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ON SOME RECURSION RELATIONS FOR HORN'S HYPERGEOMETRIC FUNCTIONS OF THREE VARIABLES

P. AGARWAL[A,B,C,D], A. SHEHATA[E,F,*], S. I. MOUSTAFA[F,G] AND K. CHAKRABORTY[H]

Abstract. The principal aim of this paper to study the recursion formulas for the Horns hypergeometric functions of three variables. Earlier in [Shehata, A.; and Moustafa, S.I. Some new results for Horn's hypergeometric functions Γ_1 and Γ_2. *Journal of Mathematics and Computer Science*, (2021), 23 (1), 26–35.], and Pathan et al. [Pathan, M.A.; Shehata, A.; and Moustafa, S.I. Certain new formulas for the Horns hypergeometric functions. Acta Universitatis Apulensis, (2020)] have studied the new results for Horns hypergeometric functions. Motivated by the above works here we will derive some contiguous relation for the families of Horn hypergeometric functions $\mathbf{G}_A$, $\mathbf{G}_B$, $\mathbf{G}_C$, $\mathbf{G}_D$ and $\mathbf{G}^*_C$ of three variables. After that we will establish the differential reclusion relations and differential operators for $\mathbf{G}_A$, $\mathbf{G}_B$, $\mathbf{G}_C$, $\mathbf{G}_D$ and $\mathbf{G}^*_C$ of three variables, respectively.

1. Introduction and notations

The Horn's functions of three variables are defined by (see, e.g.,[34, 35, 36])

$$\mathbf{G}_A(a,b,c;d;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{n+p-m}(b)_{m+p}(c)_n}{m!n!p!(d)_{n+p-m}}x^my^nz^p, \tag{1.1}$$
$$(d\neq 0,-1,-2,\dots)$$

$$\mathbf{G}_B(a,b,c,d;e;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{n+p-m}(b)_m(c)_n(d)_p}{m!n!p!(e)_{n+p-m}}x^my^nz^p, \tag{1.2}$$
$$(e\neq 0,-1,-2,\dots)$$

$$\mathbf{G}_C(a,b,c,d;e;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{p-m}(b)_n(c)_{m+p}(d)_n}{m!n!p!(e)_{n+p-m}}x^my^nz^p, \tag{1.3}$$
$$(e\neq 0,-1,-2,\dots)$$

$$\mathbf{G}_D(a,b,c,d,e;f;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{p-m}(b)_n(c)_m(d)_n(e)_p}{m!n!p!(f)_{n+p-m}}x^my^nz^p, \tag{1.4}$$
$$(f\neq 0,-1,-2,\dots)$$

and

$$\mathbf{G}^*_C(a,b,c;d;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+p}(b)_{m+n}(c)_{n-p}}{m!n!p!(d)_{n+m-p}}x^my^nz^p, \tag{1.5}$$
$$(d\neq 0,-1,-2,\dots)$$

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* Corresponding author.

where $|x| < 1$, $|y| < 1$ and $|z| < 1$, for more convergence conditions of above functions we recommend to see [10]. Here $(\alpha)_n$ denotes the Pochhammer symbol defined (for $\alpha, n \in \mathbb{C}$), in terms of the Gamma function Γ, by (see [20])

$$(\alpha)_n := \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \begin{cases} 1 & (n=0; \alpha \in \mathbb{C} \setminus \{0\}) \\ \alpha(\alpha+1)\ldots(\alpha+n-1) & (n \in \mathbb{N}; \alpha \in \mathbb{C}). \end{cases}$$

$$(\alpha)_{-n} = \frac{(-1)^n}{(1-\alpha)_n}, (\alpha \neq 0, \pm 1, \pm 2, \pm 3, \ldots, n \in \mathbb{N})$$

and

$$(\alpha)_{m-n} = \frac{(-1)^n (\alpha)_m}{(1-m-\alpha)_n}, 0 \leq n \leq m, m, n \in \mathbb{N}.$$

For our purpose, we recall some functions and notations as follows (see, e.g., [20])

$$\begin{aligned} (\alpha)_{n+1} &= \alpha(\alpha+1)_n = (\alpha+1)(\alpha)_n, \\ (\alpha+1)_n &= \left(1+\frac{n}{\alpha}\right)(\alpha)_n; (\alpha \neq 0), \\ \frac{1}{(\beta-1)_n} &= \left(1+\frac{n}{\beta-1}\right)\frac{1}{(\beta)_n}; (\beta \neq 1, 0, -1, -2, \ldots). \end{aligned} \tag{1.6}$$

Throughout the study, to simplify the notations, we write $\mathbf{G}_A$ for the function $\mathbf{G}_A(a, b, c; d; x, y, z)$, $\mathbf{G}_A(a \pm n)$ for the function $\mathbf{G}_A(a \pm n, b, c; d; x, y, z)$, $\mathbf{G}_A(b \pm n)$ for the function $\mathbf{G}_A(a, b \pm n, c; d; x, y, z)$, $\ldots$, and $\mathbf{G}^*_C(d \pm n)$ stands for the function $\mathbf{G}^*_C(a, b, c; d \pm n; x, y, z)$ etc.

Sahin [21, 22] has studied recursion formulas of Srivastava hypergeometric functions with respect to its numerator and denominator parameters, followed by the work of Sahai and Verma [23, 24, 25, 26] on derivatives of Srivastava-Daoust hypergeometric function with respect to its parameters. These works were extended to recursion formulas of Appell and Horn hypergeometric functions with respect to parameters in Ancarani et al. [1], Bezrodnykh [2], Brychkov [3], Brychkov and Saad [4, 5, 6, 7], Buschman [8, 9], Opps et al. [16, 17], Sharma [29], Wang [37]. Many researchers (for example, Dhawan [10], Exton [11], Karlsson [12], Khan and Pathan [13, 14], Mullen [15], Padmanabham [18], Saran [27, 28], Srivastava [31, 32, 33], etc.) have studied a number of recursion formulas involving a variety of special functions of mathematical physics. Shehata and Moustafa [30], and Pathan et al. [19] have earlier studied the new results for Horn's hypergeometric functions Γ_1, Γ_2, H_1, H_2, H_3, H_4, H_5, H_6, H_7, H_8, H_9, H_{10} and H_{11} of two variables. The reason of interest for this family of Horn's hypergeometric functions is due to their intrinsic mathematical importance.

It is noted that Horn's functions are a very deep mathematical object that attracts the attention of many researchers. In this paper, we aim to study of reclusion relations, differential reclusion relations, new differential operators of the Horn hypergeometric functions with respect to its parameters. This paper is constructed as follows. In Section 2, we will derive some contiguous relation for the families of Horn hypergeometric functions $\mathbf{G}_A$, $\mathbf{G}_B$, $\mathbf{G}_C$, $\mathbf{G}_D$ and $\mathbf{G}^*_C$ of three variables. In Section 3, will establish the differential reclusion relations and differential operators for $\mathbf{G}_A$, $\mathbf{G}_B$, $\mathbf{G}_C$, $\mathbf{G}_D$ and $\mathbf{G}^*_C$ of three variables, respectively.

