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A GENERALIZATION OF LIE'S THEOREM TO CERTAIN NON-LIE SOLVABLE GROUPS

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Abstract. We introduce a class of solvable groups containing the class of connected solvable Lie groups and such that every irreducible finite-dimensional representation of a group of this class is one-dimensional. Therefore, every finite-dimensional representation of a group of this class admits a basis in the representation space in which all representation operators have upper triangular matrices.

§ 1. Introduction

As is well known, by the famous Lie theorem (see, e.g., Theorem 3.7.3 of [1] (in the Lie algebra form) or Theorem V.3.1 of [2] (in the Lie group form)), every continuous irreducible finite-dimensional complex representation of a connected solvable Lie group is one-dimensional. This implies immediately that every continuous finite-dimensional complex representation of a connected solvable Lie group admits a basis in the representation space in which all representation operators have upper triangular matrices. This theorem was extended in [3] to not necessarily continuous representations of connected solvable Lie groups.

Another recent generalization [4] of the Lie theorem (based on the Mal'cev theorem in [5]) enables us to significantly extend the class of solvable groups all of whose finite-dimensional complex representations have a triangular form.

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§ 2. Preliminaries

Definition 1. A set X of a group G is said to be *divisible* if it is formed by divisible elements of G, where an element $g \in X$ is said to be *divisible* if, for every positive integer n, there is an element $h \in G$ for which $h^n = g$.

The solvable (disconnected, discrete) Lie group S_3 is not divisible. A connected solvable Lie group need not be divisible; for details, see [6]. According to [7] (see also [8]), an analytic subgroup of a Lie group is divisible if and only if its exponential function is surjective. Obviously, every connected Lie group is generated by a divisible set; for example, one can take for X a small open neighborhood, of the identity element e of G, covered by the exponential mapping in a one-to-one way.

Definition 2. A solvable group G is said to be a *group of class* $\mathcal{G}$ if there is a chain of normal subgroups

$$G = G_0 \rhd G_1 \rhd \cdots \rhd G_{m-1} \rhd G_m = \{e\}, \tag{1}$$

where every G_i is contained in the commutator subgroup of G_{i-1}, $i = 1, 2, \ldots, m$, and a family of indices

$$m_0 = 0 < m_1 < \cdots < m_r = m, \qquad m_i \in \{1, \ldots, m\}, \quad i = 1, \ldots, r,$$

such that all quotient groups $G_{m_{j-1}}/G_{m_j}$, $j = 1, \ldots, r$, have a generating subset formed by divisible elements.

§ 3. Main results

Theorem. *Let G be a group of class $\mathcal{G}$. Then every irreducible finite-dimensional representation of a group of this class is one-dimensional. Therefore, every finite-dimensional representation of a group of this class admits a basis in the representation space in which all representation operators have upper triangular matrices.*

Proof. Let us prove the theorem by induction on r mentioned in Definition 2. For $r = 1$, the group G has a generating subset formed by divisible elements. In this case, the main result of [4] claims that every representation of G in a finite-dimensional vector space admits a triangular form. Let the assertion of the theorem be valid for $r - 1$, where $r \geq 2$, and let G be a group whose chain $\{G_{m_i}\}$ mentioned in Definition 2 contains r groups. Let π be an irreducible representation of G in a finite-dimensional vector space E, and let σ

be the restriction of π to the subgroup $H = G_1$. By assumption, the representation σ has a one-dimensional subrepresentation, and thus a common eigenvector $\xi \in E$. Thus, in particular,

$$\sigma(h)\xi = \lambda(h)\xi \quad \text{for any} \quad h \in H,$$

where $\lambda(h) \in \mathbb{C}^*$, $h \in H$. In this case,

$$\sigma(h)\pi(g)\xi = \pi(g)\sigma(g^{-1}hg)\xi = \lambda(g^{-1}hg)\pi(g)\xi, \qquad h \in H, \quad g \in G, \tag{2}$$

