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# A NOTE ON $q$-ANALOGUE OF CATALAN NUMBERS ARISING FROM FERMIONIC $p$-ADIC $q$-INTEGRAL ON $\mathbb{Z}_p$

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ABSTRACT. Catalan numbers appear in many different contexts in combinatorics and some practical problems. In this paper, we introduce $q$-analogues of Catalan numbers arising from a fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$, and derive explicit expressions and some identities for those numbers.

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## 1. INTRODUCTION

Catalan numbers $C_n$ appear in a variety of contexts in combinatorics and some practical problems. Indeed, [20] presents 66 different interpretations of the Catalan numbers, including the enumeration of certain types of lattice paths, permutations, sequences and of binary trees and the counting of triangulations of convex polygons. The Catalan numbers were named after the Belgian mathematician Eugène Charles Catalan (1814-1894), even though they were first introduced by the Mongolian mathematician Ming Antu in around 1730.

The aim of this paper is to introduce the $q$-analogues of the Catalan numbers $C_{n,q}$ with the help of a fermionic $p$-adic $q$-integral of $\mathbb{Z}_p$ (see (7)) and derive explicit expressions and some identities for those numbers. In more detail, we deduce explicit expressions of $C_{n,q}$, as a rational function in $q$, in terms of $q$-Euler numbers and Stirling numbers of the first kind, as a fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$, and involving $(q,\lambda)$-Changhee numbers. In additon, we consider a polynomial extension of the $q$-analogues of Catalan numbers, namely the $q$-analogues of Catalan polynomials $C_{n,q}(x)$ (see (26)), and derive explicit expressions in terms of Catalan numbers and Stirling numbers of the first kind and of $q$-Euler polynomials and Stirling numbers of the first kind. For the rest of this section, we will recall some necessary things that are needed in this paper.

Let $p$ be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p,\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. The $p$-adic norm $|\cdot|_p$ is normalized as $|p|_p=\frac{1}{p}$. The notations $[x]_q$ and $[x]_{-q}$ respectively denote $[x]_q=\frac{1-q^x}{1-q}$ and $[x]_{-q}=\frac{1-(-q)^x}{1+q}$, (see [4,5,17]).

It is well known that the Euler numbers are defined by

$$\frac{2}{e^t+1}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!},\quad (\text{see } [1-18]). \tag{1}$$

Let $q$ be an indeterminate in $\mathbb{C}_p$ with $|1-q|_p<1$. The $q$-analogues of Euler numbers are given by

$$\frac{[2]_q}{1+qe^t}=\sum_{n=0}^{\infty}E_{n,q}\frac{t^n}{n!},\quad \text{(see [4,17])}. \tag{2}$$

Note that $\lim_{q\to 1}E_{n,q}=E_n$, $(n\geq 0)$.

The $q$-analogues of Changhee numbers are given by

$$\frac{[2]_q}{[2]_q+t}=\sum_{n=0}^{\infty}\mathrm{Ch}_{n,q}\frac{t^n}{n!}. \tag{3}$$

In addition, for $\lambda\in\mathbb{C}_p$ with $|\lambda|_p<1$, the $(q,\lambda)$-Changhee numbers are given by

$$\frac{[2]_q}{q(1+t)^{\lambda}+1}=\sum_{n=0}^{\infty}\mathrm{Ch}_{n,q,\lambda}\frac{t^n}{n!},\quad \text{(see [14-16])}.$$

An explicit formula for the Catalan numbers is given by

$$C_0=1,\quad C_1=1,\quad C_n=\frac{1}{n+1}\binom{2n}{n},\quad (n\geq 2),\quad \text{(see [3,6,7])}. \tag{4}$$

The first few Catalan numbers are $1,1,2,5,14,42,132,429,1430,4862,16796,\ldots$.

