Proceedings of the Jangjeon Mathematical Society
24 (2021), No. 2, pp. 223 – 230
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2021.24.2.223

SOME RESULTS ON ORDER PRIME GRAPHS AND GENERAL ORDER PRIME GRAPHS

R. RAJENDRA AND P. SIVA KOTA REDDY

Abstract. The order prime graph $OP(\Gamma)$ of a finite group Γ is a graph with the vertex set $V(OP(\Gamma)) = \Gamma$ and any two distinct vertices a and b are adjacent in $OP(\Gamma)$ if and only if $(o(a), o(b)) = 1$. The general order prime graph $GOP(\Gamma)$ of Γ is a graph with vertex set $V(GOP(\Gamma)) = \Gamma$ and any two distinct vertices a and b are adjacent in $GOP(\Gamma)$ if and only if $(o(a), o(b)) = 1$ or p, where p is a prime and $p < n$. In this paper, we discuss some results involving eigenvalues and energy of order prime graphs and general order prime graphs of finite groups. Further, we define the order prime graphs of a finite group Γ with respect to subsets(subgroups) of Γ and investigate some properties.

2010 Mathematics Subject Classification. 05C25, 05C50.

Keywords and Phrases. Eigenvalues, Energy, Order prime graph, General order prime graph.

1. Introduction

For standard terminology and notion in group theory and graph theory, we refer the reader to the text-books of Herstein [9] and Harary [8] respectively. The non-standard will be given in this paper as and when required.

Throughout this section, Γ denotes a finite group and we denote the identity element of Γ by e. The group of residue classes modulo n is denoted by $\mathbb{Z}_n$. The order of an element a in a group Γ is denoted by $o_\Gamma(a)$ and order of Γ is denoted by $o(\Gamma)$. The greatest common divisor (gcd) of two integers x and y is denoted by $\gcd(x, y)$.

Many concepts related to chemistry involve groups and graphs. It is interesting to study eigenvalues and energy of graphs. We recall the following basic definitions and results to discuss eigenvalues and energy of general order prime graphs of finite groups:

Let G be a graph with n vertices $v_1, v_2, \ldots, v_n$. The adjacency matrix $A = A(G)$ is a square matrix of order n whose (i, j)-entry is defined as

$$A_{ij} = \begin{cases} 1, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

[1]Corresponding author:pskreddy@jssstuniv.in; pskreddy@sjce.ac.in

The eigenvalues of $A(G)$ are said to be the eigenvalues of the graph G. We denote largest and smallest eigenvalues of a graph G by λ_{max} and λ_{min} respectively.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of a graph G. The energy of G is defined as,

$$E(G) = \sum_{i=1}^{n} |\lambda_i| \tag{1}$$

For a graph G with n vertices and m edges, McClelland inequality [11] is

$$E(G) \leq \sqrt{2mn} \tag{2}$$

The following inequalities (3) and (4) are given by Koolen and Moulton [8, 9]: If G is a graph with n vertices,

$$E(G) \leq \frac{n(\sqrt{n}+1)}{2} \tag{3}$$

and if G is a bipartite graph with n vertices, where $n > 2$,

$$E(G) \leq \frac{n(\sqrt{n}+\sqrt{2})}{\sqrt{8}} \tag{4}$$

The following result is given by I. Gutman [6, Theorem 5.4]: If G is an n-vertex graph without isolated vertices, then

$$E(G) \geq 2\sqrt{n-1} \tag{5}$$

with equality if and only if G is the n-vertex star. By this result, it follows that, for the complete bipartite graph $K_{1,n-1}$,

$$E(K_{1,n-1}) = 2\sqrt{n-1} \tag{6}$$

Also, it is computed in [6] that, for the complete graph K_n,

$$E(K_n) = 2n-2. \tag{7}$$

The concept of order prime graph was introduced by M. Sattanathan and R. Kala [19]. Further, Ma et al [12] and Dorbidi [4] studied order prime graphs of finite groups but by calling them as coprime graphs. The following definition can be found in [19]: The order prime graph $OP(\Gamma)$ of a finite group Γ is a graph with the vertex set $V(OP(\Gamma)) = \Gamma$ and any two distinct vertices a and b are adjacent in $OP(\Gamma)$ if and only if $(o(a), o(b)) = 1$.

The concept of general order prime graph was introduced by R. Rajendra and P. S.K.Reddy [14, 15]. The general order prime graph $GOP(\Gamma)$ of Γ is a graph with vertex set $V(GOP(\Gamma)) = \Gamma$ and any two distinct vertices a and b are adjacent in $GOP(\Gamma)$ if and only if $(o(a), o(b)) = 1$ or p, where p is a prime and $p < n$. Clearly, $OP(\Gamma)$ is a subgraph of $GOP(\Gamma)$.

