\sum_{n=0}^{\infty}S_1(n,k)\frac{t^n}{n!} \tag{2}$$

with $S_1(n,k)=0$ if $k>n$, and $k\in\mathbb{N}_0$ (*cf.* [2]-[39]; and references therein).

The Leibnitz numbers, $\boldsymbol{l}(n,k)$, are defined by

$$\boldsymbol{l}(n,k)=\frac{1}{(n+1)\binom{n}{k}} \tag{3}$$

whose generating function is given as follows:

$$\mathcal{G}_l(t,u):=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\boldsymbol{l}(n,k)t^ku^n=\frac{\log(1-u)+\log(1-ut)}{(1-u)(1-tu)-1}, \tag{4}$$

where $|u|<1$; $k=0,1,2,\ldots,n$ and $n\in\mathbb{N}_0$ (*cf.* [2, Exercise 16, p. 127]).

As seen from the equation (4), the function $\mathcal{G}_l(t,u)$ is the generating function for the polynomials:

$$L_n(t):=\sum_{k=0}^{n}\boldsymbol{l}(n,k)t^k$$

whose coefficients are the Leibnitz numbers and also whose degree is n. That is, the ordinary generating function for the polynomials $L_n(t)$ is given as follows:

$$\mathcal{G}_l(t,u)=\sum_{n=0}^{\infty}L_n(t)u^n. \tag{5}$$

Observe that

$$L_n(1)=\sum_{k=0}^{n}\boldsymbol{l}(n,k),$$

and

$$L_n(0)=\boldsymbol{l}(n,0)=\frac{1}{n+1}.$$

The Leibnitz numbers, $\boldsymbol{l}(n,k)$, are also given by following finite combinatorial sum:

$$\boldsymbol{l}(n,k)=\sum_{v=0}^{k}(-1)^{k-v}\frac{1}{n-v+1}\binom{k}{v}, \tag{6}$$

where $k=0,1,2,\ldots,n$ and $n\in\mathbb{N}_0$ (*cf.* [2, Exercise 16, p. 127]).

With the initial condition

$$\boldsymbol{l}(n,0)=\frac{1}{n+1},$$

the Leibnitz numbers satisfy the following recurrence relation:

$$\boldsymbol{l}\left(n,k\right)=\frac{k}{n+1}\boldsymbol{l}\left(n-1,k-1\right),$$

where $k=1,2,\ldots,n$ and $n\in\mathbb{N}$ (*cf.* [2, Exercise 16, p. 127]).

In [19], whose content could not be reached but whose existence is known, Zhao and Wuyungaowa claimed in its abstract that they gave a series of identities involving Leibniz numbers, Stirling numbers, harmonic numbers, and arctan numbers by making use of generating functions. They also claimed that give the asymptotic expansion of certain sums related to Leibniz numbers by the Laplace method. On the other hand, there is no data to comment on whether the Leibniz numbers mentioned there and the *Leibnitz numbers* discussed in this study point to the same concept or the relationship of the results. Due to the expression *Leibniz numbers* in the abstract of the relevant study, we cite it here.

The Daehee numbers, D_n, are defined by

$$\mathcal{G}_D\left(u\right):=\frac{\log\left(1+u\right)}{u}=\sum_{n=0}^{\infty}D_n\frac{u^n}{n!}\tag{7}$$

where

$$D_n=(-1)^n\frac{n!}{n+1}\tag{8}$$

(*cf.* [14]).

Combining (1) and (7), one has the following novel identity:

$$\sum_{j=0}^{n}B_jS_1(n,j)=\frac{(-1)^n n!}{n+1}\tag{9}$$

(*cf.* [3, p. 117], [14], [22, p. 45, Exercise 19 (b)]).

The Changhee numbers, Ch_n, are defined by

$$\mathcal{G}_{Ch}\left(u\right):=\frac{2}{2+u}=\sum_{n=0}^{\infty}Ch_n\frac{u^n}{n!}\tag{10}$$

where

$$Ch_n=(-1)^n\frac{n!}{2^n}\tag{11}$$

(*cf.* [15]).

