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A NEW TYPE OF DEGENERATE POLY-TYPE 2 EULER POLYNOMIALS AND DEGENERATE UNIPOLY-TYPE 2 EULER POLYNOMIALS

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ABSTRACT. The polylogarithm function was introduced by Kim-Kim (Russ. J. Math. Phys. 27(2), 227-235(2020)). In addition, many mathematicians have been studying various degenerate versions of special polynomials and numbers in some arithmetic and combinatorial aspects. Our main focus here is the degenerate poly-type 2 Euler polynomials and numbers. This focus stems from their importance for applications in combinatorics, number theory and in other aspects of applied mathematics. we construct a new type of degenerate poly-type 2 Euler polynomials by using the degenerate polylogarithm function, called degenerate poly-type 2 Euler polynomials and numbers. We also show several combinatorial identities related to the degenerate poly-type 2 Euler polynomials and numbers. We also construct the degenerate unipoly-type 2 Euler polynomials attached to an arithmetic function. By using the degenerate polylogarithm function, we also give some new explicit expressions and identities related to degenerate unipoly-type 2 Euler polynomials and some special numbers and polynomials.

1. Introduction

Recently, many mathematicians have been studying various degenerate versions of special polynomials and numbers in some arithmetic and combinatorial aspects [11, 15-25]. These degenerate versions began when Carlitz introduced the degenerate Bernoulli polynomials and the degenerate Euler polynomials [5]. These polynomials appear in combinatorial mathematics and play a very important role in the theory and application of mathematics, thus many number theory and combination experts have studied their properties, and obtained a series of interesting results. Kim et al. introduced degenerate gamma random variables as well as new Jindalrae and Gaenari numbers and polynomials, and developed above mentioned polynomials and numbers related to Jindalrae and Gaenari numbers and polynomials; discrete harmonic numbers [23-26]. Motivated by their importance and potential for applications in number theory, combinatorics and other fields of applied mathematics, in particular, we are interested in degenerate poly-type 2 Euler polynomials and numbers. The goal of this paper is to demonstrate many explicit computational formulas and relations, involving

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a new type of the degenerate poly-type 2 Euler polynomials and numbers by using Kim-Kim's the degenerate polylogarithm functions.

Now, we give some definitions and properties needed in this paper. As is known, the type 2 Euler polynomials and the type 2 Bernoulli polynomials are usually defined by the following generating functions with parallel structures [5-6],

$$\frac{t}{e^t-e^{-t}}e^{xt}=\sum_{n=0}^{\infty}B_n^*(x)\frac{t^n}{n!},\quad and\quad \frac{2}{e^t+e^{-t}}e^{xt}=\sum_{n=0}^{\infty}E_n^*(x)\frac{t^n}{n!}. \tag{1}$$

While $B_n(x)$ and $E_n(x)$ are the ordinary Bernoulli polynomials and the ordinary Euler polynomials respectively given by

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},\quad and\quad \frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}.$$

For any nonzero $\lambda\in\mathbb{R}$ (or $\mathbb{C}$), the degenerate exponential function is defined by

$$e_\lambda^x(t)=(1+\lambda t)^{\frac{x}{\lambda}},\quad e_\lambda(t)=(1+\lambda t)^{\frac{1}{\lambda}}=e_\lambda^1(t),\qquad \text{(see [17-20, 25])}. \tag{2}$$

By Taylor expansion, we get

$$e_\lambda^x(t)=\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!},\qquad \text{(see [17-20])}, \tag{3}$$

where $(x)_{0,\lambda}=1,\ (x)_{n,\lambda}=x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda),\ (n\geq 1)$.

Note that

$$\lim_{\lambda\to 0}e_\lambda^x(t)=\sum_{n=0}^{\infty}\frac{x^nt^n}{n!}=e^{xt}.$$

In [11], Jang-Kim introduced the degenerate type 2 Bernoulli polynomials and the degenerate type 2 Euler polynomials, respectively given by

$$\frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^x(t)=\sum_{n=0}^{\infty}E_{n,\lambda}^*(x)\frac{t^n}{n!}, \tag{4}$$

and

$$\frac{t}{e_\lambda(t)-e_\lambda^{-1}(t)}e_\lambda^x(t)=\sum_{n=0}^{\infty}B_{n,\lambda}^*(x)\frac{t^n}{n!}. \tag{5}$$

When $x=0$, $E_{n,\lambda}^*=E_{n,\lambda}^*(0)$ are called the degenerate type 2 Euler numbers, and $B_{n,\lambda}^*=B_{n,\lambda}^*(0)$ are called the degenerate type 2 Bernoulli numbers. Note that $\lim_{\lambda\to 0}B_{n,\lambda}^*(x)=B_n^*(x),(n\geq 0)$ and $\lim_{\lambda\to 0}E_{n,\lambda}^*(x)=E_n^*(x),(n\geq 0)$.

Carlitz introduced the ordinary degenerate Bernoulli polynomials and the degenerate Euler polynomials, respectively given by

(6)
$$\frac{2}{e_\lambda(t)+1}e_\lambda^x(t)=\sum_{n=0}^{\infty}E_{n,\lambda}(x)\frac{t^n}{n!},\quad and \quad \frac{t}{e_\lambda(t)-1}e_\lambda^x(t)=\sum_{n=0}^{\infty}B_{n,\lambda}(x)\frac{t^n}{n!}.$$

In 2020, Kim-Kim [15] introduced the degenerate polylogarithm function defined by

(7) $$l_{k,\lambda}(x)=\sum_{n=1}^{\infty}\frac{(-\lambda)^{n-1}(1)_{n,1/\lambda}}{(n-1)!n^k}x^n, k\in\mathbb{Z}\quad (|x|<1).$$

We note that

$$\lim_{\lambda\to 0}l_{k,\lambda}(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^k}=\mathrm{Li}_k(x).$$

From (7), we have

(8) $$l_{1,\lambda}(x)=\sum_{n=1}^{\infty}\frac{(-\lambda)^{n-1}(1)_{n,1/\lambda}}{n!}x^n=-\log_\lambda(1-x).$$

Kim-Kim also studied the new type degenerate poly Bernoulli polynomials and numbers, by using the degenerate polylogarithm function as follows:

(9) $$\frac{l_{k,\lambda}(1-e_\lambda(-t))}{1-e_\lambda(-t)}e_\lambda^x(-t)=\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(k)}(x)\frac{t^n}{n!},\qquad \text{(see [15])}.$$

When $x=0$, $\beta_{n,\lambda}^{(k)}=\beta_{n,\lambda}^{(k)}(0)$ are called the degenerate poly-Bernoulli numbers.
They show that
(10)
$$\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(1)}\frac{t^n}{n!}=\frac{1}{1-e_\lambda(-t)}l_{1,\lambda}(1-e_\lambda(-t))=\frac{-t}{e_\lambda(-t)-1}=\sum_{n=0}^{\infty}(-1)^nB_{n,\lambda}\frac{t^n}{n!},\qquad \text{(see [15])}.$$

For $n\geq 0$, the Stirling numbers of the first kind are defined by

(11) $$(x)_n=\sum_{l=0}^{n}S_1(n,l)x^l,\qquad \text{(see [14-27])}.$$

where $(x)_0=1$, $(x)_n=x(x-1)\dots(x-n+1)$, $(n\geq 1)$, and

(12) $$\frac{1}{k!}(\log(1+t))^k=\sum_{n=k}^{\infty}S_1(n,k)\frac{t^n}{n!},\qquad \text{(see [29])}.$$

In the inverse expression to (11), for $n\geq 0$, the Stirling numbers of the second kind are given by

(13) $$x^n=\sum_{l=0}^{n}S_2(n,l)(x)_l,\qquad \text{(see [27, 29])},$$

and

$$\frac{1}{k!}(e^t-1)^k=\sum_{n=k}^{\infty}S_2(n,k)\frac{t^n}{n!}, \quad \text{(see [6])}. \tag{14}$$

Kim et al. [18] introduced the degenerate Stirling numbers of the second kind as follows:

$$(x)_{n,\lambda}=\sum_{l=0}^{n}S_{2,\lambda}(n,l)(x)_l \quad (n\geq 0). \tag{15}$$

As an inversion formula of (15), the degenerate Stirling numbers of the first kind are given by

$$(x)_n=\sum_{l=0}^{n}S_{1,\lambda}(n,l)(x)_{l,\lambda} \quad (n\geq 0), \quad \text{(see [15])}. \tag{16}$$

From (15) and (16), it is well known that

$$\frac{1}{k!}\big(e_\lambda(t)-1\big)^k=\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!} \quad (k\geq 0), \quad \text{(see [15, 18])}, \tag{17}$$

and

$$\frac{1}{k!}\big(\log_\lambda(1+t)\big)^k=\sum_{n=k}^{\infty}S_{1,\lambda}(n,k)\frac{t^n}{n!} \quad (k\geq 0), \quad \text{(see [15])}. \tag{18}$$

In 1997, Kaneko [12] introduced the poly-Bernoulli numbers $B_n^{(k)}$ represented by the following exponential generating function

$$\frac{Li_k(1-e^{-t})}{1-e^{-t}}=\sum_{n=0}^{\infty}B_n^{(k)}(x)\frac{t^n}{n!}, \quad (|e^{-t}|<1).$$

If $k=1$, we get $B_n^{(1)}=(-1)^nB_n$ for $n\geq 0$, where B_n are the Bernoulli numbers.

Ohno and Sasaki [28] defined poly-Euler numbers as

$$\frac{Li_k(1-e^{-4t})}{4tcosh(t)}=\sum_{n=0}^{\infty}E_n^{(k)}(x)\frac{t^n}{n!}, \quad (|e^{-4t}|<1).$$

It was recently extended to

$$\frac{2Li_k(1-e^{-t})}{1+e^t}e^{xt}=\sum_{n=0}^{\infty}E_n^{(k)}(x)\frac{t^n}{n!}, \quad (|e^{-t}|<1).$$

in polynomial form by Yoshinori [30].

The outline of this paper is as follows. In Section 2, we construct a new type of degenerate poly-type 2 Euler polynomials, called degenerate poly-type 2 Euler polynomials and numbers, by using the degenerate polylogarithm function. We also show several combinatorial identities related to the degenerate poly-type 2 Euler polynomials and numbers. Some of them include special polynomials and numbers such as the Stirling number of the first kind, the degenerate Stirling number of the second kind, degenerate type 2 Euler numbers, degenerate type 2 Bernoulli polynomials, degenerate poly Bernoulli numbers and type 2 Euler numbers, etc. In Section 3, we also consider the degenerate unipoly-type 2 Euler polynomials attached to an arithmetic function, by using the degenerate polylogarithm function and investigate some identities for those polynomials. We also give some new explicit expressions and identities related to degenerate unipoly-type 2 Euler polynomials and other special numbers and polynomials.

2. A new type degenerate poly-type 2 Euler numbers and polynomials

In this section, we construct a new type of degenerate poly-type 2 Euler polynomials, called degenerate poly-type 2 Euler polynomials and numbers, by using the degenerate polylogarithm function. We also show several combinatorial identities related to the degenerate poly-type 2 Euler polynomials and numbers.

We define a new type of degenerate poly-type 2 Euler polynomials, called degenerate poly-type 2 Euler polynomials, by using the degenerate polylogarithm function as follows:

$$\frac{l_{k,\lambda}\big(1-e_\lambda(-2t)\big)}{t(e_\lambda(t)+e_\lambda^{-1}(t))}e_\lambda^x(t)=\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{*(k)}(x)\frac{t^n}{n!}. \tag{19}$$

When $x=0$, $\mathcal{E}_{n,\lambda}^{*(k)}=\mathcal{E}_{n,\lambda}^{*(k)}(0)$ are called the degenerate poly-type 2 Euler numbers.
When $k=1$, from (8), we see that $\mathcal{E}_{n,\lambda}^{*(1)}(x)=E_{n,\lambda}^{*}(x)$ $(n\geq 0)$ are the degenerate type 2 Euler polynomials.