2. Recursion formulas of functions $\mathbf{G}_A$, $\mathbf{G}_B$, $\mathbf{G}_C$, $\mathbf{G}_D$ and $\mathbf{G}^*_C$

Here we present some recursion formulas for the functions $\mathbf{G}_A$, $\mathbf{G}_B$, $\mathbf{G}_C$, $\mathbf{G}_D$ and $\mathbf{G}^*_C$ with respect to parameters. We start with the following Theorem.

Theorem 2.1. *If $n \in \mathbb{N}$ and $d \neq 1, 0, -1, -2, \ldots$. Then the following recursion formulas for the function $\mathbf{G}_A$ holds*

$$\begin{aligned}\mathbf{G}_A(a+n) =& \mathbf{G}_A + \frac{cy}{d}\sum_{k=1}^{n}\mathbf{G}_A(a+k,b,c+1;d+1;x,y,z)\\ &+ \frac{bz}{d}\sum_{k=1}^{n}\mathbf{G}_A(a+k,b+1,c;d+1;x,y,z)\\ &- b(d-1)x\sum_{k=1}^{n}\frac{\mathbf{G}_A(a+k-2,b+1,c;d-1;x,y,z)}{(a+k-2)(a+k-1)}, (a \neq 1-k, 2-k, k \in \mathbb{N})\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}\mathbf{G}_A(a-n) =& \mathbf{G}_A - \frac{cy}{d}\sum_{k=1}^{n}\mathbf{G}_A(a-k+1,b,c+1;d+1;x,y,z)\\ &- \frac{bz}{d}\sum_{k=1}^{n}\mathbf{G}_A(a-k+1,b+1,c;d+1;x,y,z)\\ &+ b(d-1)x\sum_{k=1}^{n}\frac{\mathbf{G}_A(a-k-1,b+1,c;d-1;x,y,z)}{(a-k)(a-k-1)}, (a \neq k, k+1, k \in \mathbb{N})\end{aligned} \tag{2.2}$$

Proof. Using the following transformation in the Definition of the function $\mathbf{G}_A$ (1.1)

$$(a+1)_{n+p-m} = (a)_{n+p-m}\left(1 + \frac{n+p-m}{a}\right),$$

we obtain

$$\begin{aligned}\mathbf{G}_A(a+1) = \mathbf{G}_A &+ \frac{cy}{d}\mathbf{G}_A(a+1,b,c+1;d+1;x,y,z) + \frac{bz}{d}\mathbf{G}_A(a+1,b+1,c;d+1;x,y,z)\\ &- \frac{b(d-1)x}{a(a-1)}\mathbf{G}_A(a-1,b+1,c;d-1;x,y,z), (a \neq 0, 1, d \neq 1, 0, -1, -2, \ldots).\end{aligned} \tag{2.3}$$

By applying this contiguous relation to the function $\mathbf{G}_A$ (1.1) with the parameter $a = a + n$ which, in view of the relation (2.3) for n times, leads us easily to the (2.1).

Replacing a by $a - 1$, from the contiguous relation (2.3), we get

$$\begin{aligned}\mathbf{G}_A(a-1) = \mathbf{G}_A &- \frac{cy}{d}\mathbf{G}_A(a,b,c+1;d+1;x,y,z) - \frac{bz}{d}\mathbf{G}_A(a,b+1,c;d+1;x,y,z)\\ &+ \frac{b(d-1)x}{(a-1)(a-2)}\mathbf{G}_A(a-2,b+1,c;d-1;x,y,z), (a \neq 1, 2, d \neq 1, 0, -1, -2, \ldots).\end{aligned} \tag{2.4}$$

Apply this relation to the function $\mathbf{G}_A$ with the parameter $a - n$ for n times, we easily obtain the recursion formula (2.2). □

Theorem 2.2. *The recurrence relation hold true for the function $\mathbf{G}_A$, if $n \in \mathbb{N}$, $a \neq 1$ and $d \neq 1, 0, -1, -2, \ldots$.*

$$\begin{aligned}\mathbf{G}_A(b+n) =& \mathbf{G}_A + \frac{(d-1)x}{a-1}\sum_{k=1}^{n}\mathbf{G}_A(a-1,b+k,c;d-1;x,y,z)\\ &+ \frac{az}{d}\sum_{k=1}^{n}\mathbf{G}_A(a+1,b+k,c;d+1;x,y,z)\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\mathbf{G}_A(b-n) =&\mathbf{G}_A - \frac{(d-1)x}{a-1}\sum_{k=1}^{n}\mathbf{G}_A(a-1,b-k+1,c;d-1;x,y,z)\\&-\frac{az}{d}\sum_{k=1}^{n}\mathbf{G}_A(a+1,b-k+1,c;d+1;x,y,z).\end{aligned}\tag{2.6}$$

Proof. Using the definition of the function $\mathbf{G}_A$ (1.1) and the relation

$$(b+1)_{m+p} = (b)_{m+p}\left(1+\frac{m+p}{b}\right)$$

we obtain the contiguous function

$$\begin{aligned}\mathbf{G}_A(b+1) =&\mathbf{G}_A + \frac{(d-1)x}{a-1}\mathbf{G}_A(a-1,b+1,c;d-1;x,y,z)\\&+\frac{az}{d}\mathbf{G}_A(a+1,b+1,c;d+1;x,y,z), (a\neq 1, d\neq 1,0,-1,-2,\ldots).\end{aligned}\tag{2.7}$$

By iterating this method on $\mathbf{G}_A$ with the parameter $b+n$ for n times, leads us to the (2.5).

Replacing b by $b-1$ in contiguous relation (2.7), we get

$$\begin{aligned}\mathbf{G}_A(b-1) =&\mathbf{G}_A - \frac{(d-1)x}{a-1}\mathbf{G}_A(a-1,b,c;d-1;x,y,z)\\&-\frac{az}{d}\mathbf{G}_A(a+1,b,c;d+1;x,y,z), a\neq 1, d\neq 1,0,-1,-2,\ldots.\end{aligned}\tag{2.8}$$

If we compute the function $\mathbf{G}_A$ with the parameter $b-n$ by the contiguous relation (2.8) for n times, we obtain the recursion formula (2.6). □

Theorem 2.3. *Let $d\neq 0,-1,-2,\ldots$. Then the function $\mathbf{G}_A$ satisfies the following identity*

$$\mathbf{G}_A(c+n) = \mathbf{G}_A + \frac{ay}{d}\sum_{k=1}^{n}\mathbf{G}_A(a+1,b,c+k;d+1;x,y,z)\tag{2.9}$$

and

$$\mathbf{G}_A(c-n) = \mathbf{G}_A - \frac{ay}{d}\sum_{k=1}^{n}\mathbf{G}_A(a+1,b,c-k+1;d+1;x,y,z).\tag{2.10}$$

Proof. By using the definition of the function $\mathbf{G}_A$ (1.1) and transformation

$$(c+1)_n = (c)_n\left(1+\frac{n}{c}\right),$$

we obtain the contiguous function

$$\mathbf{G}_A(c+1) = \mathbf{G}_A + \frac{ay}{d}\mathbf{G}_A(a+1,b,c+1;d+1;x,y,z), (d\neq 0,-1,-2,\ldots.)\tag{2.11}$$

If we compute the function $\mathbf{G}_A$ with the parameter $c+n$. Then by relation (2.11) for n times, we find the formula given in (2.9).