The numbers, $Y_n\left(\lambda\right)$, are defined by

$$\mathcal{G}_Y\left(u,\lambda\right):=\frac{2}{\lambda\left(1+\lambda u\right)-1}=\sum_{n=0}^{\infty}Y_n\left(\lambda\right)\frac{u^n}{n!}\tag{12}$$

where

$$Y_n\left(\lambda\right)=(-1)^n\frac{2n!}{\lambda-1}\left(\frac{\lambda^2}{\lambda-1}\right)^n\tag{13}$$

(*cf.* [36], [40]).

The relation between Changhee numbers and the numbers $Y_n(\lambda)$ is given as follows:

$$Ch_n = (-1)^{n+1} Y_n(-1) \tag{14}$$

(*cf.* [36], [40]). The numbers $Y_n(-1)$ are also related to the other combinatorial numbers (*cf.* for detail, see, [2], [3], [8], [22]).

Next, we summarize the contents of this paper as follows:

In Section 2, we give combinatorial identities and relations related to the Bernoulli numbers, the Stirling numbers, the Leibnitz numbers, the Daehee numbers, and the Changhee numbers. We also give some computational formulas for these numbers.

In Section 3, we give further remarks and observations on the Leibnitz numbers. Moreover, by using finite sums derived from application of the integral to the modification for the Bernstein basis functions, we introduce a generalization of the Leibnitz numbers.

In Section 4, we give Mathematica implementation of the generalized Leibnitz numbers and by this implementation, we compute and present a few values of these numbers with their tables.

In Section 5, by applying the Volkenborn integral on the set of p-adic integers, we derive some novel formulas involving the Leibnitz numbers.

In Section 6, we give further remarks and observations with two open questions. Finnaly, we give acknowledgement about Professor Lee Chae Jang.

2. Combinatorial identities and relations involving the Leibnitz numbers, Daehee numbers and Changhee numbers

By using functional equations of the generating functions for the Leibnitz numbers, the Daehee numbers and special series, we find many formulas, identities and relations involving the Changhee numbers and combinatorial numbers and sums.

By using (4), we get

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \boldsymbol{l}(n,k)\, t^k u^n = \frac{1}{t+1}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{n-k+1}\left(\frac{t}{t+1}\right)^k\left(1+t^{n-k+1}\right)u^n. \tag{15}$$

Comparing the coefficients u^n on both-sides of the equation (15), we get the following theorem:

Theorem 2.1. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{k=0}^{n} \boldsymbol{l}(n,k)\, t^k = \frac{1}{t+1}\sum_{k=0}^{n}\frac{1}{n-k+1}\left(\frac{t}{t+1}\right)^k\left(1+t^{n-k+1}\right). \tag{16}$$

Substituting $t = 1$ into (16), we get the finite summation of the Leibnitz numbers as in the following corollary:

Corollary 2.2. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{k=0}^{n} \boldsymbol{l}(n,k) = \sum_{k=0}^{n}\frac{1}{(n-k+1)\,2^k}. \tag{17}$$

Theorem 2.3. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{k=0}^{n} \boldsymbol{l}\left(n,k\right)t^k = \frac{1}{1+t}\sum_{j=0}^{n}(-1)^j \frac{D_j}{j!}\left(\frac{t}{1+t}\right)^{n-j}\left(1+t^{j+1}\right). \tag{18}$$

Proof. By using (4) and (7), we get the following functional equation of generating functions:

$$\mathcal{G}_l\left(t,u\right) = \frac{\mathcal{G}_D\left(-u\right)+t\mathcal{G}_D\left(-ut\right)}{1+t-ut} \tag{19}$$

which yields

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \boldsymbol{l}\left(n,k\right)t^k u^n = \frac{1}{1+t}\sum_{n=0}^{\infty}u^n\frac{t^n}{(1+t)^n}\left(\sum_{n=0}^{\infty}\frac{(-1)^n D_n}{n!}u^n + t\sum_{n=0}^{\infty}\frac{(-1)^n D_n u^n}{n!}t^n\right).$$