We note that

$$\frac{d}{dx}l_{k,\lambda}(x)=\frac{1}{x}l_{k-1,\lambda}(x), \tag{20}$$

and

$$l_{k,\lambda}(x)=\int_0^x\frac{1}{t}\underbrace{\int_0^t\frac{1}{t}\cdots\cdots\int_0^t\frac{1}{t}}_{(k-2)-times}l_{1,\lambda}(x)\ dt\ dt\cdots dt,\ (k\geq 2). \tag{21}$$

Theorem 2.1. *For $n\geq 0,\ k\in\mathbb{Z}$, we have*

$$\mathcal{E}_{n,\lambda}^{*(k)}(x)=\sum_{l=0}^{n}\binom{n}{l}\mathcal{E}_{l,\lambda}^{*(k)}(x)_{n-l,\lambda},$$
$$\mathcal{E}_{n,\lambda}^{*(k)}(-x)=\sum_{l=0}^{n}\binom{n}{l}(-1)^{n-l}\mathcal{E}_{l,\lambda}^{*(k)}(x)_{n-l,-\lambda}.$$

Proof. From (3) and (19), we have

$$
\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{*(k)}(x)\frac{t^n}{n!} &= \frac{l_{k,\lambda}\big(1-e_\lambda(-2t)\big)}{t(e_\lambda(t)+e_\lambda^{-1}(t))}e_\lambda^x(t)\\
&= \sum_{l=0}^{\infty}\mathcal{E}_{l,\lambda}^{*(k)}\frac{t^l}{l!}\sum_{m=0}^{\infty}(x)_{m,\lambda}\frac{t^m}{m!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\mathcal{E}_{l,\lambda}^{*(k)}(x)_{n-l,\lambda}\right)\frac{t^n}{n!}.
\end{aligned}\tag{22}
$$

On the other hand,

$$
\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{*(k)}(-x)\frac{t^n}{n!} &= \frac{l_{k,\lambda}\big(1-e_\lambda(-2t)\big)}{t(e_\lambda(t)+e_\lambda^{-1}(t))}e_\lambda^{-x}(t)\\
&= \frac{l_{k,\lambda}\big(1-e_\lambda(-2t)\big)}{t(e_\lambda(t)+e_\lambda^{-1}(t))}e_{-\lambda}^{x}(-t)\\
&= \sum_{l=0}^{\infty}\mathcal{E}_{l,\lambda}^{*(k)}\frac{t^l}{l!}\sum_{m=0}^{\infty}(x)_{m,-\lambda}(-1)^m\frac{t^m}{m!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(-1)^{n-l}\mathcal{E}_{l,\lambda}^{*(k)}(x)_{n-l,-\lambda}\right)\frac{t^n}{n!}.
\end{aligned}\tag{23}
$$

Therefore, by comparing the coefficients of (22) and (23) respectively, we get what we desired. □

The next lemma is intended to be used conveniently in proving some of the theorems below.

Lemma 2.2. *For $n\ge 1,\ k\in\mathbb{Z}$, we have*

$$
l_{k,\lambda}\big(1-e_\lambda(-2t)\big)=\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\frac{(1)_{m,1/\lambda}(-1)^{n-1}2^n\lambda^{m-1}}{m^{k-1}}S_{2,\lambda}(n,m)\right)\frac{t^n}{n!}.
$$

Proof. From (7) and (17), we get

$$
\begin{aligned}
l_{k,\lambda}\big(1-e_\lambda(-2t)\big) &= \sum_{m=1}^{\infty}\frac{(1)_{m,1/\lambda}(-\lambda)^{m-1}}{(m-1)!m^k}\big(1-e_\lambda(-2t)\big)^m\\
&= \sum_{m=1}^{\infty}\frac{(1)_{m,1/\lambda}(-1)^{m-1}\lambda^{m-1}}{m^{k-1}}\frac{(-1)^m\big(e_\lambda(-2t)-1\big)^m}{m!}\\
&= \sum_{m=1}^{\infty}\frac{(1)_{m,1/\lambda}(-1)^{-1}\lambda^{m-1}}{m^{k-1}}\sum_{n=m}^{\infty}S_{2,\lambda}(n,m)(-1)^n\frac{2^nt^n}{n!}\\
&= \sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\frac{(1)_{m,1/\lambda}(-1)^{n-1}2^n\lambda^{m-1}}{m^{k-1}}S_{2,\lambda}(n,m)\right)\frac{t^n}{n!}.
\end{aligned}
$$

□

Theorem 2.3. *For $n \geq 0,\ k \in \mathbb{Z}$, we have*

$$\mathcal{E}_{n,\lambda}^{*(k)}(1) + \mathcal{E}_{n,\lambda}^{*(k)}(-1) = \sum_{m=1}^{n+1} \frac{(1)_{m,1/\lambda}(-1)^n 2^{n+1}\lambda^{m-1}}{m^{k-1}(n+1)} S_{2,\lambda}(n+1,m).$$

Proof. By using (3), (19) and Theorem 1, we observe that

(24)

$$\begin{aligned}
\frac{1}{t} l_{k,\lambda}\big(1-e_\lambda(-2t)\big) &= \big(e_\lambda(t)+e_\lambda^{-1}(t)\big)\sum_{l=0}^{\infty}\mathcal{E}_{l,\lambda}^{*(k)}\frac{t^l}{l!} \\
&= \left(\sum_{m=0}^{\infty}(1)_{m,\lambda}\frac{t^m}{m!} + \sum_{m=0}^{\infty}(1)_{m,-\lambda}(-1)^m\frac{t^m}{m!}\right)\sum_{l=0}^{\infty}\mathcal{E}_{l,\lambda}^{*(k)}\frac{t^l}{l!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\mathcal{E}_{l,\lambda}^{*(k)}(1)_{n-l,\lambda} + \sum_{l=0}^{n}\binom{n}{l}\mathcal{E}_{l,\lambda}^{*(k)}(1)_{n-l,-\lambda}(-1)^{n-l}\right)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\mathcal{E}_{n,\lambda}^{*(k)}(1)+\mathcal{E}_{n,\lambda}^{*(k)}(-1)\right)\frac{t^n}{n!}.
\end{aligned}$$