It is easy to show that

$$\sqrt{1-4t}=\sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}(-4t)^k=-\sum_{k=0}^{\infty}\binom{2k}{k}\frac{1}{2k-1}t^k. \tag{5}$$

From (5), we can derive the generating function of Catalan numbers as follows:

$$\begin{aligned}\frac{2}{1+\sqrt{1-4t}} &= \frac{1-\sqrt{1-4t}}{2t}\\ &= \frac{1}{2t}\left(1+\sum_{n=0}^{\infty}\frac{1}{2n-1}\binom{2n}{n}t^n\right)\\ &= \frac{1}{2t}\sum_{n=0}^{\infty}\frac{1}{2n+1}\binom{2n+2}{n+1}t^{n+1}\\ &= \sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}t^n = \sum_{n=0}^{\infty}C_nt^n,\quad \text{(see [3,6,7])}.\end{aligned} \tag{6}$$

It is known that the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by Kim as

$$\begin{aligned}I_{-q}(f) &= \int_{\mathbb{Z}_p}f(x)d\mu_{-q}(x) = \lim_{N\to\infty}\sum_{x=0}^{p^N-1}f(x)\mu_{-q}\left(x+p^N\mathbb{Z}_p\right)\\ &= \lim_{N\to\infty}\frac{1}{[p^N]_{-q}}\sum_{x=0}^{p^N-1}f(x)(-q)^x,\quad \text{(see [4])},\end{aligned} \tag{7}$$

where $f$ is any continuous function on $\mathbb{Z}_p$.

From (7), we note that

$$qI_{-q}(f_1)+I_{-q}(f)=[2]_qf(0),\quad \text{(see [17])}, \tag{8}$$

where $f_1(x)=f(x+1)$.

For $n \geq 0$, the Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k, \ \frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \quad \text{(see [12])}, \tag{9}$$

where $(x)_0 = 1,\ (x)_n = x(x-1)\cdots(x-n+1),\ (n \geq 1)$.

Further, for $n \geq 0$, the Stirling numbers of the second kind are given by

$$x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \ \frac{1}{k!}(e^t-1)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}, \quad \text{(see [12])}. \tag{10}$$

## 2. $q$-ANALOGUES OF CATALAN NUMBERS

In this section we assume that $q,t \in \mathbb{C}_p$ with $|1-q|_p < 1$ and $|t|_p < p^{-1/p-1}$. Let us apply (8) with $f(x) = (1-4t)^{\frac{x}{2}}$. Then we have

$$\int_{\mathbb{Z}_p} (1-4t)^{\frac{x}{2}} d\mu_{-q}(x) = \frac{[2]_q}{q\sqrt{1-4t}+1} = \frac{[2]_q}{1-q^2+4q^2t}\left(1-q\sqrt{1-4t}\right). \tag{11}$$

Now, we consider the $q$-analogues of Catalan numbers which are given by

$$\frac{[2]_q}{1-q^2+4q^2t}\left(1-q\sqrt{1-4t}\right) = \sum_{n=0}^{\infty} C_{n,q}t^n. \tag{12}$$

From (6) and (12), we note that

$$\sum_{n=0} \lim_{q\to 1} C_{n,q}t^n = \frac{1}{2t}\left(1-\sqrt{1-4t}\right) = \sum_{n=0}^{\infty} C_n t^n. \tag{13}$$

Thus, by (13), we get

$$\lim_{q\to 1} C_{n,q} = C_n, \quad (n \geq 0).$$

From (5), we derive the following equation.

(14)

$$\begin{aligned}
&\frac{[2]_q}{1-q^2+4q^2t}\left(1-q\sqrt{1-4t}\right) = \frac{1+q}{1-q^2+4q^2t}\left(1+q\sum_{l=0}^{\infty}\binom{2l}{l}\frac{1}{2l-1}t^l\right)\\
&= \frac{1+q}{1-q^2}\frac{1}{1+\frac{4q^2}{1-q^2}t}\left(1+q\sum_{l=0}^{\infty}\binom{2l}{l}\left(\frac{1}{2l-1}\right)t^l\right)\\
&= \frac{1+q}{1-q^2}\sum_{m=0}^{\infty}\left(\frac{4q^2}{1-q^2}\right)^m(-1)^m t^m\left(1+q\sum_{l=0}^{\infty}\binom{2l}{l}\frac{t^l}{2l-1}\right)\\
&= \frac{1+q}{1-q^2}\left(\sum_{n=0}^{\infty} q\sum_{l=0}^{n}\binom{2l}{l}\frac{(-1)^{n-l}}{2l-1}\left(\frac{4q^2}{1-q^2}\right)^{n-l} t^n + \sum_{n=0}^{\infty}\left(\frac{4q^2}{1-q^2}\right)^n(-1)^n t^n\right)\\
&= \sum_{n=0}^{\infty}\frac{1}{1-q}\left(q\sum_{l=0}^{n}\binom{2l}{l}\frac{(-1)^{n-l}}{2l-1}\left(\frac{4q^2}{1-q^2}\right)^{n-l} + \left(-\frac{4q^2}{1-q^2}\right)^n\right)t^n.
\end{aligned}$$

Therefore, by (12) and (14), we obtain the following theorem.