Replacing c by $c-1$ in the contiguous relation (2.11), we get

$$\mathbf{G}_A(c-1) = \mathbf{G}_A - \frac{ay}{d}\mathbf{G}_A(a+1,b,c;d+1;x,y,z), d\neq 0,-1,-2,\ldots.\tag{2.12}$$

If we apply this relation to the function $\mathbf{G}_A$ with the parameter $c-n$ for n times, we easily obtain the recursion formula (2.10). □

Theorem 2.4. *The function* $\mathbf{G}_A$ *satisfies the following identities*

$$\begin{aligned}\mathbf{G}_A(d-n) = \mathbf{G}_A &+ acy\sum_{k=1}^{n}\frac{\mathbf{G}_A(a+1,b,c+1;d-k+2;x,y,z)}{(d-k+1)(d-k)}\\ &+ abz\sum_{k=1}^{n}\frac{\mathbf{G}_A(a+1,b+1,c;d-k+2;x,y,z)}{(d-k+1)(d-k)}\\ &- \frac{bx}{a-1}\sum_{k=1}^{n}\mathbf{G}_A(a-1,b+1,c;d-k;x,y,z),\\ &\qquad (a\neq 1, d\neq k,k-1,k-2,\ldots,k\in\mathbb{N})\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}\mathbf{G}_A(d+n) = \mathbf{G}_A &- acy\sum_{k=1}^{n}\frac{\mathbf{G}_A(a+1,b,c+1;d+k+1;x,y,z)}{(d+k)(d+k-1)}\\ &- abz\sum_{k=1}^{n}\frac{\mathbf{G}_A(a+1,b+1,c;d+k+1;x,y,z)}{(d+k)(d+k-1)}\\ &+ \frac{bx}{a-1}\sum_{k=1}^{n}\mathbf{G}_A(a-1,b+1,c;d+k-1;x,y,z),\\ &\qquad (a\neq 1, d\neq 1-k,-k,-k-1,-k-2,\ldots,k\in\mathbb{N}.)\end{aligned} \tag{2.14}$$

Proof. Start from the definition of the function $\mathbf{G}_A$ (1.1) and transformation

$$\frac{1}{(d-1)_{n+p-m}} = \frac{1}{(d)_{n+p-m}} + \frac{n+p-m}{(d-1)(d)_{n+p-m}}$$

we get

$$\begin{aligned}\mathbf{G}_A(d-1) =&\mathbf{G}_A + \frac{acy}{d(d-1)}\mathbf{G}_A(a+1,b,c+1;d+1;x,y,z)\\ &+ \frac{abz}{d(d-1)}\mathbf{G}_A(a+1,b+1,c;d+1;x,y,z)\\ &- \frac{bx}{a-1}\mathbf{G}_A(a-1,b+1,c;d-1;x,y,z),\\ &\qquad (a\neq 1, d\neq 1,0,-1,-2,\ldots).\end{aligned} \tag{2.15}$$

By iterating this method on $\mathbf{G}_A$ with $d-n$ for n times, we obtain (2.13).

Replacing d by $d+1$ in contiguous relation (2.15), we get

$$\begin{aligned}\mathbf{G}_A(d+1) =&\mathbf{G}_A - \frac{acy}{(d+1)d}\mathbf{G}_A(a+1,b,c+1;d+2;x,y,z)\\ &- \frac{abz}{(d+1)d}\mathbf{G}_A(a+1,b+1,c;d+2;x,y,z)\\ &+ \frac{bx}{a-1}\mathbf{G}_A(a-1,b+1,c;d;x,y,z),\\ &\qquad (a\neq 1, d\neq 0,-1,-2,\ldots.)\end{aligned} \tag{2.16}$$

If we apply this relation (2.16) to the function $\mathbf{G}_A$ with the parameter $d+n$ for n times, we obtain the recursion formulas (2.14). □

By the similar method in the Theorems 2.1-2.4, we establish some recursion formulas for functions $\mathbf{G}_B$, $\mathbf{G}_C$, $\mathbf{G}_D$ and $\mathbf{G}_C^*$.

Theorem 2.5. *Recursion formulas for the function $\mathbf{G}_B$ holds true for $n \in \mathbb{N}$ and $e \neq 1, 0, -1, -2, \ldots$.*

$$\begin{aligned}\mathbf{G}_B(a+n) =&\mathbf{G}_B + \frac{cy}{e}\sum_{k=1}^{n}\mathbf{G}_B(a+k, b, c+1, d; e+1; x, y, z)\\&+\frac{dz}{e}\sum_{k=1}^{n}\mathbf{G}_B(a+k, b, c, d+1; e+1; x, y, z)\\&-b(e-1)x\sum_{k=1}^{n}\frac{\mathbf{G}_B(a+k-2, b+1, c, d; e-1; x, y, z)}{(a+k-1)(a+k-2)},\\&(a \neq 1-k, 2-k, k \in \mathbb{N})\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}\mathbf{G}_B(a-n) =&\mathbf{G}_B - \frac{cy}{e}\sum_{k=1}^{n}\mathbf{G}_B(a-k+1, b, c+1, d; e+1; x, y, z)\\&-\frac{dz}{e}\sum_{k=1}^{n}\mathbf{G}_B(a-k+1, b, c, d+1; e+1; x, y, z)\\&+b(e-1)x\sum_{k=1}^{n}\frac{\mathbf{G}_B(a-k-1, b+1, c, d; e-1; x, y, z)}{(a-k)(a-k-1)},\\&(a \neq k, k+1, k \in \mathbb{N}).\end{aligned} \tag{2.18}$$

Theorem 2.6. *If, $a \neq 1$, $e \neq 1, 0, -1, -2, \ldots$. Then the recurrence relations hold true for the function $\mathbf{G}_B$*

$$\mathbf{G}_B(b+n) = \mathbf{G}_B + \frac{(e-1)x}{a-1}\sum_{k=1}^{n}\mathbf{G}_B(a-1, b+k, c, d; e-1; x, y, z) \tag{2.19}$$

and

$$\mathbf{G}_B(b-n) = \mathbf{G}_B + \frac{(e-1)x}{a-1}\sum_{k=1}^{n}\mathbf{G}_B(a-1, b-k+1, c, d; e-1; x, y, z). \tag{2.20}$$

Theorem 2.7. *Let $e \neq 0, -1, -2, \ldots$. Then the following recursion formulas hold true for the function $\mathbf{G}_B$*

$$\mathbf{G}_B(c+n) = \mathbf{G}_B + \frac{ay}{e}\sum_{k=1}^{n}\mathbf{G}_B(a+1, b, c+k, d; e+1; x, y, z) \tag{2.21}$$

and

$$\mathbf{G}_B(c-n) = \mathbf{G}_B - \frac{ay}{e}\sum_{k=1}^{n}\mathbf{G}_B(a+1, b, c-k+1, d; e+1; x, y, z). \tag{2.22}$$