By using Lemma 2,

(25)

$$\begin{aligned}
\frac{1}{t} l_{k,\lambda}\big(1-e_\lambda(-2t)\big) &= \frac{1}{t}\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\frac{(1)_{m,\frac{1}{\lambda}}(-1)^{n-1}2^n\lambda^{m-1}}{m^{k-1}}S_{2,\lambda}(n,m)\right)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=1}^{n+1}\frac{(1)_{m,\frac{1}{\lambda}}(-1)^{n}2^{n+1}\lambda^{m-1}}{m^{k-1}(n+1)}S_{2,\lambda}(n+1,m)\right)\frac{t^n}{n!}.
\end{aligned}$$

Therefore, by comparing the coefficients of (24) and (25), we get what we wanted. □

Corollary 2.4. *For $n \geq 0,\ k = 1$, we have*

$$\sum_{m=1}^{n+1}\frac{(1)_{m,1/\lambda}(-1)^n 2^{n+1}\lambda^{m-1}}{(n+1)}S_{2,\lambda}(n+1,m) = \begin{cases} 2, & \text{if } n=0, \\ 0, & \text{otherwise.}\end{cases}$$

Moreover, we get

$$\mathcal{E}_{n,\lambda}^{*}(1)+\mathcal{E}_{n,\lambda}^{*}(-1) = \begin{cases} 2, & \text{if } n=0, \\ 0, & \text{otherwise.}\end{cases}$$

Proof. We note that $l_{1,\lambda}\big(1-e_\lambda(-2t)\big) = 2t$. Thus, from Lemma 2, we have

(26)

$$\begin{aligned}
2 &= \frac{1}{t}l_{1,\lambda}\big(1-e_\lambda(-2t)\big) \\
&= \sum_{n=0}^{\infty}\Big(\sum_{m=1}^{n+1}\frac{(1)_{m,1/\lambda}(-1)^n 2^{n+1}\lambda^{m-1}}{(n+1)}S_{2,\lambda}(n+1,m)\Big)\frac{t^n}{n!}.
\end{aligned}$$

Therefore, by comparing the coefficients on both sides of (26), we get the desired result. □

Theorem 2.5. *For $n \geq 0$, $k \in \mathbb{Z}$, we have*

$$\mathcal{E}_{n,\lambda}^{*(k)}(x) = \sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}(x)_m S_{2,\lambda}(l,m)\mathcal{E}_{n-l}^{*(k)}.$$

Proof. From (17) and (19), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{*(k)}(x)\frac{t^n}{n!} &= \frac{l_{k,\lambda}\big(1-e_\lambda(-2t)\big)}{t(e_\lambda(t)+e_\lambda^{-1}(t))}\big(e_\lambda(t)-1+1\big)^x \\ &= \sum_{i=0}^{\infty}\mathcal{E}_{i,\lambda}^{*(k)}\frac{t^i}{i!}\sum_{m=0}^{\infty}(x)_m\frac{(e_\lambda(t)-1)^m}{m!} \\ &= \sum_{i=0}^{\infty}\mathcal{E}_{i,\lambda}^{*(k)}\frac{t^i}{i!}\sum_{l=0}^{\infty}\bigg(\sum_{m=0}^{l}(x)_m S_{2,\lambda}(l,m)\bigg)\frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty}\bigg(\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}(x)_m S_{2,\lambda}(l,m)\mathcal{E}_{n-l}^{*(k)}\bigg)\frac{t^n}{n!}.\end{aligned} \tag{27}$$

Therefore, by comparing the coefficients on both sides of (27), we have the desired result. □

Theorem 2.6. *For $n \geq 0$, $k \in \mathbb{Z}$, we have*

$$\mathcal{E}_{n,\lambda}^{*(k)}(x+1) - \mathcal{E}_{n,\lambda}^{*(k)}(x) = \begin{cases}\displaystyle\sum_{m=0}^{n-1}\binom{n}{m}(1)_{n-m,\lambda}\mathcal{E}_{m,\lambda}^{*(k)}(x), & \textit{otherwise},\\ 0, & \textit{if } n=0.\end{cases}$$

Proof. From (3) and (19), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\big(\mathcal{E}_{n,\lambda}^{*(k)}(x+1) - \mathcal{E}_{n,\lambda}^{*(k)}(x)\big)\frac{t^n}{n!} = \frac{l_{k,\lambda}(1-e_\lambda(-2t))}{t(e_\lambda(t)+e_\lambda^{-1}(t))}(e_\lambda^{x+1}(t)-e_\lambda^{x}(t)) \\ &= \sum_{m=0}^{\infty}\mathcal{E}_{m,\lambda}^{*(k)}(x)\frac{t^m}{m!}\sum_{l=1}^{\infty}(1)_{l,\lambda}\frac{t^l}{l!} = \sum_{n=1}^{\infty}\bigg(\sum_{m=0}^{n-1}\binom{n}{m}(1)_{n-m,\lambda}\mathcal{E}_{m,\lambda}^{*(k)}(x)\bigg)\frac{t^n}{n!}.\end{aligned} \tag{28}$$

Thus, by comparing the coefficients on both sides of (28), we have the desired result. □

Now, we note that

$$\frac{d}{dx}e_\lambda(x) = e_\lambda^{1-\lambda}(x). \tag{29}$$

From (21) and (29), for $k \geq 2$, we have

$$\frac{d}{dx}l_{k,\lambda}\big(1-e_\lambda(-2x)\big) = \frac{-2e_\lambda^{1-\lambda}(-2x)}{e_\lambda(-2x)-1}l_{k-1,\lambda}\big(1-e_\lambda(-2x)\big),$$

and

(30)

$$l_{k,\lambda}\big(1-e_\lambda(-2x)\big)=2\int_0^x\underbrace{\frac{-2e_\lambda^{1-\lambda}(-2t)}{e_\lambda(-2t)-1}\int_0^t\frac{-2e_\lambda^{1-\lambda}(-2t)}{e_\lambda(-2t)-1}\cdots\int_0^t\frac{-2te_\lambda^{1-\lambda}(-2t)}{e_\lambda(-2t)-1}}_{(k-2)-times}\,dt\cdots dt.$$

Theorem 2.7. *For $n\geq 1$, and $k=2$, we have*

$$\mathcal{E}_{n,\lambda}^{*(2)}=\sum_{l=0}^{n}\binom{n}{l}\frac{(-2)^l}{l+1}B_{l,\lambda}(1-\lambda)E_{n-l,\lambda}^{*}.$$