Hence, we get

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{n} \boldsymbol{l}\left(n,k\right)t^k u^n &= \frac{1}{1+t}\sum_{n=0}^{\infty}\sum_{j=0}^{n}(-1)^j\frac{D_j}{j!}\left(\frac{t}{1+t}\right)^{n-j}u^n \\ &\quad +\frac{t}{1+t}\sum_{n=0}^{\infty}\sum_{j=0}^{n}(-1)^j\frac{D_j}{j!}t^j\left(\frac{t}{1+t}\right)^{n-j}u^n.\end{aligned}$$

Comparing the coefficients of u^n on both sides of the above equation, we arrive at the desired result. □

Combining (18) with (8) and (9), we arrive at the following theorem:

Theorem 2.4. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{k=0}^{n} \boldsymbol{l}\left(n,k\right)t^k = \sum_{j=0}^{n}\sum_{v=0}^{j}(-1)^j\frac{\left(t^{n-j}+t^{n+1}\right)B_v S_1(j,v)}{(1+t)^{n-j+1}j!}.$$

Remark 1. *Combining* (18) *with* (8)*, we also arrive at the equation* (16)*.*

By combining (3) with (16), we get the following corollary:

Corollary 2.5. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{k=0}^{n}\frac{t^k}{(n+1)\binom{n}{k}} = \frac{1}{1+t}\sum_{j=0}^{n}\frac{1+t^{j+1}}{j+1}\left(\frac{t}{1+t}\right)^{n-j}. \tag{20}$$

Substituting $t=1$ into (20) yields the following result:

Corollary 2.6. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{k=0}^{n}\frac{1}{(n+1)\binom{n}{k}} = \sum_{j=0}^{n}\frac{2^{j-n}}{j+1}. \tag{21}$$

Observe that the combination of (3) with equation (21) is equivalent to equation (17).
Combining (21) with (11), we have the following result:

Theorem 2.7. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{k=0}^{n} \frac{1}{(n+1)\binom{n}{k}} = \sum_{j=0}^{n} \frac{(-1)^{n-j}\, Ch_{n-j}}{(j+1)\,(n-j)!}. \tag{22}$$

Substituting (14) into (22), we arrive at the following corollary:

Corollary 2.8. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{k=0}^{n} \frac{1}{(n+1)\binom{n}{k}} = -\sum_{j=0}^{n} \frac{Y_{n-j}\,(-1)}{(j+1)\,(n-j)!}.$$

Theorem 2.9. *Let $n \in \mathbb{N}$. Then we have*

$$\sum_{k=0}^{n} \boldsymbol{l}\,(n,k) - \frac{1}{2}\sum_{k=0}^{n-1} \boldsymbol{l}\,(n-1,k) = \frac{(-1)^n}{n!} D_n. \tag{23}$$

Proof. We set

$$\mathcal{G}_D\,(-u) = \left(1 - \frac{u}{2}\right) \mathcal{G}_l\,(1,u)\,. \tag{24}$$

By using the above equation, we obtain

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \boldsymbol{l}\,(n,k)\,u^n - \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n} \boldsymbol{l}\,(n,k)\,u^{n+1} = \sum_{n=0}^{\infty} (-1)^n D_n \frac{u^n}{n!}.$$

Therefore

$$\sum_{n=1}^{\infty}\sum_{k=0}^{n} \boldsymbol{l}\,(n,k)\,u^n - \frac{1}{2}\sum_{n=1}^{\infty}\sum_{k=0}^{n-1} \boldsymbol{l}\,(n-1,k)\,u^{n} = \sum_{n=1}^{\infty} (-1)^n D_n \frac{u^n}{n!}.$$

Comparing the coefficients of u^n on both sides of the above equation yields the desired result. □

Combining (23) with (8), we obtain the following result:

Theorem 2.10. *Let $n \in \mathbb{N}$. Then we have*

$$\sum_{k=0}^{n} \boldsymbol{l}\,(n,k) - \frac{1}{2}\sum_{k=0}^{n-1} \boldsymbol{l}\,(n-1,k) = \frac{1}{n+1}.$$