**Theorem 1.** *For $n \geq 0$, we have*

$$\begin{aligned} C_{n,q} &= \frac{1}{1-q}\left(q\sum_{l=0}^{n}\binom{2l}{l}\frac{(-1)^{n-l}}{2l-1}\left(\frac{4q^2}{1-q^2}\right)^{n-l}+\left(-\frac{4q^2}{1-q^2}\right)^{n}\right) \\ &= \frac{q}{1-q}\sum_{l=1}^{n}\binom{2l}{l}\frac{(-1)^{n-l}}{2l-1}\left(\frac{4q^2}{1-q^2}\right)^{n-l}+\left(-\frac{4q^2}{1-q^2}\right)^{n}. \end{aligned}$$

From Theorem 1, for instance we have

$$\begin{aligned} C_{0,q} &= 1, \\ C_{1,q} &= \frac{2q}{1-q}-\frac{4q^2}{1-q^2}=\frac{2q}{1+q}, \\ C_{2,q} &= \frac{q}{1-q}\sum_{l=1}^{2}\binom{2l}{l}\frac{(-1)^{2-l}}{2l-1}\left(\frac{4q^2}{1-q^2}\right)^{2-l}+\left(\frac{-4q^2}{1-q^2}\right)^{2} \\ &= \frac{2q}{1-q}\left(-\frac{4q^2}{1-q^2}+1\right)+\left(\frac{4q^2}{1-q^2}\right)^{2} \\ &= \frac{6q^4-10q^3+2q^2+2q}{(1-q^2)^2} \\ &= \frac{(1-q)^2(6q^2+2q)}{(1-q^2)^2} = \frac{6q^2+2q}{(1+q)^2}, \ldots. \end{aligned}$$

Note that

$$\begin{aligned} &\lim_{q\to 1}C_{0,q}=1,\ \lim_{q\to 1}C_{1,q}=1,\ \lim_{q\to 1}C_{2,q}=2,\ \ldots, \\ &\lim_{q\to 1}C_{n,q}=\lim_{q\to 1}\frac{1}{1-q}\left(q\sum_{l=0}^{n}\binom{2l}{l}\frac{(-1)^{n-l}}{2l-1}\left(\frac{4q^2}{1-q^2}\right)^{n-l}+\left(-\frac{4q^2}{1-q^2}\right)^{n}\right) \\ &\qquad =\binom{2n}{n}\frac{1}{n+1}=C_n, \quad (n\geq 0). \end{aligned}$$

From (8), we note that

$$\int_{\mathbb{Z}_p} e^{xt}\,d\mu_{-q}(x)=\frac{[2]_q}{qe^t+1}=\sum_{n=0}^{\infty}E_{n,q}\frac{t^n}{n!}. \tag{15}$$

Thus, by (15), we get

$$\int_{\mathbb{Z}_p} x^n\,d\mu_{-q}(x)=E_{n,q}, \quad (n\geq 0). \tag{16}$$

From (11) and (12), we obtain

$$\int_{\mathbb{Z}_p}(1-4t)^{\frac{x}{2}}\,d\mu_{-q}(x)=\sum_{n=0}^{\infty}(-1)^n 2^{2n}\int_{\mathbb{Z}_p}\binom{\frac{x}{2}}{n}d\mu_{-q}(x)t^n=\sum_{n=0}^{\infty}C_{n,q}t^n. \tag{17}$$

Thus, by (17), we get

$$\int_{\mathbb{Z}_p}\binom{\frac{x}{2}}{n}d\mu_{-q}(x)=(-1)^n\frac{C_{n,q}}{2^{2n}}, \quad (n\geq 0). \tag{18}$$