and therefore all vectors of the form $\pi(g)\xi$, $g \in G$, are common eigenvectors of the representation σ with the eigenvalues $\lambda(g^{-1}hg)$, respectively. All vectors $\pi(g)\xi$, $g \in G$, span a finite-dimensional subspace $F \subset E$ (namely, the linear span of all vectors of the form $\pi(g)\xi$, $g \in G$, in the subspace E) and, by the construction of the subspace F, this subspace has a basis of the eigenvectors of the representation σ. Certainly, every set of linearly independent vectors of the form $\sigma(g)\xi$, $g \in G$, is finite. It follows from formula (2) that the nonzero subspace F is invariant with respect to the representation π of G. By assumption, the representation π is irreducible, and thus $F = E$. We see that the set of linearly independent functions of the form λ_g, $g \in G$, is finite, where

$$\lambda_g(n) = \lambda(g^{-1}ng), \qquad g \in G, \quad n \in N.$$

Thus the set of eigenvalues is finite, and therefore there is only a finite set of characters of the form $\{\lambda_g \mid g \in G\}$. Since λ_g, $g \in G$, are complex-valued characters of the normal subgroup H, these functions are invariant with respect to the inner automorphisms of the group H. Therefore, $\lambda_h = \lambda$, $h \in H$. Thus, the group G/H acts transitively on the (nonempty) finite set $\{\lambda_g \mid g \in G\}$ by permutations of this finite set and, if the number of elements in this finite set is equal to p, then the image of the group G/H is a subgroup of the symmetric group S_p, and therefore the order of this subgroup is a divisor of the number $p!$. By assumption, there is a divisible subset X of G/H generating G/H, and thus, for any $x \in X$, there is an element $y \in G$ such that $y^{p!} = x$. Hence, the permutation corresponding to the element x is the $p!$th power of the permutation corresponding to the element y, and therefore this is the identity permutation. Since X generates G/H, it follows that all permutations defining the transitive action of G/H on $\{\lambda_g \mid g \in G\}$ are identity permutations. Consequently, the set $\{\lambda_g \mid g \in G\}$ is a singleton, we have $\lambda_g = \lambda$ for any $g \in G$, and thus all operators of the representation σ

are scalar multiples of the identity operator on E (operators of multiplication by a number, namely, $\sigma(h) = \lambda(h)1_E$, $h \in H$).

Since the commutator subgroup of G contains $G_1 \supset G_{m_1}$, it follows that all operators of the form $\pi(g_1)$, $g_1 \in G_{m_1}$, are products of operators of the form $\sigma(ghg^{-1}h^{-1})$, $g, h \in G$, and, at the same time, scalar multiples of the identity operator on E (the operator of multiplication by the number $\lambda(g_1)$). However, the determinants of all operators of the form $\sigma(ghg^{-1}h^{-1}) = \pi(g)\pi(h)\pi(g^{-1})\pi(h^{-1})$, $g, h \in G$, are equal to 1, and therefore

$$\det \sigma(ghg^{-1}h^{-1}) = 1 \quad \text{for any} \quad g, h \in G;$$

thus, $\det \sigma(g_1) = 1$ for all $g_1 \in G_1$, or

$$(\lambda(g_1)^{\dim E} = 1, \qquad g_1 \in G_1.$$

Hence, the set of values of the function λ is finite (it is contained in the set of roots of unity of degree n) on the whole group G_{m_1}. However, now $m \geq 2$, and the function λ is equal to one on G_{m_2}, and the λ-image of the group G_{m_1} can be regarded as the image of a group G_{m_1}/G_{m_2} (generated by a divisible set) under a homomorphism defined by the passage to the quotient group by G_{m_2}. Since the image of a divisible set under any homomorphism is divisible and a finite group is generated by a divisible set if and only if it is trivial, it follows that the λ-image of the group G_{m_1} is trivial. Thus,

$$\lambda(g_1) \equiv 1 \quad \text{for any} \quad g_1 \in G_{m_1}.$$

Therefore, the restriction of the representation ρ to G_{m_1} is a multiple of the trivial representation, and we can assume that π is a representation of the quotient group of G by the normal subgroup G_{m_1} of G. Moreover, by assumption, the representation π in question is irreducible, and hence, by the main result of [4], it is one-dimensional.

§ 4. Comments

There is a natural question.

Question. Is it true that every group of class $\mathcal{G}$ admits a generating subset formed by divisible elements?

If the answer is "yes," then our main theorem is equivalent to the main result of [4]. However, in this case, it can be simpler to verify the very conditions of the above theorem.

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