Theorem 2.8. *The Horn function $\mathbf{G}_B$ satisfy the identity, if $e \neq 0, -1, -2, \ldots$.*

$$\mathbf{G}_B(d+n) = \mathbf{G}_B + \frac{az}{e}\sum_{k=1}^{n}\mathbf{G}_B(a+1, b, c, d+k; e+1; x, y, z) \tag{2.23}$$

and

$$\mathbf{G}_B(d-n) = \mathbf{G}_B - \frac{az}{e}\sum_{k=1}^{n}\mathbf{G}_B(a+1,b,c,d-k+1;e+1;x,y,z). \tag{2.24}$$

Theorem 2.9. *The function* $\mathbf{G}_B$ *satisfy the following relations*

$$\begin{aligned}\mathbf{G}_B(e-n) =&\mathbf{G}_B + acy\sum_{k=1}^{n}\frac{\mathbf{G}_B(a+1,b+1,c,d;e-k+2;x,y,z)}{(e-k+1)(e-k)}\\ &+ adz\sum_{k=1}^{n}\frac{\mathbf{G}_B(a+1,b,c,d+1;e-k+2;x,y,z)}{(e-k+1)(e-k)}\\ &-\frac{bx}{a-1}\sum_{k=1}^{n}\mathbf{G}_B(a-1,b+1,c,d;e-k;x,y,z),\\ &(a\neq 1, e\neq k,k-1,k-2,\ldots,k\in\mathbb{N})\end{aligned} \tag{2.25}$$

and

$$\begin{aligned}\mathbf{G}_B(e+n) =&\mathbf{G}_B - acy\sum_{k=1}^{n}\frac{\mathbf{G}_B(a+1,b+1,c,d;e+k+1;x,y,z)}{(e+k)(e+k-1)}\\ &+ adz\sum_{k=1}^{n}\frac{\mathbf{G}_B(a+1,b,c,d+1;e+k+1;x,y,z)}{(e+k)(e+k-1)}\\ &-\frac{bx}{a-1}\sum_{k=1}^{n}\mathbf{G}_B(a-1,b+1,c,d;e+k-1;x,y,z),\\ &(a\neq 1, e\neq 1-k,-k,-1-k,-2-k,\ldots,k\in\mathbb{N}).\end{aligned} \tag{2.26}$$

Theorem 2.10. *If* $n\in\mathbb{N}$ *and* $e\neq 1,0,-1,-2,\ldots$. *Then the recursion formulas for the function* $\mathbf{G}_C$ *holds true*

$$\begin{aligned}\mathbf{G}_C(a+n) = \mathbf{G}_C &+ \frac{cz}{e}\sum_{k=1}^{n}\mathbf{G}_C(a+k,b,c+1,d;e+1;x,y,z)\\ &- c(e-1)x\sum_{k=1}^{n}\frac{\mathbf{G}_C(a+k-1,b+1,c,d;e-1;x,y,z)}{(a+k-2)(a+k-1)},\\ &\qquad (a\neq 1-k,2-k,k\in\mathbb{N})\end{aligned} \tag{2.27}$$

and

$$\begin{aligned}\mathbf{G}_C(a-n) = \mathbf{G}_C &- \frac{cz}{e}\sum_{k=1}^{n}\mathbf{G}_C(a-k+1,b,c+1,d;e+1;x,y,z)\\ &+ c(e-1)x\sum_{k=1}^{n}\frac{\mathbf{G}_C(a-k-1,b,c+1,d;e-1;x,y,z)}{(a-k)(a-k-1)},\\ &\qquad (a\neq k,k+1,k\in\mathbb{N})\end{aligned} \tag{2.28}$$

Theorem 2.11. *Let* $e\neq 0,-1,-2,\ldots$. *Then the recurrence relations hold true for the function* $\mathbf{G}_C$

$$\mathbf{G}_C(b+n) = \mathbf{G}_C + \frac{d\,y}{e}\sum_{k=1}^{n}\mathbf{G}_C(a,b+k,c,d+1;e+1;x,y,z), \tag{2.29}$$

and

$$\mathbf{G}_C(b-n)=\mathbf{G}_C-\frac{d\,y}{e}\sum_{k=1}^{n}\mathbf{G}_C(a,b-k+1,c,d+1;e+1;x,y,z). \tag{2.30}$$

Theorem 2.12. *If $a\neq 1$ and $e\neq 1,0,-1,-2,\ldots$. Then the recursion formulas for the function $\mathbf{G}_C$ holds*

$$\begin{aligned}\mathbf{G}_C(c+n)=\mathbf{G}_C&+\frac{(e-1)x}{a-1}\sum_{k=1}^{n}\mathbf{G}_C(a-1,b,c+k,d;e-1;x,y,z)\\&+\frac{a\,z}{e}\sum_{k=1}^{n}\mathbf{G}_C(a+1,b,c+k,d;e+1;x,y,z)\end{aligned} \tag{2.31}$$

and

$$\begin{aligned}\mathbf{G}_C(c-n)=\mathbf{G}_C&-\frac{(e-1)x}{a-1}\sum_{k=1}^{n}\mathbf{G}_C(a-1,b,c-k+1,d;e-1;x,y,z)\\&-\frac{a\,z}{e}\sum_{k=1}^{n}\mathbf{G}_C(a+1,b,c-k+1,d;e+1;x,y,z).\end{aligned} \tag{2.32}$$

Theorem 2.13. *The function $\mathbf{G}_C$ satisfies the following recursion formulas for $e\neq 0,-1,-2,\ldots$.*

$$\mathbf{G}_C(d+n)=\mathbf{G}_C+\frac{b\,y}{e}\sum_{k=1}^{n}\mathbf{G}_C(a,b+1,c,d+k;e+1;x,y,z) \tag{2.33}$$

and

$$\mathbf{G}_C(d-n)=\mathbf{G}_C-\frac{b\,y}{e}\sum_{k=1}^{n}\mathbf{G}_C(a,b+1,c,d-k+1;e+1;x,y,z). \tag{2.34}$$

Theorem 2.14. *The function $\mathbf{G}_C$ satisfies the identity*

$$\begin{aligned}\mathbf{G}_C(e-n)=&\mathbf{G}_C+bdy\sum_{k=1}^{n}\frac{\mathbf{G}_C(a,b+1,c,d+1;e-k+2;x,y,z)}{(e-k+1)(e-k)}\\&+acz\sum_{k=1}^{n}\frac{\mathbf{G}_C(a+1,b,c+1,d;e-k+2;x,y,z)}{(e-k+1)(e-k)}\\&-\frac{cx}{a-1}\sum_{k=1}^{n}\mathbf{G}_C(a-1,b,c+1,d;e-k;x,y,z),\\&\qquad(a\neq 1,e\neq k,k-1,k-2,\ldots,k\in\mathbb{N}),\end{aligned} \tag{2.35}$$

and

$$\begin{aligned}\mathbf{G}_C(e+n)=&\mathbf{G}_C-bdy\sum_{k=1}^{n}\frac{\mathbf{G}_C(a+1,b+1,c,d+1;e+k+1;x,y,z)}{(e+k)(e+k-1)}\\&-acz\sum_{k=1}^{n}\frac{\mathbf{G}_C(a+1,b,c+1,d;e+k+1;x,y,z)}{(e+k)(e+k-1)}\\&+\frac{cx}{a-1}\sum_{k=1}^{n}\mathbf{G}_C(a-1,b,c+1,d;e+k-1;x,y,z),\\&\qquad(a\neq 1,e\neq 1-k,-k,-k-1,-k-2,\ldots,k\in\mathbb{N}).\end{aligned} \tag{2.36}$$