On the other hand, by using (16) we also have

(19)

$$\int_{\mathbb{Z}_p}\binom{\frac{x}{2}}{n}d\mu_{-q}(x)=\frac{1}{n!}\int_{\mathbb{Z}_p}\left(\frac{x}{2}\right)_n d\mu_{-q}(x)=\frac{1}{n!}\sum_{l=0}^{n}S_1(n,l)2^{-l}\int_{\mathbb{Z}_p}x^l d\mu_{-q}(x)$$
$$=\frac{1}{n!}\sum_{l=0}^{n}\frac{S_1(n,l)}{2^l}E_{l,q},\quad (n\geq 0).$$

Now, combining (18) and (19), we get the following corollary.

**Corollary 2.** *For $n\geq 0$, we have*

$$\int_{\mathbb{Z}_p}\binom{\frac{x}{2}}{n}d\mu_{-q}(x)=(-1)^n\frac{C_{n,q}}{2^{2n}}=\frac{1}{n!}\sum_{l=0}^{n}\frac{S_1(n.l)}{2^l}E_{l,q}.$$

For $t,\lambda\in\mathbb{C}_p$ with $|\lambda|_p<1$ and $|t|_p<p^{-1/p-1}$, from (8) we have

$$\int_{\mathbb{Z}_p}(1+t)^{\lambda x}d\mu_{-q}(x)=\frac{[2]_q}{q(1+t)^{\lambda}+1}=\sum_{n=0}^{\infty}\mathrm{Ch}_{n,q,\lambda}\frac{t^n}{n!}. \tag{20}$$

Thus, by (20), we get

$$\int_{\mathbb{Z}_p}(1+t)^{\frac{x}{2}}d\mu_{-q}(x)=\frac{[2]_q}{q\sqrt{1+t}+1}=\sum_{n=0}^{\infty}\mathrm{Ch}_{n,q,1/2}\frac{t^n}{n!}. \tag{21}$$

From (11), (12) and (21), we note that

$$\sum_{n=0}^{\infty}C_{n,q}t^n=\frac{[2]_q}{q\sqrt{1-4t}+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}2^{2n}\mathrm{Ch}_{n,q,1/2}t^n. \tag{22}$$

Comparing the coefficients on both sides of (22), we have the next result.

**Theorem 3.** *For $n\geq 0$, we have*

$$C_{n,q}=\frac{(-1)^n}{n!}2^{2n}\mathrm{Ch}_{n,q,1/2}$$

Replacing $t$ by $\frac{1}{4}(1-e^{2t})$ in (12), we get

$$\begin{aligned}\int_{\mathbb{Z}_p}e^{xt}d\mu_{-q}(x)&=\sum_{l=0}^{\infty}C_{l,q}\left(\frac{1}{4}(1-e^{2t})\right)^l\\&=\sum_{l=0}^{\infty}l!C_{l,q}2^{-2l}(-1)^l\sum_{n=l}^{\infty}S_2(n,l)2^n\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}l!C_{l,q}(-1)^l2^{n-2l}S_2(n.l)\right)\frac{t^n}{n!}.\end{aligned} \tag{23}$$

Now, from (23) and (15), we obtain the following theorem.

**Theorem 4.** *For $n\geq 0$, we have*

$$E_{n,q}=\sum_{l=0}^{n}(-1)^l l!2^{n-2l}S_2(n,l)C_{l,q}.$$

Now, we consider the $q$-analogues of Catalan polynomials which are given by

$$\int_{\mathbb{Z}_p}(1-4t)^{\frac{x+y}{2}}d\mu_{-q}(y)=\frac{[2]_q}{1+q\sqrt{1-4t}}(1-4t)^{\frac{x}{2}}=\sum_{n=0}^{\infty}C_{n,q}(x)t^n. \tag{24}$$

From (24), we note that

$$\frac{[2]_q}{1+q\sqrt{1-4t}}(1-4t)^{\frac{x}{2}} = \sum_{l=0}^{\infty} C_{l,q}t^l \sum_{m=0}^{\infty}\binom{\frac{x}{2}}{m}(-1)^m 2^{2m} t^m = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{\frac{x}{2}}{m}(-1)^m 2^{2m} C_{n-m,q}\right)t^n. \tag{25}$$