Remark 2. *By using* (24)*, assuming that* $|u| < 1$*, we obtain*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \boldsymbol{l}\,(n,k)\,u^n = \sum_{n=0}^{\infty} \frac{u^n}{n+1} \sum_{n=0}^{\infty} \frac{u^n}{2^n}.$$

Therefore

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \boldsymbol{l}\,(n,k)\,u^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{2^{k-n}}{k+1} u^n.$$

Comparing the coefficients of u^n on both sides of the above equation yields the equation (21)*.*

3. Generalized Leibnitz type numbers

In this section, we give further remarks and observations on the Leibnitz numbers and their relations with finite sums derived from application of the integral to the modification for the Bernstein basis functions. By using the following well-known beta functions:

$$B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt=B(\beta,\alpha)$$

where $\mathrm{Re}(\alpha)>0$ and $\mathrm{Im}(\beta)>0$ (*cf.* [20], [39]), one has the following novel integral formula for the function $B(\alpha,\beta)$:

$$\int_a^b (x-a)^{\alpha-1}(b-x)^{\beta-1}\,dx=(b-a)^{\alpha+\beta-1}B(\alpha,\beta) \tag{25}$$

(*cf.* [39, p.10, Eq. (69)]).

Let $x\in[a,b]$ and $k=0,1,2,\ldots,n$. The modification for the Bernstein basis functions are defined by

$$B_k^n(x;a,b)=\binom{n}{k}\left(\frac{x-a}{b-a}\right)^k\left(\frac{b-x}{b-a}\right)^{n-k};\quad a<b \tag{26}$$

(*cf.* [6], [31]; see references therein).

By applying (25) to (26), in [31], we not only found the following combinatorial type sums

$$\sum_{j=0}^{k}\sum_{v=0}^{n-k}(-1)^{n-j-v}\binom{k}{j}\binom{n-k}{v}\frac{a^{k-j}b^{n+j-k+1}-a^{n-v+1}b^v}{n+j-k-v+1}, \tag{27}$$

$a<b$ $(a,b\in[0,\infty))$, but also gave many identities and relations involving other combinatorial numbers and special numbers.

With the aid of (27), we introduce generalized Leibnitz type numbers by the following definition:

Definition 3.1. *Let a and b be nonnegative real parameters with $a<b$. Let $k=0,1,2,\ldots,n$ and $n\in\mathbb{N}_0$. Then generalized Leibnitz type numbers $\mathcal{L}(n,k;a,b)$ are defined by*

$$\mathcal{L}(n,k;a,b)=\sum_{j=0}^{k}\sum_{v=0}^{n-k}(-1)^{n-j-v}\binom{k}{j}\binom{n-k}{v}\frac{a^{k-j}b^{n+j-k+1}-a^{n-v+1}b^v}{n+j-k-v+1}. \tag{28}$$

Note that in (28) we assume that $0^0=1$.

Remark 3. *Setting $a=0$ and $b=1$ in (28), we have (6). Combining (3) with (6), we have*

$$\boldsymbol{l}(n,k)=\mathcal{L}(n,k;0,1).$$

On the other hand,

$$\frac{1}{(n+1)\binom{n}{k}}=\sum_{v=0}^{n-k}(-1)^{n-k-v}\binom{n-k}{v}\frac{1}{n-v+1}.$$

With the help of integration of Bernstein basis functions, recently, we have studied on the above combinatorial sums (cf. [26, 28, 30], [31, Eq. (29)]*). Observe that further identities and new number families may be discovered by using the methods in paper* [28, 26, 30, 31]*, of the generalized Leibnitz numbers, which seem to be closely related to the integral applications for the modification and unification of the Bernstein basis functions.*

4. Mathematica implementation of the numbers $\mathcal{L}(n, k; a, b)$

In this section, in order to give some applications of the computation formulas given in the previous sections, we present Mathematica implementation (see: Implementation 1) for the numbers $\mathcal{L}(n, k; a, b)$ by coding (28) in Mathematica 12.0.