Theorem 2.15. *Recursion formulas for the function* $\mathbf{G}_D$ *hold true for* $n \in \mathbb{N}$ *and* $f \neq 1, 0, -1, -2, \ldots$.

$$\begin{aligned}\mathbf{G}_D(a+n) =&\mathbf{G}_D + \frac{e\,z}{f}\sum_{k=1}^{n}\mathbf{G}_D(a+k,b,c,d,e+1;f+1;x,y,z)\\&- c(f-1)x\sum_{k=1}^{n}\frac{\mathbf{G}_D(a+k-2,b,c+1,d,e;f-1;x,y,z)}{(a+k-2)(a+k-1)},\\&(a \neq 1-k, 2-k, k \in \mathbb{N})\end{aligned} \tag{2.37}$$

and

$$\begin{aligned}\mathbf{G}_D(a-n) =&\mathbf{G}_D - \frac{e\,z}{f}\sum_{k=1}^{n}\mathbf{G}_D(a-k+1,b,c,d,e+1;f+1;x,y,z)\\&+ c(f-1)x\sum_{k=1}^{n}\frac{\mathbf{G}_D(a-k-1,b,c+1,d,e;f-1;x,y,z)}{(a-k)(a-k-1)},\\&(a \neq k, k+1, k \in \mathbb{N})\,.\end{aligned} \tag{2.38}$$

Theorem 2.16. *If* $f \neq 0, -1, -2, \ldots$. *Then the recurrence relation hold true for the function* $\mathbf{G}_D$*:*

$$\mathbf{G}_D(b+n) = \mathbf{G}_D + \frac{bd\,y}{f}\sum_{k=1}^{n}\mathbf{G}_D(a,b+k,c,d+1,e;f+1;x,y,z) \tag{2.39}$$

and

$$\mathbf{G}_D(b-n) = \mathbf{G}_D - \frac{bd\,y}{f}\sum_{k=1}^{n}\mathbf{G}_D(a,b-k+1,c,d+1,e;f+1;x,y,z). \tag{2.40}$$

Theorem 2.17. *For* $a \neq 1$ *and* $f \neq 1, 0, -1, -2, \ldots$, *the recursion formula holds true for the parameter* c *of* $\mathbf{G}_D$

$$\mathbf{G}_D(c+n) = \mathbf{G}_D + \frac{(f-1)x}{a-1}\sum_{k=1}^{n}\mathbf{G}_D(a-1,b,c+k,d,e;f-1;x,y,z) \tag{2.41}$$

and

$$\mathbf{G}_D(c-n) = \mathbf{G}_D - \frac{(f-1)x}{a-1}\sum_{k=1}^{n}\mathbf{G}_D(a-1,b,c-k+1,d,e;f-1;x,y,z). \tag{2.42}$$

Theorem 2.18. *If* $f \neq 0, -1, -2, \ldots$. *Then the recursion formulas hold true for the parameters* d *of* $\mathbf{G}_D$

$$\mathbf{G}_D(d+n) = \mathbf{G}_D + \frac{bd\,y}{f}\sum_{k=1}^{n}\mathbf{G}_D(a,b+1,c,d+k,e;f+1;x,y,z) \tag{2.43}$$

and

$$\mathbf{G}_D(d-n) = \mathbf{G}_D - \frac{bd\,y}{f}\sum_{k=1}^{n}\mathbf{G}_D(a,b,c,d-k+1,e;f+1;x,y,z). \tag{2.44}$$

Theorem 2.19. *Let* $f \neq 0, -1, -2, \ldots$. *Then the recursion formulas hold true for the parameter* e *of* $\mathbf{G}_D$

$$\mathbf{G}_D(e+n) = \mathbf{G}_D + \frac{a\,z}{f}\sum_{k=1}^{n}\mathbf{G}_D(a+1,b,c,d,e+k;f+1;x,y,z) \tag{2.45}$$

and

$$\mathbf{G}_D(e-n)=\mathbf{G}_D-\frac{a\,z}{f}\sum_{k=1}^{n}\mathbf{G}_D(a+1,b,c,d,e-k+1;f+1;x,y,z). \tag{2.46}$$

Theorem 2.20. *For $a\neq 1$, the function $\mathbf{G}_D$ satisfies the following formulas*

$$\begin{aligned}\mathbf{G}_D(f-n)=&\mathbf{G}_D+bd\,y\sum_{k=1}^{n}\frac{\mathbf{G}_D(a,b+1,c,d+1,e;f-k+2;x,y,z)}{(f-k+1)(f-k)}\\&+ae\,z\sum_{k=1}^{n}\frac{\mathbf{G}_D(a+1,b,c,d,e+1;f-k+2;x,y,z)}{(f-k+1)(f-k)}\\&-\frac{cx}{a-1}\sum_{k=1}^{n}\mathbf{G}_D(a-1,b,c+1,d,e;f-k;x,y,z),\\&\qquad(f\neq k,k-1,k-2,\ldots,k\in\mathbb{N})\end{aligned} \tag{2.47}$$

and

$$\begin{aligned}\mathbf{G}_D(f+n)=&\mathbf{G}_D-bd\,y\sum_{k=1}^{n}\frac{\mathbf{G}_D(a,b+1,c,d+1,e;f+k+1;x,y,z)}{(f+k)(f+k-1)}\\&-ae\,z\sum_{k=1}^{n}\frac{\mathbf{G}_D(a+1,b,c,d,e+1;f+k+1;x,y,z)}{(f+k)(f+k-1)}\\&+\frac{cx}{a-1}\sum_{k=1}^{n}\mathbf{G}_D(a-1,b,c+1,d,e;f+k-1;x,y,z),\\&\qquad(f\neq 1-k,-k,-k-1,-k-2,\ldots,k\in\mathbb{N}).\end{aligned} \tag{2.48}$$