In [17], the authors investigated some properties of prime graphs of finite rings and established a formula for finding the number of edges in the prime graph of the ring of residue classes modulo n by using the gcd-sum function. Further, discussed some results regarding eigenvalues and energy of prime graphs of finite rings.

Let S be a non-empty set of positive integers. We define the set-prime graph $G_S(\Gamma)$ of a given finite group Γ of order n with respect to S, as a graph with vertex set $V(G_S(\Gamma)) = \Gamma$ and any two vertices a and b are adjacent in $G_S(\Gamma)$ if and only if $(o(a), o(b)) \in S$. Rajendra et al. [18] observed that order prime and general order prime graphs are special cases of set-prime graphs and they investigated some properties of set-prime graphs of finite groups.

2. Some Results Involving Eigenvalues and Energy

Theorem 2.1. *If G is a group of order n, then*

$$(8)\quad max\left\{\frac{2n-2}{n},\ \sqrt{n-1}\right\} \leq \lambda_{max}(OP(\Gamma)) \leq \lambda_{max}(GOP(\Gamma)) \leq n-1$$

In particular, if $n \geq 3$, $\sqrt{n-1} \leq \lambda_{max}(OP(\Gamma)) \leq \lambda_{max}(GOP(\Gamma)) \leq n-1$.

Proof. We have, if H is a subgraph of a graph G,

$$(9)\qquad \lambda_{max}(G) \geq \lambda_{max}(H)$$

Since $OP(\Gamma)$ is a subgraph of $GOP(\Gamma)$, from (9) it follows that

$$(10)\qquad \lambda_{max}(GOP(\Gamma)) \geq \lambda_{max}(OP(\Gamma))$$

For any graph G, $\lambda_{max} \leq d_{max}$, where d_{max} is the maximum vertex degree of G. Hence

$$(11)\qquad \lambda_{max}(GOP(\Gamma)) \leq n-1$$

Also by [16, Theorem 1], we have

$$(12)\qquad max\left\{\frac{2n-2}{n},\ \sqrt{n-1}\right\} \leq \lambda_{max}(OP(\Gamma)) \leq n-1$$

Then (8) follows from (10), (11) and (12).

Clearly, for $n \geq 3$,

$$max\left\{\frac{2n-2}{n},\ \sqrt{n-1}\right\} = \sqrt{n-1}$$

and from (8), we have, $\sqrt{n-1} \leq \lambda_{max}(OP(\Gamma)) \leq \lambda_{max}(GOP(\Gamma)) \leq n-1$. □

Note: From the inequality (8), it follows that,

$$\lambda_{max}(OP(\Gamma)) = \lambda_{max}(GOP(\Gamma)) = \begin{cases} 0, & \text{for } n = 1; \\ 1, & \text{for } n = 2. \end{cases}$$

Theorem 2.2. *Let Γ be a group of finite order n. Then*

(i) *$o(\Gamma)$ is a prime if and only if for each eigenvalue λ of $GOP(\Gamma)$, $-\lambda$ is an eigenvalue with the same multiplicity.*
(ii) *$o(\Gamma)$ is a prime if and only if $\lambda_{min}(GOP(\Gamma)) = -\lambda_{max}(GOP(\Gamma))$.*

Proof. We have, by [15, Theorem 3.4], $o(\Gamma) = n$ is a prime if and only if $GOP(\Gamma) \cong K_{1,n-1}$, a bipartite graph. Hence by [1, Proposition 3.4.1, p.38], the proof of (i) and (ii) follows. □

Theorem 2.3. *Let Γ be a finite group.*

(i) *If Γ is of order 2, then $GOP(\Gamma)$ has two distinct eignvalues, namely, $+1$ and -1.*
(ii) *If Γ is of odd prime order $n \geq 3$, then $GOP(\Gamma)$ has atleast three distinct eignvalues.*

Proof.

(i) Follows by direct computation.
(ii) A connected graph with diameter d, has at least $d+1$ distinct eigenvalues [1, Proposition 1.3.3, p.5]. Since $o(\Gamma) = n \geq 3$, by [15, Corollary 2.5], it follows that the diameter of $OP(\Gamma)$ is 2. Hence $GOP(\Gamma)$ has atleast three distinct eigenvaues.