Implementation 1. The following Mathematica code returns the values of the generalized Leibnitz numbers $\mathcal{L}(n, k; a, b)$.

```
Unprotect[Power];
Power[0,0]=1;
Protect[Power];
GLeibnitzNum[n_,k_,a_,b_]:=Sum[Sum[((−1)^(n−j−v))∗Binomial[k,j]∗Binomial[n−k,v]∗((a^(k−j))∗(b^(n
    +j−k+1))−(a^(n−v+1))(b^v))/(n+j−k−v+1), {v,0,n−k}], {j,0,k}]
```

Then, by (1), we compute few values of the numbers $\mathcal{L}(n, k; a, b)$, and give their tables as follows:

Table 1. For $k = 1$ and $n \in \{0, 1, 2, 3, 4\}$, few values of the generalized Leibnitz numbers $\mathcal{L}(n, k; a, b)$.

$L(0, 1; a, b)$	0
$L(1, 1; a, b)$	$a^2 - a b + \frac{1}{2}(-a^2 + b^2)$
$L(2, 1; a, b)$	$a^2 b - a b^2 + \frac{1}{2}(-a^3 + a b^2) + \frac{1}{3}(a^3 - b^3) + \frac{1}{2}(-a^2 b + b^3)$
$L(3, 1; a, b)$	$-a^3 b + a^2 b^2 + \frac{1}{3}(a^4 - a b^3) + \frac{1}{4}(-a^4 + b^4) - \frac{2}{3}(-a^3 b + b^4) + \frac{1}{2}(-a^2 b^2 + b^4)$
$L(4, 1; a, b)$	$a^4 b + a^3 b^2 + a^2 b^3 - 2 a b^4 - b^5 + \frac{1}{4}(-a^5 + a b^4) + \frac{3}{2}(-a^3 b^2 + a b^4) + \frac{1}{5}(a^5 - b^5) + \frac{3}{4}(-a^4 b + b^5) + \frac{1}{2}(-a^2 b^3 + b^5)$

Table 2. For $k = 2$ and $n \in \{0, 1, 2, 3, 4\}$, few values of the generalized Leibnitz numbers $\mathcal{L}(n, k; a, b)$.

$L(0, 2; a, b)$	0
$L(1, 2; a, b)$	0
$L(2, 2; a, b)$	$a^2 b - a b^2 + \frac{1}{3}(-a^3 + b^3)$
$L(3, 2; a, b)$	$a^2 b^2 - a b^3 + \frac{1}{2}(a^4 - a^2 b^2) + \frac{2}{3}(-a^4 + a b^3) + \frac{1}{4}(a^4 - b^4) + \frac{1}{3}(-a^3 b + b^4)$
$L(4, 2; a, b)$	$a^4 b - a b^4 + \frac{1}{3}(-a^5 + a^2 b^3) + \frac{1}{2}(a^5 - a b^4) + \frac{4}{3}(-a^4 b + a b^4) + \frac{1}{2}(a^4 b - b^5) + \frac{1}{5}(-a^5 + b^5) + \frac{1}{3}(-a^3 b^2 + b^5)$

Table 3. For $n \in \{0, 1, 2, 3\}$ and $k \in \{0, 1, 2\}$, few values of the generalized Leibnitz numbers $\mathcal{L}(n, k; a, b)$.

	$k = 0$	$k = 1$	$k = 2$
$L(0, k; a, b)$	$-a + b$	0	0
$L(1, k; a, b)$	$-a b + b^2 + \frac{1}{2}(a^2 - b^2)$	$a^2 - a b + \frac{1}{2}(-a^2 + b^2)$	0
$L(2, k; a, b)$	$a^2 b - a b^2 + \frac{1}{3}(-a^3 + b^3)$	$a^2 b - a b^2 + \frac{1}{2}(-a^3 + a b^2) + \frac{1}{3}(a^3 - b^3) + \frac{1}{2}(-a^2 b + b^3)$	$a^2 b - a b^2 + \frac{1}{3}(-a^3 + b^3)$

Substituting $a = 0$ and $b = 1$, the numbers $\mathcal{L}(n, k; a, b)$ are reduced to the classical Leibnitz numbers $\boldsymbol{l}(n, k)$.