By (24) and (25), we get

$$C_{n,q}(x) = \sum_{m=0}^{n}(-1)^m 2^{2m} C_{n-m,q}\binom{\frac{x}{2}}{m}, \quad (n \geq 0). \tag{26}$$

Now, we observe that

$$\begin{aligned}(1-4t)^{\frac{x}{2}} &= \sum_{l=0}^{\infty}\left(\frac{x}{2}\right)^l \frac{1}{l!}\Big(\log(1-4t)\Big)^l \\ &= \sum_{l=0}^{\infty}\left(\frac{x}{2}\right)^l \sum_{k=l}^{\infty} S_1(k,l)(-4)^k \frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty}\left(\sum_{l=0}^{k} S_1(k,l)\frac{(-1)^k}{k!}2^{2k-l}x^l\right)t^k\end{aligned}$$

Thus, by (24), we get

$$\begin{aligned}\sum_{n=0}^{\infty} C_{n,q}(x)t^n &= \frac{[2]_q}{1+q\sqrt{1-4t}}(1-4x)^{\frac{x}{2}} \\ &= \sum_{m=0}^{\infty} C_{m,q}t^m \sum_{k=0}^{\infty}\left(\sum_{l=0}^{k} S_1(k,l)\frac{(-1)^k}{k!}2^{2k-l}x^l\right)t^k \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k} S_1(k,l)\frac{(-1)^k}{k!}2^{2k-l}C_{n-k}x^l\right)t^n.\end{aligned} \tag{27}$$

Thus, by comparing the coefficients on both sides of (27), we obtain the following theorem,

**Theorem 5.** *For $n \geq 0$, we have*

$$C_{n,q}(x) = \sum_{k=0}^{n}\sum_{l=0}^{k}\frac{(-1)^k}{k!}S_1(k,l)2^{2k-l}C_{n-k}x^l.$$

From (8), we note that

$$\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}(x+y)^n d\mu_{-q}(y)\frac{t^n}{n!} = \int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-q}(y) = \frac{[2]_q}{qe^t+1}e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}, \tag{28}$$

where $E_{n,q}(x)=\sum_{l=0}^{n}\binom{n}{l}E_{l,q}x^{n-l}=\int_{\mathbb{Z}_p}(x+y)^n d\mu_{-q}(y)$ are the $q$-Euler polynomials.

$$\begin{aligned}(29)\quad \frac{[2]_q}{1+q\sqrt{1-4t}}(1-4t)^{\frac{x}{2}} &= \int_{\mathbb{Z}_p}(1-4t)^{\frac{x+y}{2}}d\mu_{-q}(y)\\ &= \sum_{m=0}^{\infty}2^{-m}\frac{1}{m!}\Big(\log(1-4t)\Big)^m\int_{\mathbb{Z}_p}(x+y)^m d\mu_{-q}(y)\\ &= \sum_{m=0}^{\infty}2^{-m}E_{m,q}(x)\sum_{n=m}S_1(n,m)\frac{(-1)^n 2^{2n}}{n!}t^n\\ &= \sum_{n=0}^{\infty}\left(\frac{(-1)^n}{n!}\sum_{m=0}^{n}2^{2n-m}E_{m,q}(x)S_1(n,m)\right)t^n.\end{aligned}$$

Thus, by (27) and (29), we get the following theorem.

**Theorem 6.** *For $n\geq 0$, we have*

$$C_{n,q}(x)=\frac{(-1)^n}{n!}\sum_{m=0}^{n}2^{2n-m}S_1(n,m)E_{m,q}(x).$$

## 3. Conclusion

Catalan numbers $C_n$ appear in many interesting counting problems in combinatorics and some practical problems.

In this paper, the $q$-analogues of the Catalan numbers $C_{n,q}$ were introduced with the help of a fermionic $p$-adic $q$-integral of $\mathbb{Z}_p$. We derived explicit expressions of $C_{n,q}$, as a rational function in $q$, in terms of $q$-Euler numbers and Stirling numbers of the first kind, as a fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$, and involving $(q,\lambda)$-Changhee numbers. In additon, we considered a polynomial extension of the $q$-analogues of Catalan numbers, namely the $q$-analogues of Catalan polynomials $C_{n,q}(x)$, and deduced explicit expressions in terms of Catalan numbers and Stirling numbers of the first kind and of $q$-Euler polynomials and Stirling numbers of the first kind.