□

Theorem 2.4. *Let Γ be a finite group of order pq where p and q are odd primes. Then $GOP(\Gamma)$ has*

(i) *at least three distinct eignvalues, when Γ is a cyclic group,*
(ii) *at least two distinct eignvalues, when Γ is not a cyclic group,*

Proof. By [15, Proposition 2.6],

$$diam(GOP(\Gamma)) = \begin{cases} 2, & \text{when } \Gamma \text{ is a cyclic group;} \\ 1, & \text{when } \Gamma \text{ is not a cyclic group.} \end{cases}$$

Therefore by [1, Proposition 1.3.3, p.5] the result follows. □

Definition 2.5. *The order prime energy of a finite group Γ, denoted by $OPE(\Gamma)$, is defined as the energy of the order prime graph $OP(\Gamma)$. That is, $OPE(\Gamma) = E(OP(\Gamma))$.*

Definition 2.6. *The general order prime energy of a finite group Γ, denoted by $GOPE(\Gamma)$, is defined as the energy of the general order prime graph $GOP(\Gamma)$. That is, $GOPE(\Gamma) = E(GOP(\Gamma))$.*

Theorem 2.7. *If Γ_1 and Γ_2 are isomorphic finite groups then $GOPE(\Gamma_1) = GOPE(\Gamma_2)$.*

Proof. If $\Gamma_1 \cong \Gamma_2$, then $GOP(\Gamma_1) \cong GOP(\Gamma_2)$ and hence $GOPE(\Gamma_1) = GOPE(\Gamma_2)$. □

There are non-isomrphic finite groups such that their general order prime graphs are isomorphic [14]. Hence converse of the Theorem 2.7 is not true.

Theorem 2.8. *Let Γ be a finite group.*

(i) *If $o(\Gamma) = 2$, then $GOP(\Gamma) \cong K_2$ and $GOPE(\Gamma) = 2$.*
(ii) *If $o(\Gamma) = n$, then*

$$GOPE(\Gamma) \leq \frac{n(\sqrt{n}+1)}{2} \tag{13}$$

(iii) *If $o(\Gamma) = p$, where p is a prime then*

$$GOPE(\Gamma) \leq \frac{p(\sqrt{p}+\sqrt{2})}{\sqrt{8}} \tag{14}$$

Proof.

(i) Follows by direct computation.
(ii) Follows from the Koolen and Moulton inequality (3).
(iii) If Γ is a group of order p, a prime then $GOP(\Gamma) \cong K_{1,p-1}$, a bipartite graph [15, Theorem 2.4]. Then the inequality (14) follows from the Koolen and Moulton inequality (4). □

Proposition 2.9. *If Γ is a group of order 2, then*

$$OPE(\Gamma) = GOPE(\Gamma) = 2.$$

Proof. If Γ is a group of order 2, then $OP(\Gamma) \cong GOP(\Gamma) \cong K_2$ and from (7), $OPE(\Gamma) = GOPE(\Gamma) = 2$. □

Theorem 2.10. *Let Γ be a finite group of order $n > 2$.*
(a) *If $n = p^\alpha$, where p is a prime and $\alpha \in \mathbb{N}$, then $OPE(\Gamma) = 2\sqrt{n-1}$*
(b) *If n is a prime, then $GOPE(\Gamma) = 2\sqrt{n-1}$.*

Proof. For a finite group Γ of order n, the following results (i) and (ii) can be found in [19] and [14], respectively:
(i) $OP(\Gamma) \cong K_{1,n-1}$ if and only if $n = p^\alpha$, where p is a prime and $\alpha \in \mathbb{N}$.
(ii) $GOP(\Gamma) \cong K_{1,n-1}$ if and only if n is a prime.

Using (6) with (i) and (ii), we obtain (a) and (b), respectively. □

3. Order Prime Graphs of Groups with respect to Subsets

We define order prime graphs of subsets(subgroups) of a finite group as the induced subgraph of the order prime graph. In this section we investigate some properties order prime graphs of subsets and subgroups of finite groups.

Definition 3.1. [3] *Let $G = (V, E)$ be a graph, and let S be a subset of V. Then the induced subgraph $G[S]$ is the graph whose vertex set is S and whose edge set consists of all of the edges in E that have both endpoints in S.*

Definition 3.2. *Let Γ be a finite group and S be a subset of Γ. The order prime graph $OP_\Gamma(S)$ of S with respect to Γ is the induced subgraph $OP(\Gamma)[S]$.*

The order prime graph $OP_\Gamma(S)$ of S with respect to Γ is the graph with vertex set S and two vertices are adjacent in $OP_\Gamma(S)$ if and only if $\gcd(o_\Gamma(a), o_\Gamma(b)) = 1$ and $a \neq b$.