Table 4. For $n \in \{0, 1, 2, \ldots, 8\}$ and $k \in \{0, 1, 2, \ldots, 8\}$, few values of the generalized Leibnitz numbers $\mathcal{L}(n, k; 0, 1)$, namely $\boldsymbol{l}(n, k)$.

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$L(0, k; 0, 1)$	1	0	0	0	0	0	0	0	0
$L(1, k; 0, 1)$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0	0	0
$L(2, k; 0, 1)$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$	0	0	0	0	0	0
$L(3, k; 0, 1)$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	0	0	0	0	0
$L(4, k; 0, 1)$	$\frac{1}{5}$	$\frac{1}{20}$	$\frac{1}{30}$	$\frac{1}{20}$	$\frac{1}{5}$	0	0	0	0
$L(5, k; 0, 1)$	$\frac{1}{6}$	$\frac{1}{30}$	$\frac{1}{60}$	$\frac{1}{60}$	$\frac{1}{30}$	$\frac{1}{6}$	0	0	0
$L(6, k; 0, 1)$	$\frac{1}{7}$	$\frac{1}{42}$	$\frac{1}{105}$	$\frac{1}{140}$	$\frac{1}{105}$	$\frac{1}{42}$	$\frac{1}{7}$	0	0
$L(7, k; 0, 1)$	$\frac{1}{8}$	$\frac{1}{56}$	$\frac{1}{168}$	$\frac{1}{280}$	$\frac{1}{280}$	$\frac{1}{168}$	$\frac{1}{56}$	$\frac{1}{8}$	0
$L(8, k; 0, 1)$	$\frac{1}{9}$	$\frac{1}{72}$	$\frac{1}{252}$	$\frac{1}{504}$	$\frac{1}{630}$	$\frac{1}{504}$	$\frac{1}{252}$	$\frac{1}{72}$	$\frac{1}{9}$

5. The Volkenborn integral representation of the Leibnitz numbers on the set of p-adic integers

In this section, by applying Volkenborn integral on the set of p-adic integers not only to the Mahler coefficients, but also to uniformly differential function on the set of p-adic integers, we obtain some novel formulas involving the Leibnitz numbers.

Here, we follow notations of the following references: [10, 13, 24, 37]; and the references cited therein.

Some notations and definitions for p-adic integrals are given as follows:

Let $m \in \mathbb{N}$. Let $ord_p(m)$ denote the greatest integer k ($k \in \mathbb{N}_0$) such that p^k divides m in $\mathbb{Z}$. If $m = 0$, then $ord_p(m) = \infty$.

$|.|_p$ is a norm on $\mathbb{Q}$. This norm is given by

$$|x|_p = \begin{cases} p^{-ord_p(x)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Let $\mathbb{Z}_p$ be a set of p-adic integers which is given by

$$\mathbb{Z}_p = \left\{ x \in \mathbb{Q}_p : |x|_p \leq 1 \right\}.$$

Let f be defined on $\mathbb{Z}_p$. The function f is called a uniformly differential function at a point $a \in \mathbb{Z}_p$ if f satisfies the following conditions:

If the difference quotients $\Phi_f : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{C}_p$ such that

$$\Phi_f(x, y) = \frac{f(x) - f(y)}{x - y}$$

have a limit $f'(a)$; $(x, y) \to (a, a)$. A set of uniformly differential functions is indicated by $f \in UD(\mathbb{Z}_p)$ or $f \in C^1(\mathbb{Z}_p \to \mathbb{C}_p)$.

The well-known Volkenborn integral (bosonic p-integral) is given by

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_1(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N - 1} f(x),$$

where $\mu_1(x) = \mu_1(x + p^N \mathbb{Z}_p)$ denotes the Haar distribution, which is defined by

$$\mu_1(x) = \frac{1}{p^N}$$

(*cf.* [7]-[15], [24]; see also the references cited in each of these earlier works).

In order to achieve the results of this section, we let

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n} \in C^1(\mathbb{Z}_p \to \mathbb{C}_p), \tag{29}$$

where $\binom{x}{n} = \frac{x_{(n)}}{n!}$ denotes the Mahler coefficients. Applying the Volkenborn integral to the function $f(x)$ in terms of the Mahler coefficients $\binom{x}{n}$, we have the following well-known formula:

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} a_n, \tag{30}$$

(*cf.* [24, p. 168-Proposition 55.3]).