Proof. By using (6), (10) and (30), we get

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{*(2)}\frac{x^n}{n!}&=\frac{2}{x(e_\lambda(x)+e_\lambda^{-1}(x))}\int_0^x\frac{-2t}{e_\lambda(-2t)-1}e_\lambda^{1-\lambda}(-2t)\;dt\\
&=\frac{2}{x(e_\lambda(x)+e_\lambda^{-1}(x))}\int_0^x\sum_{l=0}^{\infty}B_{l,\lambda}(1-\lambda)\frac{(-2t)^l}{l!}dt\\
&=\frac{2}{(e_\lambda(x)+e_\lambda^{-1}(x))}\sum_{l=0}^{\infty}\frac{(-2)^lB_{l,\lambda}(1-\lambda)}{l+1}\frac{x^l}{l!}\\
&=\sum_{m=0}^{\infty}E_{m,\lambda}^{*}\frac{x^m}{m!}\sum_{l=0}^{\infty}\frac{(-2)^lB_{l,\lambda}(1-\lambda)}{l+1}\frac{x^l}{l!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\frac{(-2)^l}{l+1}B_{l,\lambda}(1-\lambda)E_{n-l,\lambda}^{*}\right)\frac{x^n}{n!}.
\end{aligned}\tag{31}$$

Therefore, by comparing the coefficients on both sides of (31), we get what we wanted.

□

Theorem 2.8. *For $n\geq 0$, $k\in\mathbb{Z}$, we have*

$$\begin{aligned}
\mathcal{E}_{n,\lambda}^{*(k)}=&\sum_{n_1+n_2+\cdots+n_{k-1}=m}\binom{n}{m}(-2)^m\binom{m}{n_1,n_2,\ldots,n_{k-1}}\\
&\times\frac{B_{n_1,\lambda}(1-\lambda)}{n_1+1}\cdots\cdots\frac{B_{n_{k-1},\lambda}(1-\lambda)}{n_1+n_2+\cdots+n_{k-1}+1}E_{n-m,\lambda}^{*},
\end{aligned}$$

where $\binom{m}{n_1,n_2,\ldots,n_{k-1}}=\frac{m!}{n_1!\;n_2!\;\ldots,\;n_{k-1}!}$.

Proof. By using (6), (30), we have

(32)

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{*(k)}\frac{x^n}{n!} &= \frac{l_{k,\lambda}\big(1-e_\lambda(-2x)\big)}{x(e_\lambda(x)+e_\lambda^{-1}(x))}\\ &= \frac{2}{e_\lambda(x)+e_\lambda^{-1}(x)}\sum_{m=0}^{\infty}\sum_{n_1+n_2+\cdots+n_{k-1}=m}(-2)^m\binom{m}{n_1,n_2,\ldots,n_{k-1}}\\ &\quad\times\frac{B_{n_1,\lambda}(1-\lambda)}{n_1+1}\cdots\cdots\frac{B_{n_{k-1},\lambda}(1-\lambda)}{n_1+n_2+\cdots+n_{k-1}+1}\frac{x^m}{m!}\\ &= \sum_{n=0}^{\infty}\Bigg(\sum_{n_1+n_2+\cdots+n_{k-1}=m}\binom{n}{m}(-2)^m\binom{m}{n_1,n_2,\ldots,n_{k-1}}\\ &\quad\times\frac{B_{n_1,\lambda}(1-\lambda)}{n_1+1}\cdots\cdots\frac{B_{n_{k-1},\lambda}(1-\lambda)}{n_1+n_2+\cdots+n_{k-1}+1}E_{n-m,\lambda}^{*}\Bigg)\frac{x^m}{m!}.\end{aligned}$$

Therefore, by comparing the coefficients on both sides of (32), we get what we desired.

□

Theorem 2.9. *For $n\geq 0, k\in\mathbb{Z}$, we have*

$$\mathcal{E}_{n,\lambda}^{*(k)}(x)=\sum_{l=0}^{n}\sum_{m=1}^{l+1}\binom{n}{l}\frac{(1)_{m,1/\lambda}(-2)^l\lambda^{m-1}}{(l+1)m^{k-1}}S_{2,\lambda}(l+1,m)E_{n-l,\lambda}^{*}(x).$$

Proof. From (4), (17) and Lemma 2, we have

(33)

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{*(k)}(x)\frac{t^n}{n!} &= \frac{e_\lambda^x(t)}{t(e_\lambda(t)+e_\lambda^{-1}(t))}l_{k,\lambda}\big(1-e_\lambda(-2t)\big)\\ &= \frac{2te_\lambda^x(t)}{t(e_\lambda(t)+e_\lambda^{-1}(t))}\sum_{l=0}^{\infty}\Bigg(\sum_{m=1}^{l+1}\frac{(1)_{m,1/\lambda}(-1)^l\lambda^{m-1}2^l}{m^{k-1}(l+1)}S_{2,\lambda}(l+1,m)\Bigg)\frac{t^l}{l!}\\ &= \sum_{i=0}^{\infty}E_{i,\lambda}^{*}(x)\frac{t^i}{i!}\sum_{l=0}^{\infty}\Bigg(\sum_{m=1}^{l+1}\frac{(1)_{m,1/\lambda}(-1)^l\lambda^{m-1}2^l}{m^{k-1}(l+1)}S_{2,\lambda}(l+1,m)\Bigg)\frac{t^l}{l!}\\ &= \sum_{n=0}^{\infty}\Bigg(\sum_{l=0}^{n}\sum_{m=1}^{l+1}\binom{n}{l}\frac{(1)_{m,1/\lambda}(-2)^l\lambda^{m-1}}{(l+1)m^{k-1}}S_{2,\lambda}(l+1,m)E_{n-l,\lambda}^{*}(x)\Bigg)\frac{t^n}{n!}.\end{aligned}$$

Therefore, by comparing the coefficients on both sides of (33), we have the desired result. □