It is one of our future projects to continue to study $q$-analogues of some special numbers and polynomials.

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## Conflicts of Interest

The authors declare no conflict of interest.

## References

[1] Araci, S.; Özer, Ö. *Extended q-Dedekind-type Daehee-Changhee sums associated with extended q-Euler polynomials,* Adv. Difference Equ. 2015, 2015:272.

[2] Hwang, S. Y.; Park, J.-W. *A note on the $(h,q)$-Changhee numbers and polynomials,* Adv. Stud. Contemp. Math. (Kyungshang) **29** (2019), no. 1, 15-23.

[3] Kim, D. S.; Kim, T. *A new approach to Catalan numbers using differential equations,* Russ. J. Math. Phys. **24** (2017), no. 4, 465-475.

[4] Kim, T. *q-Euler numbers and polynomials associated with p-adic q-integrals,* J. Nonlinear Math. Phys. **14** (2007), no. 1, 1527.
[5] Kim, T. *Analytic continuation of q-Euler numbers and polynomials,* Appl. Math. Lett. **21** (2008), no. 12, 13201323.
[6] Kim, T.; Kim, D. S. *Some identities of Catalan-Daehee polynomials arising from umbral calculus,* Appl. Comput. Math. **16** (2017), no. 2, 177-189.
[7] Kim, T.; Kim, D. S. *Differential equations associated with Catalan-Daehee numbers and their applications,* Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Mat. RACSAM **111** (2017), no. 4, 1071-1081.
[8] Kim, Y.; Kwon, J.; Sohn, G.; Lee, J. *Some identities of the partially degenerate Changhee-Genocchi polynomials and numbers,* Adv. Stud. Contemp. Math. (Kyungshang) **29** (2019), no. 4, 537-550.
[9] Kucukoglu,I. ; Simsek, B.; Simsek, Y. *New classes of Catalan-type numbers and polynomials with their applications related to p-adic integrals and computational algorithms,* Turkish J. Math. **44** (2020), no. 6, 2337-2355.
[10] Kucukoglu, I.; Simsek, Y. *Observations on identities and relations for interpolation fucntions and special numbers,* Adv. Stud. Contemp. Math. (Kyungshang) **28** (2018), no. 1, 41-56.
[11] Kwon, J.; Kim, W. J.; Rim, S.-H. *On the some identities of the type 2 Daehee and Changhee polynomials arising from p-adic integrals on $\mathbb{Z}_p$,* Proc. Jangjeon Math. Soc. **22** (2019), no. 3, 487-497.
[12] Lee, D. S.; Kim, H. K. *On the new type of degenerate poly-Genocchi numbers and polynomials,* Adv. Difference Equ. 2020, 2020:431.
[13] Ma, M.; Lim, D. *On finite times degenerate Changhee numbers and polynomials with some applications,* Adv. Stud. Contemp. Math. (Kyungshang) **30** (2020), no. 2, 207-222.
[14] Moon, E.-J.; Park, J.-W. *A note on the generalized q-Changhee numbers of higher order,* J. Comput. Anal. Appl. **20** (2016), no. 3, 470-479.
[15] Pak, H.-K.; Jeong, J.; Kang, D.-J.; Rim, S.-H. *Changhee-Genocchi numbers and their applications,* Ars Combin. **136** (2018), 153-159.
[16] Park, J.-W. *On the twisted q-Changhee polynomials of higher order,* J. Comput. Anal. Appl. **20** (2016), no. 3, 424-431.
[17] Rim, S.-H.; Kim, T. *A note on q-Euler numbers associated with the basic q-zeta function,* Appl. Math. Lett. **20** (2007), no. 4, 366-369.
[18] Roman, S. *The umbral calculus,* Pure and Applied Mathematics 111, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1984.
[19] So, J. S.; Simsek, Y. *Derivation of computational formulas for Changhee polynomials and their functional and differential equations,* J. Inequal. Appl. 2020, 2020:149.
[20] Stanley, Richard P. *Enumerative combinatorics* Vol. 2., Cambridge Studies in Advanced Mathematics 62, Cambridge University Press, Cambridge, 1999.

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