In order to give generating function (4), we apply the Volkenborn integral to the following uniformly differential function on $\mathbb{Z}_p$:

$$f(x, u; t) = (1-u)^x (1-tu)^x,$$

where u, $x \in \mathbb{Z}_p$.

Substituting the above function into the following well-known integral equation:

$$\int_{\mathbb{Z}_p} f(x+1) d\mu_1(x) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x) + f'(0)$$

(*cf.* [24]), we get

$$\int_{\mathbb{Z}_p} (1-u)^x (1-tu)^x\, d\mu_1(x) = \frac{\log\left[(1-u)(1-tu)\right]}{(1-u)(1-tu)-1}. \tag{31}$$

Consequently, the function on the right of equation (31) gives the generating function given in equation (4) for the Leibnitz numbers.

Combining (31) with the binomial series

$$\sum_{n=0}^{\infty} x_{(n)} \frac{u^n}{n!} = (1+u)^x,$$

and using (30), we obtain

$$\sum_{n=0}^{\infty} (-1)^n \frac{u^n}{n!} \sum_{k=0}^{n} \binom{n}{k} t^{n-k} \int_{\mathbb{Z}_p} x_{(k)} x_{(n-k)} d\mu_1(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \boldsymbol{l}(n,k)\, t^k u^n.$$

Comparing the coefficients of u^n on both sides of the above equation yields the following theorem:

Theorem 5.1. *Let $n \in \mathbb{N}_0$. Then we have*

$$\frac{(-1)^n}{n!}\sum_{k=0}^{n}\binom{n}{k}t^{n-k}\int_{\mathbb{Z}_p} x_{(k)}x_{(n-k)}d\mu_1(x) = \sum_{k=0}^{n}\boldsymbol{l}(n,k)\,t^k. \tag{32}$$

Combining (32) with the following formula:

$$\int_{\mathbb{Z}_p} x_{(n)}x_{(m)}d\mu_1(x) = \sum_{k=0}^{m}(-1)^{m+n-k}\binom{n}{k}\binom{m}{k}\frac{k!(n+m-k)!}{n+m-k+1}$$

where $m, n \in \mathbb{N}_0$ (*cf.* [38]), we arrive at the following theorem:

Theorem 5.2. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{k=0}^{n}\boldsymbol{l}(n,k)\,t^k = \frac{1}{n!}\sum_{k=0}^{n}\sum_{j=0}^{n-k}(-1)^j\binom{n}{k}\binom{n-k}{j}\binom{k}{j}\frac{j!\,(n-j)!}{n-j+1}t^{n-k}. \tag{33}$$

Substituting $t=1$ into (33), we get the following corollary:

Corollary 5.3.

$$\sum_{k=0}^{n}\boldsymbol{l}(n,k) = \frac{1}{n!}\sum_{k=0}^{n}\sum_{j=0}^{n-k}(-1)^j\binom{n}{k}\binom{n-k}{j}\binom{k}{j}\frac{j!\,(n-j)!}{n-j+1}. \tag{34}$$

6. Remarks and Open Questions

The factorials involving binomial coefficients and combinatorial sums have many important applications in theory of combinatorial analysis, in theory of discrete probability, and in theoretical computer science related to finite differences. Especially in the calculus of finite differences, in combinatorial analysis, and in discrete mathematics, factorials involving binomial coefficients and special numbers such as the Lebnitz numbers have also used to construct mathematical models and their applications (*cf.* [1]-[42]).

In order to study applications of the special numbers involving Lebnitz numbers in analytic number theory, not only generating functions, but also interpolation functions related to zeta-type functions are very useful and efficient areas.

Therefore, the following two open problems involving the Leibnitz numbers are come up with at the and of this section:

1- How can we construct generating functions for the (generalized) Leibnitz numbers?

2- Are there any zeta-type functions, on the set of complex numbers, which interpolates the (generalized) Leibnitz numbers?

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