Theorem 2.10. *For $n\geq 0,\ k\in\mathbb{Z}$, we have*

$$\mathcal{E}_{n,\lambda}^{*(k)}(x)=\sum_{j=0}^{n}\sum_{l=0}^{[\frac{n-j}{2}]}\sum_{m=1}^{j+1}\binom{n}{j}\frac{\lambda^{n-j-2l+m-1}(1)_{m,1/\lambda}(-2)^j}{(j+1)m^{k-1}}S_1(n-j,2l)S_{2,\lambda}(j+1,m)E_{2l}^{*}.$$

Proof. From (1), (12), (17) and Lemma 2, we get

(34)
$$\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{*(k)}(x)\frac{t^n}{n!}=\frac{l_{k,\lambda}\big(1-e_\lambda(-2t)\big)}{t(e^{\frac{1}{\lambda}log(1+\lambda t)}+e^{-\frac{1}{\lambda}log(1+\lambda t)})}$$
$$=\frac{1}{2t}\sum_{l=0}^{\infty}E_{2l}^{*}\frac{(\frac{1}{\lambda}log(1+\lambda t))^{2l}}{(2l)!}\sum_{j=1}^{\infty}\left(\sum_{m=1}^{j}\frac{(1)_{m,1/\lambda}(-1)^{j-1}2^j\lambda^{m-1}}{m^{k-1}}S_{2,\lambda}(j,m)\right)\frac{t^j}{j!}$$
$$=\sum_{i=0}^{\infty}\left(\sum_{l=0}^{[\frac{i}{2}]}E_{2l}^{*}\lambda^{i-2l}S_1(i,2l)\right)\frac{t^i}{i!}\sum_{j=0}^{\infty}\left(\sum_{m=1}^{j+1}\frac{(1)_{m,1/\lambda}(-2)^j\lambda^{m-1}}{m^{k-1}(j+1)}S_{2,\lambda}(j+1,m)\right)\frac{t^j}{j!}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}\sum_{l=0}^{[\frac{n-j}{2}]}\sum_{m=1}^{j+1}\frac{\lambda^{n-j-2l+m-1}(1)_{m,1/\lambda}(-2)^j}{(j+1)m^{k-1}}S_1(n-j,2l)S_{2,\lambda}(j+1,m)E_{2l}^{*}\right)\frac{t^n}{n!}.$$

Therefore, by comparing the coefficients on both sides of (34), we have the desired result.

□

For next the theorem, we observe that

(35)
$$\begin{aligned}e_\lambda^m(t)=e^{\frac{m}{\lambda}\log(1+\lambda t)}&=\sum_{k=0}^{\infty}\left(\frac{m}{\lambda}\right)^k\frac{(\log(1+\lambda t))^k}{k!}\\&=\sum_{k=0}^{\infty}\left(\frac{m}{\lambda}\right)^k\sum_{l=k}^{\infty}S_1(l,k)\frac{\lambda^l t^l}{l!}=\sum_{l=0}^{\infty}\left(\sum_{k=0}^{l}m^k\lambda^{l-k}S_1(l,k)\right)\frac{t^l}{l!}.\end{aligned}$$

From (35), we get

(36)
$$\begin{aligned}e_\lambda(t)-e_\lambda^{-1}(t)&=\sum_{l=0}^{\infty}\left(\sum_{k=0}^{l}(1^k-(-1)^k)\lambda^{l-k}S_{1,\lambda}(l,k)\right)\frac{t^l}{l!}\\&=\sum_{l=1}^{\infty}\left(\sum_{k=0}^{[\frac{l-1}{2}]}2\lambda^{l-2k-1}S_1(l,2k+1)\right)\frac{t^l}{l!}.\end{aligned}$$

Theorem 2.11. *For $n\geq 1, k\in\mathbb{Z}$, we have*

$$n\mathcal{E}_{n-1,\lambda}^{*(k)}(x)=\sum_{l=1}^{n}\sum_{i=1}^{l}\sum_{m=1}^{i}\sum_{\upsilon=0}^{[\frac{l-i}{2}]}\binom{n}{l}\binom{l}{i}\frac{(1)_{m,1/\lambda}(-1)^{i-1}2^{n-l+i+1}\lambda^{m+l-i-2\upsilon-1}}{(l-i+1)m^{k-1}}$$
$$\times S_1(l-i+1,2\upsilon+1)S_{2,\lambda}(i,m)B_{n-l,\frac{\lambda}{2}}^{*}\left(\frac{x}{2}\right).$$

Proof. From (3), (5), (17), (36) and Lemma 2, we get

(37)
$$
\begin{aligned}
t\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{*(k)}(x)\frac{t^n}{n!} &= \frac{e_\lambda^x(t)}{e_\lambda(t)+e_\lambda^{-1}(t)}l_{k,\lambda}(1-e_\lambda(-2t)) \\
&= \frac{2te_\lambda^x(t)}{e_\lambda^2(t)-e_\lambda^{-2}(t)}\frac{1}{2t}\Big(e_\lambda(t)-e_\lambda^{-1}(t)\Big)l_{k,\lambda}(1-e_\lambda(-2t)) \\
&= \frac{1}{2}\sum_{\alpha=0}^{\infty}B_{\alpha,\frac{\lambda}{2}}^{*}(\frac{x}{2})\frac{2^\alpha t^\alpha}{\alpha!}\sum_{j=0}^{\infty}\Bigg(\sum_{\upsilon=0}^{[\frac{j}{2}]}2\lambda^{j-2\upsilon}S_1(j+1,2\upsilon+1)\Bigg)\frac{t^j}{(j+1)!} \\
&\quad\times\sum_{i=1}^{\infty}\Bigg(\sum_{m=1}^{i}\frac{(1)_{m,1/\lambda}(-1)^{i-1}2^i\lambda^{m-1}}{m^{k-1}}S_{2,\lambda}(i,m)\Bigg)\frac{t^i}{i!} \\
&= \sum_{\alpha=0}^{\infty}B_{\alpha,\frac{\lambda}{2}}^{*}(\frac{x}{2})\frac{2^\alpha t^\alpha}{\alpha!}\sum_{l=1}^{\infty}\Bigg(\sum_{i=1}^{l}\sum_{m=1}^{i}\sum_{\upsilon=0}^{[\frac{l-i}{2}]}\binom{l}{i}\lambda^{l-i-2\upsilon} \\
&\quad\times\frac{(1)_{m,1/\lambda}(-1)^{i-1}2^{i+1}\lambda^{m-1}}{(l-i+1)m^{k-1}}S_1(l-i+1,2\upsilon+1)S_{2,\lambda}(i,m)\Bigg)\frac{t^l}{l!} \\
&= \sum_{n=1}^{\infty}\Bigg(\sum_{l=1}^{n}\sum_{i=1}^{l}\sum_{m=1}^{i}\sum_{\upsilon=0}^{[\frac{l-i}{2}]}\binom{n}{l}\binom{l}{i}\frac{(1)_{m,1/\lambda}(-1)^{i-1}2^{n-l+i+1}\lambda^{m+l-i-2\upsilon-1}}{(l-i+1)m^{k-1}} \\
&\quad\times S_1(l-i+1,2\upsilon+1)S_{2,\lambda}(i,m)B_{n-l,\frac{\lambda}{2}}^{*}\left(\frac{x}{2}\right)\Bigg)\frac{t^n}{n!}.
\end{aligned}
$$