Observation: From the Definition 3.2, we have the following:
(i) $OP_\Gamma(\Gamma) = OP(\Gamma)$.
(ii) If S is a subgroup of Γ, then $OP_\Gamma(S) = OP(S)$.
(iii) If $e \in S$, then $OP_\Gamma(S)$ is connected and $\Delta(OP_\Gamma(S)) = o(S) - 1$. If $e \notin S$, then $OP_\Gamma(S)$ need not be connected and $\Delta(OP_\Gamma(S)) \leq o(S) - 1$.
(iv) If $e \in S$, then $dia(OP_\Gamma(S)) \leq 2$.
(v) Let Γ_1 and Γ_2 be two finite groups such that S is a subset of both Γ_1 and Γ_2. Then $OP_{\Gamma_1}(S)$ need not be isomorphic to $OP_{\Gamma_2}(S)$. For e.g. consider the group $\Gamma_1 = (\mathbb{Z}_4, +_4)$ and the group $\Gamma_2 = (U(\mathbb{Z}_4), \times_4)$ of units under multiplication modulo 4 in the ring $(\mathbb{Z}_4, +_4, \times_4)$. If

$S = \{1, 3\}$, then $S = U(\mathbb{Z}_4) \subset \mathbb{Z}_4$. We see that $OP_{\Gamma_2}(S) \cong K_2$, a complete graph, but $OP_{\Gamma_1}(S)$ is a null graph with two vertices.

The following propositions are immediate from the Definition 3.2:

Proposition 3.3. *For a subset S of Γ the following are equivalent:*

(i) *$OP_\Gamma(S)$ is totally disconnected.*
(ii) *the elements of S have orders(as elements in Γ) mutually not relatively prime.*

Proposition 3.4. *For a subset S of Γ the following are equivalent:*

(i) *$OP_\Gamma(S)$ is a complete graph.*
(ii) *the elements of S have orders(as elements in Γ) mutually relatively prime.*

Proposition 3.5. *Let H and K be subsets of Γ. We have*

(i) *If $H \subseteq K$, $OP_\Gamma(H) \subseteq OP_\Gamma(K)$.*
(ii) *$OP_\Gamma(H \cap K) = OP_\Gamma(H) \cap OP_\Gamma(K)$.*
(iii) *If for all $a \in H$ and all $b \in K$, $\gcd(o_\Gamma(a), o_\Gamma(b)) \neq 1$, then $OP_\Gamma(H \cup K) = OP_\Gamma(H) \cup OP_\Gamma(K)$.*

Remark 3.6. *If H and K are subsets of Γ, then $OP_\Gamma(H \cup K)$ need not be equal to $OP_\Gamma(H) \cup OP_\Gamma(K)$. To see this it is enough to consider a group Γ with at least two elements, and the subsets $H = \{e\}$ and $K = \{x\}$, $x \neq e$ in Γ. In this case $OP_\Gamma(H \cup K) \cong K_2$, a complete graph, but $OP_\Gamma(H) \cup OP_\Gamma(K)$ is a null graph with two vertices.*

Proposition 3.7. *For any finite group Γ,*

(i) *$OP_\Gamma(\Gamma - \{e\}) = OP(\Gamma) - \{e\}$*
(ii) *$OP(\Gamma) =$ Corona of $OP_\Gamma(\Gamma - \{e\})$ (obtained by adding the vertex e and drawing edges from e to all vertices in $OP_\Gamma(\Gamma - \{e\})$*

Theorem 3.8. *Let H be a subgroup of Γ and let p be a prime dividing $o(\Gamma)$. Then*

(i) *$OP_\Gamma(H)$ is complete if and only if $o(H) = 2$.*
(ii) *If $o(H) \geq 3$, then $OP_\Gamma(H)$ cannot be regular.*
(iii) *$OP_\Gamma(H)$ can never be a unicyclic group.*
(iv) *H is a p-subgroup of Γ of order p^α if and only if $OP_\Gamma(H) \cong K_{1,p^\alpha-1}$, where α is a positive integer.*

Proof. Follows by [19, Theorem 2.7] and the observation (3) in section 1. □

By Theorem 3.8, the following two corollaries are immediate:

Corollary 3.9. *Let H be a subgroup of Γ and let p be a prime dividing $o(\Gamma)$. Then, H is a Sylow p-subgroup of Γ if and only if $OP_\Gamma(H) \cong K_{1,p^\alpha-1}$, where α is a positive integer such that $p^\alpha | o(\Gamma)$ and $p^{\alpha+1} \nmid o(\Gamma)$.*

Corollary 3.10. *Let H be a subgroup of Γ. Then $OP_\Gamma(H)$ is a tree if and only if H is a p-subgroup, where p is a prime.*