Theorem 2.21. *Recursion formulas for the function $\mathbf{G}^*_C$ holds true for $n\in\mathbb{N}$, $c\neq 1$ and $d\neq 1,0,-1,-2,\ldots$.*

$$\begin{aligned}\mathbf{G}^*_C(a+n)=&\mathbf{G}^*_C+\frac{z}{d(c-1)}\sum_{k=1}^{n}\mathbf{G}^*_C(a+k,b,c-1;d+1;x,y,z)\\&+b(d-1)x\sum_{k=1}^{n}\mathbf{G}^*_C(a+k,b+1,c;d-1;x,y,z)\end{aligned} \tag{2.49}$$

and

$$\begin{aligned}\mathbf{G}^*_C(a-n)=&\mathbf{G}^*_C-\frac{z}{d(c-1)}\sum_{k=1}^{n}\mathbf{G}^*_C(a-k+1,b,c-1;d+1;x,y,z)\\&+b(d-1)x\sum_{k=1}^{n}\mathbf{G}^*_C(a-k+1,b+1,c;d-1;x,y,z).\end{aligned} \tag{2.50}$$

Theorem 2.22. *The recurrence relations hold true for the function $\mathbf{G}^*_C$, if $d\neq 1,0,-1,-2,\ldots$.*

$$\begin{aligned}\mathbf{G}^*_C(b+n)=&\mathbf{G}^*_C+a(d-1)x\sum_{k=1}^{n}\mathbf{G}^*_C(a+1,b+k,c;d-1;x,y,z)\\&+\frac{c\,y}{d}\sum_{k=1}^{n}\mathbf{G}^*_C(a,b+k,c+1;d+1;x,y,z)\end{aligned} \tag{2.51}$$

and

$$\begin{aligned}\mathbf{G}_C^*(b-n) =&\mathbf{G}_C^* - a(d-1)x\sum_{k=1}^{n}\mathbf{G}_C^*(a+1,b-k+1,c;d-1;x,y,z)\\&-\frac{c\,y}{d}\sum_{k=1}^{n}\mathbf{G}_C^*(a,b-k+1,c+1;d+1;x,y,z).\end{aligned}\tag{2.52}$$

Theorem 2.23. *Let $d \neq 0,-1,-2,\ldots$. Then the recursion formulas hold true for the numerator parameter c of $\mathbf{G}_C^*$*

$$\begin{aligned}\mathbf{G}_C^*(c+n) =&\mathbf{G}_C^* + \frac{by}{d}\sum_{k=1}^{n}\mathbf{G}_C^*(a,b+1,c+k;d+1;x,y,z)\\&-\frac{az}{d}\sum_{k=1}^{n}\frac{\mathbf{G}_C^*(a+1,b,c+k-2;d+1;x,y,z)}{(c+k-1)(c+k-2)},\\&\qquad(c\neq 1-k,2-k,k\in\mathbb{N})\end{aligned}\tag{2.53}$$

and

$$\begin{aligned}\mathbf{G}_C^*(c-n) =&\mathbf{G}_C^* - \frac{by}{d}\sum_{k=1}^{n}\mathbf{G}_C^*(a,b+1,c-k+1;d+1;x,y,z)\\&+\frac{az}{d}\sum_{k=1}^{n}\frac{\mathbf{G}_C^*(a+1,b,c-k-1;d+1;x,y,z)}{(c-k)(c-k-1)},\\&\qquad(c\neq k,k+1,k\in\mathbb{N}).\end{aligned}\tag{2.54}$$

Theorem 2.24. *The function $\mathbf{G}_C^*$ satisfies the following formulas*

$$\begin{aligned}\mathbf{G}_C^*(d-n) =&\mathbf{G}_C^* + bcy\sum_{k=1}^{n}\frac{\mathbf{G}_C^*(a,b+1,c+1;d-k+2;x,y,z)}{(d-k+1)(d-k)}\\&+\frac{az}{c-1}\sum_{k=1}^{n}\frac{\mathbf{G}_C^*(a+1,b,c-1;d-k+2;x,y,z)}{(d-k+1)(d-k)}\\&-abx\sum_{k=1}^{n}\mathbf{G}_C^*(a+1,b+1,c;d-k;x,y,z),\\&\qquad(c\neq 1,d\neq k,k-1,k-2,\ldots,k\in\mathbb{N})\end{aligned}\tag{2.55}$$

and

$$\begin{aligned}\mathbf{G}_C^*(d+n) =&\mathbf{G}_C^* - bcy\sum_{k=1}^{n}\frac{\mathbf{G}_C^*(a+1,b+1;d+k+1;x,y,z)}{(d+k)(d+k-1)}\\&-\frac{az}{c-1}\sum_{k=1}^{n}\frac{\mathbf{G}_C^*(a+1,b+1;d+k+1;x,y,z)}{(d+k)(d+k-1)}\\&+abx\sum_{k=1}^{n}\mathbf{G}_C^*(a-1,b+1,c;d+k-1;x,y,z),\\&\qquad(c\neq 1,d\neq 1-k,-k,-k-1,-k-2,\ldots,k\in\mathbb{N}).\end{aligned}\tag{2.56}$$

3. Differential recursion formulas of Horn's functions $\mathbf{G}_A$, $\mathbf{G}_B$, $\mathbf{G}_C$, $\mathbf{G}_D$ and $\mathbf{G}_C^*$

Here we obtain differential recursion formulas satisfied by the Horn's functions $\mathbf{G}_A$, $\mathbf{G}_B$, $\mathbf{G}_C$, $\mathbf{G}_D$ and $\mathbf{G}_C^*$ most simply by using the differential operators $\theta_x = x\frac{\partial}{\partial x}$, $\theta_y = y\frac{\partial}{\partial y}$ and $\theta_z = z\frac{\partial}{\partial z}$, whose effect on the series is to multiply the mn^{th} term by m, n or p, respectively.

Theorem 3.1. *Differential recursion formulas for the function* $\mathbf{G}_A$ *are as follows*

$$\mathbf{G}_A(a+1) = \left(1 + \frac{\theta_y + \theta_z - \theta_x}{a}\right)\mathbf{G}_A, a \neq 0. \tag{3.1}$$

Proof. Defining the differential operators

$$\begin{aligned}\theta_x x^m &= x\frac{\partial}{\partial x}x^m = mx^m,\\ \theta_y y^n &= y\frac{\partial}{\partial y}y^n = ny^n,\\ \theta_z z^n &= z\frac{\partial}{\partial z}z^n = nz^n.\end{aligned}$$

By using the above differential operators, we obtain the differential recursion formula for $\mathbf{G}_A$

$$\begin{aligned}\mathbf{G}_A(a+1) &= \sum_{m,n=0}^{\infty} \frac{\left(1 + \frac{n+p-m}{a}\right)(a)_{n+p-m}(b)_{m+p}(c)_n}{m!n!p!(d)_{n+p-m}} x^m y^n z^p\\ &= \mathbf{G}_A + \frac{\theta_y}{a}\mathbf{G}_A + \frac{\theta_z}{a}\mathbf{G}_A - \frac{\theta_x}{a}\mathbf{G}_A.\end{aligned}$$

□

Theorem 3.2. *The differential recursion formulas hold true for the parameters of the Horn function* $\mathbf{G}_A$*:*

$$\mathbf{G}_A(b+1) = \left(1 + \frac{\theta_x + \theta_z}{b}\right)\mathbf{G}_A, b \neq 0, \tag{3.2}$$

$$\mathbf{G}_A(c+1) = \left(1 + \frac{\theta_y}{c}\right)\mathbf{G}_A, c \neq 0 \tag{3.3}$$

and

$$\mathbf{G}_A(d-1) = \left(1 + \frac{\theta_y + \theta_z - \theta_x}{d-1}\right)\mathbf{G}_A, d \neq 1, 0, -1, -2, \ldots. \tag{3.4}$$