On the other hand, we have

$$
t\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{*(k)}(x)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{*(k)}(x)\frac{t^{n+1}}{n!}=\sum_{n=1}^{\infty}n\mathcal{E}_{n-1,\lambda}^{*(k)}(x)\frac{t^n}{n!}. \tag{38}
$$

Therefore, by comparing the coefficients on both sides of (37) and (38), we have the desired result. □

Theorem 2.12. *For $n\geq 1,\ k\in\mathbb{Z}$, we get*

$$
\begin{aligned}
&\sum_{m=0}^{n-1}\sum_{l=0}^{m}\binom{n}{m}\lambda^{n-m-1}(1)_{n-m,1/\lambda}(-1)^{l+m+1}2^{-l-1}S_{1,\lambda}(m,l)\mathcal{E}_{l,\lambda}^{*(k)} \\
&\quad=\sum_{m=1}^{n}\sum_{l=0}^{n-m}\binom{n}{m}\frac{(-1)^{l+n-1}(1)_{m,1/\lambda}\lambda^{m-1}2^{-l-1}}{m^{k-1}}S_{1,\lambda}(n-m,l)E_{l,\lambda}^{*}.
\end{aligned}
$$

Proof. From (4), (18) and Lemma 2, we observe

(39)

$$
\begin{aligned}
&\frac{l_{k,\lambda}(t)}{e_\lambda(-\frac{1}{2}log_\lambda(1-t))+e_\lambda^{-1}(-\frac{1}{2}log_\lambda(1-t))}\\
&=\frac{1}{2}\sum_{l=0}^{\infty}E^*_{l,\lambda}\frac{(-\frac{1}{2}log_\lambda(1-t))^l}{l!}\sum_{m=1}^{\infty}\frac{(-\lambda)^{m-1}(1)_{m,1/\lambda}}{(m-1)!m^k}t^m\\
&=\frac{1}{2}\sum_{l=0}^{\infty}E^*_{l,\lambda}(-\frac{1}{2})^l\sum_{j=l}^{\infty}S_{1,\lambda}(j,l)\frac{(-t)^j}{j!}\sum_{m=1}^{\infty}\frac{(1)_{m,1/\lambda}(-\lambda)^{m-1}}{m^{k-1}}\frac{t^m}{m!}\\
&=\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\sum_{l=0}^{n-m}\binom{n}{m}(-1)^{l+n-m}2^{-l-1}\frac{(1)_{m,1/\lambda}(-\lambda)^{m-1}}{m^{k-1}}S_{1,\lambda}(n-m,l)E^*_{l,\lambda}\right)\frac{t^n}{n!}.
\end{aligned}
$$

On the other hand, from (18) and (19), (39) is equal to

(40)

$$
\begin{aligned}
&-\frac{1}{2}log_\lambda(1+t)\sum_{l=0}^{\infty}\mathcal{E}^{*(k)}_{l,\lambda}\frac{(-\frac{1}{2}log_\lambda(1-t))^l}{l!}\\
&=-\frac{1}{2}\sum_{j=1}^{\infty}\lambda^{j-1}(1)_{j,1/\lambda}\frac{t^j}{j!}\sum_{l=0}^{\infty}\mathcal{E}^{*(k)}_{l,\lambda}(-\frac{1}{2})^l\sum_{m=l}^{\infty}S_{1,\lambda}(m,l)\frac{(-t)^m}{m!}\\
&=\sum_{n=1}^{\infty}\left(\sum_{m=0}^{n-1}\sum_{l=0}^{m}\binom{n}{m}\lambda^{n-m-1}(1)_{n-m,1/\lambda}(-1)^{l+m+1}2^{-l-1}S_{1,\lambda}(m,l)\mathcal{E}^{*(k)}_{l,\lambda}\right)\frac{t^n}{n!}.
\end{aligned}
$$

Therefore, by comparing the coefficients of (39) and (40), we have the desired result.

□

Theorem 2.13. *For $n\geq 0,\ k\in\mathbb{Z}$, we have*

$$
\mathcal{E}^{*(k)}_{n,\lambda}=\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\binom{l}{m}\frac{(-1)^l2^l(1)_{m+1,\lambda}}{m+1}\beta^{(k)}_{l-m,\lambda}E^*_{n-l,\lambda}
$$

Proof. From (3), (4) and (9), we observe

$$
\begin{aligned}
&\frac{l_{k,\lambda}(1-e_\lambda(-2t)))}{t(e_\lambda(t)+e_\lambda^{-1}(t))}\frac{1-e_\lambda(-2t)}{1-e_\lambda(-2t)}\\
&=\frac{1}{2t}\frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)}\frac{l_{k,\lambda}(1-e_\lambda(-2t))}{1-e_\lambda(-2t)}(1-e_\lambda(-2t))\\
&=\frac{1}{2t}\sum_{i=0}^{\infty}E^*_{i,\lambda}\frac{t^i}{i!}\sum_{j=0}^{\infty}\beta^{(k)}_{j,\lambda}\frac{(-2t)^j}{j!}\sum_{m=1}^{\infty}(-1)^{m+1}2^m(1)_{m,\lambda}\frac{t^m}{m!}\\
&=\sum_{i=0}^{\infty}E^*_{i,\lambda}\frac{t^i}{i!}\sum_{j=0}^{\infty}\beta^{(k)}_{j,\lambda}(-2)^j\frac{t^j}{j!}\sum_{m=0}^{\infty}\frac{(-1)^{m+2}2^m(1)_{m+1,\lambda}}{m+1}\frac{t^m}{m!}\\
&=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\binom{l}{m}\frac{(-1)^{l+2}2^l(1)_{m+1,\lambda}}{m+1}\beta^{(k)}_{l-m,\lambda}E^*_{n-l,\lambda}\frac{t^n}{n!}.
\end{aligned}
\tag{41}
$$