Theorem 3.11. *Let $a \in \Gamma$, $a \neq e$. Then $OP_\Gamma(\langle a \rangle)$ has atleast two pendant vertices.*

Proof. Follows by the observation (4) in section 1 and [19, Proposition 2.9]. □

Theorem 3.12. *If H and K are conjugate subgroups of Γ, then $OP_\Gamma(H) \cong OP_\Gamma(K)$.*

Proof. Since conjugate subgroups are isomorphic, by the observation (3) it follows that $OP_\Gamma(H) \cong OP_\Gamma(K)$. □

Remark 3.13. *If H and K are subgroups of Γ such that $OP_\Gamma(H) \cong OP_\Gamma(K)$, then H need not be isomorphic to K (and so H need not be a conjugate of K). For example, consider the symmetric group S_4. Let $H = \langle(1,2,3,4)\rangle$ and $K = \langle(1,2),(3,4)\rangle$. Then $o(H) = o(K)$, and hence $OP_\Gamma(H) \cong OP_\Gamma(K) \cong K_{1,3}$. But as groups $H \ncong K$.*

Acknowledgments

The authors thank the anonymous reviewers for their careful reading of our manuscript and their many insightful comments and suggestions.

References

[1] A. E. Brouwer and W. H. Haemers, *Spectra of Graphs-Monograph*, Springer, 2011.

[2] D. M. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs*, Academic Press, 1979.

[3] R. Diestel, *Graph Theory*, Graduate texts in Mathematics-Volume 173, Springer, 2006

[4] H. R. Dorbidi, A note on the coprime graph of a group, *Int. J. Group Theory*, 5(4) (2016), 17-22.

[5] I. Gutman, The energy of a graph, *Ber. Math.-Statist. Sekt. Forschungsz. Graz*, 103 (1978), 1–22.

[6] I. Gutman, The Energy of a Graph: Old and New Results. In: Betten, A., Kohnert, A., Laue, R. and Wassermann, A., Eds., *Algebraic Combinatorics and Applications*, Springer, Berlin, (2001), 196-211.

[7] I. Gutman et al., On the energy of regular graphs, *MATCH Commun. Math, Comput. Chem.*, 57 (2007), 435–442.

[8] F. Harary, *Graph Theory*, Addison Wesley, Reading, Mass, 1972.

[9] I. N. Herstein, *Topics in Algebra*, Second Ed., John Wiley & Sons, 2003.

[10] J. H. Koolen and V. Moulton, Maximal energy graphs, *Adv. Appl. Math.*, 26 (2001), 47–52.

[11] J. H. Koolen and V. Moulton, Maximal energy bipartite graphs, *Graphs Combin*, 19 (2003), 131–135.

[12] X. Ma, H. Wei and L. Yang, The coprime graph of a group, *Int. J. Group Theory*, 3(3) (2014), 13-23.

[13] B. J. McClelland, Properties of the latent roots of a matrix: The estimation of $\pi-$electron energies, *J. chem. Phys.*, 54 (1971), 640–643.

[14] R. Rajendra and P.S.K. Reddy, On general order prime graph of a finite group, *Proc. Jangjeon Math. Soc.*, 17(4), (2014), 641-644.

[15] R. Rajendra, P.S.K. Reddy, V. M. Siddalingaswamy, On general order prime graph of a finite group-II, *Adv. Appl. Discrete Math.*, 17(4) (2016), 437-444.

[16] R. Rajendra, A.C. Chandrashekar and B.M. Chandrashekara, A note on energy of order prime graph of a finite group, *International Journal of Scientific and Engineering Research*, 7(5) (2016), 15-16.

[17] R. Rajendra, P.S.K. Reddy, H.M. Gowrandamma and P. S. Hemavathi, A Note on Prime Graph of a Finite Ring, *Proceedings of the Jangjeon Math. Soc.*, 22(3) (2019), 415-426.

[18] R. Rajendra, P.S.K. Reddy and K. V. Madhusudhan, Set-Prime Graph of a Finite Group, *Proceedings of the Jangjeon Math. Soc.*, 22(3) (2019), 455-462.

[19] M. Sattanathan and R. Kala, An Introduction to Order Prime Graph, *Int. J. Contemp. Math. Sciences*, 10(4) (2009), 467–474.

Department of Mathematics, Mangalore University, Mangalagangothri, Mangalore-574 199, INDIA
Email address: rrajendrar@gmail.com

Department of Mathematics, Sri Jayachamarajendra College of Engineering, JSS Science and Technology University, Mysuru-570 006, INDIA
Email address: pskreddy@jssstuniv.in; pskreddy@sjce.ac.in