Theorem 3.3. *Differential recursion formulas for the functions* $\mathbf{G}_B$*,* $\mathbf{G}_C$*,* $\mathbf{G}_D$ *and* $\mathbf{G}_C^*$ *are holds true as follows*

$$\begin{aligned}\mathbf{G}_B(a+1) &= \left(1 + \frac{\theta_y + \theta_z - \theta_x}{a}\right)\mathbf{G}_B, (a \neq 0),\\ \mathbf{G}_C(a+1) &= \left(1 + \frac{\theta_z - \theta_x}{a}\right)\mathbf{G}_C, (a \neq 0),\\ \mathbf{G}_D(a+1) &= \left(1 + \frac{\theta_z - \theta_x}{a}\right)\mathbf{G}_C, (a \neq 0),\\ \mathbf{G}_C^*(a+1) &= \left(1 + \frac{\theta_x + \theta_z}{a}\right)\mathbf{G}_C^*, (a \neq 0),\end{aligned} \tag{3.5}$$

$$
\begin{aligned}
\mathbf{G}_B(b+1) &= \left(1+\frac{\theta_x}{b}\right)\mathbf{G}_B, (b \neq 0)\,, \\
\mathbf{G}_C(b+1) &= \left(1+\frac{\theta_y}{b}\right)\mathbf{G}_C, (b \neq 0)\,, \\
\mathbf{G}_D(b+1) &= \left(1+\frac{\theta_y}{b}\right)\mathbf{G}_D, (b \neq 0)\,, \\
\mathbf{G}^*_C(b+1) &= \left(1+\frac{\theta_x+\theta_y}{b}\right)\mathbf{G}^*_C, (b \neq 0)\,,
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
\mathbf{G}_B(c+1) &= \left(1+\frac{\theta_y}{c}\right)\mathbf{G}_B, (c \neq 0)\,, \\
\mathbf{G}_C(c+1) &= \left(1+\frac{\theta_x+\theta_z}{c}\right)\mathbf{G}_C, (c \neq 0)\,, \\
\mathbf{G}_D(c+1) &= \left(1+\frac{\theta_x}{c}\right)\mathbf{G}_D, (c \neq 0)\,, \\
\mathbf{G}^*_C(c+1) &= \left(1+\frac{\theta_y-\theta_z}{c}\right)\mathbf{G}^*_C, (c \neq 0)\,,
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
\mathbf{G}_B(d+1) &= \left(1+\frac{\theta_z}{d}\right)\mathbf{G}_B, (d \neq 0)\,, \\
\mathbf{G}_C(d+1) &= \left(1+\frac{\theta_y}{d}\right)\mathbf{G}_C, (d \neq 0)\,, \\
\mathbf{G}_D(d+1) &= \left(1+\frac{\theta_y}{d}\right)\mathbf{G}_D, (d \neq 0)\,,
\end{aligned}
\tag{3.8}
$$

$$
\mathbf{G}_D(e+1) = \left(1+\frac{\theta_z}{e}\right)\mathbf{G}_D, (e \neq 0)
\tag{3.9}
$$

and

$$
\begin{aligned}
\mathbf{G}_B(e-1) &= \left(1+\frac{\theta_y+\theta_z-\theta_x}{e-1}\right)\mathbf{G}_B, (e \neq 1, 0, -1, -2, \ldots)\,, \\
\mathbf{G}_C(e-1) &= \left(1+\frac{\theta_y+\theta_z-\theta_x}{e-1}\right)\mathbf{G}_C, (e \neq 1, 0, -1, -2, \ldots)\,, \\
\mathbf{G}_D(f-1) &= \left(1+\frac{\theta_y+\theta_z-\theta_x}{f-1}\right)\mathbf{G}_C, (f \neq 1, 0, -1, -2, \ldots)\,, \\
\mathbf{G}^*_C(d-1) &= \left(1+\frac{\theta_x+\theta_y-\theta_z}{d-1}\right)\mathbf{G}^*_C, (d \neq 1, 0, -1, -2, \ldots)\,.
\end{aligned}
\tag{3.10}
$$

Theorem 3.4. *The derivative formulas hold true for the Horn hypergeometric function* $\mathbf{G}_A$

$$
\begin{aligned}
\frac{\partial^r}{\partial x^r}\mathbf{G}_A = \frac{(b)_r(1-d)_r}{(1-a)_r}&\mathbf{G}_A(a-r, b+r, c; d-r; x, y, z), \\
&(a \neq 1, 2, 3, \ldots, d \neq 0, -1, -2, \ldots)\,,
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
\frac{\partial^r}{\partial y^r}\mathbf{G}_A = \frac{(a)_r(c)_r}{(d)_r}&\mathbf{G}_A(a+r, b, c+r; d+r; x, y, z), \\
&(d \neq -r, -r-1, -r-2, \ldots, r \in \mathbb{N}_0)
\end{aligned}
\tag{3.12}
$$

and

$$\frac{\partial^r}{\partial z^r}\mathbf{G}_A=\frac{(a)_r(b)_r}{(d)_r}\mathbf{G}_A(a+r,b+r,c;d+r;x,y,z),\qquad(d\neq -r,-r-1,-r-2,\dots,r\in\mathbb{N}_0)\,. \tag{3.13}$$

Proof. Differentiating (1.1) with respect to x yields

$$\tfrac{\partial}{\partial x}\mathbf{G}_A=\tfrac{b(d-1)}{a-1}\mathbf{G}_A(a-1,b+1,c;d-1;x,y,z),\qquad(a\neq 1,d\neq 0,-1,-2,\dots)\,.$$

Repeating the above process, we eventually arrive at

$$\begin{aligned}\frac{\partial^r}{\partial x^r}\mathbf{G}_A&=\frac{b(b+1)\dots(b+r-1)(d-1)(d-2)\dots(d-r)}{(a-1)(a-2)\dots(a-r)}\mathbf{G}_A(a-r,b+r,c;d-r;x,y,z)\\&=\frac{b(b+1)\dots(b+r-1)(1-d)(2-d)\dots(r-d)}{(1-a)(2-a)\dots(r-a)}\mathbf{G}_A(a-r,b+r,c;d-r;x,y,z)\\&=\frac{(b)_r(1-d)_r}{(1-a)_r}\mathbf{G}_A(a-r,b+r,c;d-r;x,y,z),\\&\qquad(a\neq 1,2,3,\dots,d\neq 0,-1,-2,\dots)\,.\end{aligned}$$

The derivative (1.1) with respect to y, we proceed

$$\frac{\partial}{\partial y}\mathbf{G}_A=\frac{ac}{d}\mathbf{G}_A(a+1,b,c+1;d+1;x,y,z),(d\neq 0,-1,-2,\dots)\,.$$

Iterating for $0\leq r\leq n$, we get (3.12). The derivative (1.1) with respect to z, we get

$$\frac{\partial}{\partial z}\mathbf{G}_A=\frac{ab}{d}\mathbf{G}_A(a+1,b+1,c;d+1;x,y,z),(d\neq 0,-1,-2,\dots)\,.$$

Iterating for $0\leq r\leq n$, we obtain the relation (3.13). □

Theorem 3.5. *The derivative formulas hold true for Horn function* $\mathbf{G}_B$

$$\frac{\partial^r}{\partial x^r}\mathbf{G}_B=\frac{(b)_r(1-e)_r}{(1-a)_r}\mathbf{G}_B(a-r,b+r,c,d;e-r;x,y,z),\qquad(a\neq 1,2,3,\dots,e\neq r,r-1,r-2,\dots,r\in\mathbb{N}_0)\,, \tag{3.14}$$

$$\frac{\partial^r}{\partial y^r}\mathbf{G}_B=\frac{(a)_r(c)_r}{(e)_r}\mathbf{G}_B(a+r,b,c+r,d;e+r;x,y,z),\qquad(e\neq -r,-r-1,-r-2,\dots,r\in\mathbb{N}_0) \tag{3.15}$$

and

$$\frac{\partial^r}{\partial z^r}\mathbf{G}_B=\frac{(a)_r(d)_r}{(e)_r}\mathbf{G}_B(a+r,b,c,d+r;e+r;x,y,z),\qquad(e\neq -r,-r-1,-r-2,\dots,r\in\mathbb{N}_0)\,. \tag{3.16}$$

Theorem 3.6. *The derivative formulas hold true for the Horn function* $\mathbf{G}_C$

$$\frac{\partial^r}{\partial x^r}\mathbf{G}_C=\frac{(c)_r(1-e)_r}{(1-a)_r}\mathbf{G}_C(a-r,b,c+r,d;e-r;x,y,z),\qquad(a\neq 1,2,3,\dots,e\neq r,r-1,r-2,\dots,r\in\mathbb{N}_0)\,, \tag{3.17}$$

$$\frac{\partial^r}{\partial y^r}\mathbf{G}_C = \frac{(b)_r(d)_r}{(e)_r}\mathbf{G}_C(a, b+r, c, d+r; e+r; x, y, z), \qquad (e \neq -r, -r-1, -r-2, \ldots, r \in \mathbb{N}_0)\,, \tag{3.18}$$

and

$$\frac{\partial^r}{\partial z^r}\mathbf{G}_C = \frac{(a)_r(c)_r}{(e)_r}\mathbf{G}_A(a+r, b, c+r, d; e+r; x, y, z), \qquad (e \neq -r, -r-1, -r-2, \ldots, r \in \mathbb{N}_0)\,. \tag{3.19}$$

Theorem 3.7. *The derivative formulas hold true for the Horn function* $\mathbf{G}_D$

$$\frac{\partial^r}{\partial x^r}\mathbf{G}_D = \frac{(c)_r(1-f)_r}{(1-a)_r}\mathbf{G}_D(a-r, b, c+r, d, e; f-r; x, y, z), \qquad (a \neq 1, 2, 3, \ldots, f \neq r, r-1, r-2, \ldots, r \in \mathbb{N}_0)\,, \tag{3.20}$$

$$\frac{\partial^r}{\partial y^r}\mathbf{G}_D = \frac{(b)_r(d)_r}{(f)_r}\mathbf{G}_D(a, b+r, c, d+r, e; f+r; x, y, z), \qquad (f \neq -r, -r-1, -r-2, \ldots, r \in \mathbb{N}_0) \tag{3.21}$$

and

$$\frac{\partial^r}{\partial z^r}\mathbf{G}_D = \frac{(a)_r(e)_r}{(f)_r}\mathbf{G}_D(a+r, b, c, d, e+r; f+r; x, y, z), \qquad (f \neq -r, -r-1, -r-2, \ldots, r \in \mathbb{N}_0)\,. \tag{3.22}$$

Theorem 3.8. *The derivative formulas hold true for the Horn function* $\mathbf{G}_D$

$$\frac{\partial^r}{\partial x^r}\mathbf{G}^*_C = \frac{(a)_r(b)_r}{(d)_r}\mathbf{G}^*_C(a+r, b+r, c; d+r; x, y, z), \qquad (d \neq -r, -r-1, -r-2, \ldots, r \in \mathbb{N}_0)\,, \tag{3.23}$$

$$\frac{\partial^r}{\partial y^r}\mathbf{G}^*_C = \frac{(b)_r(c)_r}{(d)_r}\mathbf{G}^*_C(a, b+r, c+r; d+r; x, y, z), \qquad (d \neq -r, -r-1, -r-2, \ldots, r \in \mathbb{N}_0) \tag{3.24}$$

and

$$\frac{\partial^r}{\partial z^r}\mathbf{G}^*_C = \frac{(a)_r(1-d)_r}{(1-c)_r}\mathbf{G}^*_C(a+r, b, c-r; d-r; x, y, z), \qquad (c \neq 1, 2, 3, \ldots, d \neq r, r-1, r-2, \ldots, r \in \mathbb{N}_0)\,. \tag{3.25}$$

Conclusion

Here, it is important to mention that Horn's hypergeometric functions and its various generalizations, in particular such involving multiple series, appear in various branches of mathematics and its applications. This paper is focused on to established the recursion relations and differential recursion relations for the Horn's hypergeometric functions $\mathbf{G}_A$, $\mathbf{G}_B$, $\mathbf{G}_C$, $\mathbf{G}_D$ and $\mathbf{G}^*_C$ of three variables by specializing the parameters, respectively. These types of series appear very naturally in quantum field theory, in particular in the computation of analytic expressions for Feynman integrals. That'why, we conclude our investigation by remarking that the results presented here are general enough to yield and able to solve a number of known or new mathematical and its applications problems.

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[a] Department of Mathematics, Anand International College of Engineering, Jaipur 303012, India

[b]International Center for Basic and Applied Sciences, Jaipur 302029, India

[c] Harish-Chandra Research Institute (HRI), Jhusi, Uttar Pradesh 211019, India

[d] Netaji Subhas University of Technology, Dwarka, New Delhi 110078, India
E-mail address: goyal.praveen2011@gmail.com

[e] Department of Mathematics, College of Science and Arts in Unaizah, Qassim University, Qassim, Saudi Arabia.

[f] Department of Mathematics, Faculty of Science, Assiut University, 71516-Assiut, Egypt.
E-mail address: drshehata2009@gmail.com, aymanshehata@science.aun.edu.eg, A.Ahmed@qu.edu.sa

[g] Human Resource Management Department, College of Humanities and Administration, Onaizah Colleges, Qassim, Saudi Arabia.
E-mail address: shimaa1362011@yahoo.com, shimaa_m.@science.aun.edu.eg

[h] Department of Mathematics, Harish-Chandra Research Institute (HRI), Prayagraj (Allahabad), India.
E-mail address: kalyan@hri.res.in