Therefore, by comparing the coefficients on both sides of (41), we have the desired result. □

3. The degenerate unipoly type 2 Euler polynomials and numbers

In this section, we also introduce the degenerate unipoly-type 2 Euler polynomials attached to an arithmetic function. By using the degenerate polylogarithm function, we give some new explicit expressions and identities related to degenerate unipoly-type 2 Euler polynomials and special numbers and polynomials.

Let p be any arithmetic function which is real or complex valued function defined on the set of positive integers $\mathbb{N}$. Kim-Kim[14] defined the unipoly function attached to polynomials $p(x)$ by

$$u_k(x|p)=\sum_{n=1}^{\infty}\frac{p(n)x^n}{n^k}\quad (k\in\mathbb{Z}).$$

When $p(n)=1,\ \forall n\in\mathbb{N}$, we observe that

$$u_k(x|1)=\sum_{n=1}^{\infty}\frac{x^n}{n^k}=Li_k(x)$$

is the ordinary polylogarithm function.

In this paper, we define the degenerate unipoly function attached to polynomials $p(x)$ as follows:

$$u_{k,\lambda}(x|p)=\sum_{n=1}^{\infty}p(n)\frac{(-\lambda)^{n-1}(1)_{n,1/\lambda}}{n^k}x^n. \tag{42}$$

When $\Gamma(n)=(n-1)!$, we note that

$$u_{k,\lambda}(x|\frac{1}{\Gamma})=l_{k,\lambda}(x)$$

is the degenerate polylogarithm function.

We also define the degenerate unipoly type 2 Euler polynomials by

$$\frac{u_{k,\lambda}\big(1-e_\lambda(-2t)\big|p\big)}{t(e_\lambda(t)+e_\lambda^{-1}(t))}e_\lambda^x(t)=\sum_{n=0}^{\infty}\mathcal{E}^{*(k)}_{n,\lambda,p}(x)\frac{t^n}{n!}. \tag{43}$$

When $x=0$, $\mathcal{E}^{*(k)}_{n,\lambda,p}=\mathcal{E}^{*(k)}_{n,\lambda,p}(0)$ is the degenerate unipoly type 2 Euler numbers.

We note that

$$\mathcal{E}^{*(k)}_{n,\lambda,\frac{1}{\Gamma}}(x)=\sum_{n=0}^{\infty}\mathcal{E}^{*(k)}_{n,\lambda}(x).$$

The next lemma is intended to be used conveniently in proving some of the theorems below.

Lemma 3.1. *For $k \in \mathbb{Z}$, we have*

$$u_{k,\lambda}\big(1-e_\lambda(-2t)\mid p\big)=\sum_{l=1}^{\infty}\bigg(\sum_{m=1}^{l}\frac{p(m)(-1)^{l-1}\lambda^{m-1}(1)_{m,1/\lambda}m!2^l}{m^k}S_{2,\lambda}(l,m)\bigg)\frac{t^l}{l!}.$$

Proof. From (17) and (42), we have

(44)

$$\begin{aligned}u_{k,\lambda}\big(1-e_\lambda(-2t)\mid p\big)&=\sum_{m=1}^{\infty}\frac{p(m)(-\lambda)^{m-1}(1)_{m,1/\lambda}}{m^k}\big(1-e_\lambda(-2t)\big)^m\frac{m!}{m!}\\&=\sum_{m=1}^{\infty}\frac{p(m)(-1)^{2m-1}\lambda^{m-1}(1)_{m,1/\lambda}m!}{m^k}\frac{\big(e_\lambda(-2t)-1\big)^m}{m!}\\&=\sum_{m=1}^{\infty}\frac{p(m)(-1)^{2m-1}\lambda^{m-1}(1)_{m,1/\lambda}m!}{m^k}\sum_{l=m}^{\infty}S_{2,\lambda}(l,m)\frac{(-2t)^l}{l!}\\&=\sum_{l=1}^{\infty}\bigg(\sum_{m=1}^{l}\frac{p(m)(-1)^{l-1}\lambda^{m-1}(1)_{m,1/\lambda}m!2^l}{m^k}S_{2,\lambda}(l,m)\bigg)\frac{t^l}{l!}.\end{aligned}$$

Therefore, by comparing the coefficients on both sides of (44), we have what we wanted. □

Theorem 3.2. *For $n \geq 0, k \in \mathbb{Z}$, we get*

$$\mathcal{E}^{*(k)}_{n,\lambda,p}=\sum_{l=0}^{n}\sum_{m=1}^{l+1}\binom{n}{l}\frac{m!p(m)(-1)^l\lambda^{m-1}(1)_{m,1/\lambda}2^l}{m^k(l+1)}S_{2,\lambda}(l+1,m)E^*_{n-l,\lambda}.$$

Proof. From (4), (17) and Lemma 14, we have

(45)

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{E}^{*(k)}_{n,\lambda,p}\frac{t^n}{n!}&=\frac{u_{k,\lambda}\big(1-e_\lambda(-2t)\big|p\big)}{t(e_\lambda(t)+e_\lambda^{-1}(t))}\\&=\frac{1}{2t}\bigg(\frac{2}{e_\lambda(t)+e_\lambda(t)^{-1}}\bigg)\sum_{l=1}^{\infty}\bigg(\sum_{m=1}^{l}\frac{m!p(m)(-1)^{l-1}\lambda^{m-1}(1)_{m,1/\lambda}2^l}{m^k}S_{2,\lambda}(l,m)\bigg)\frac{t^l}{l!}\\&=\sum_{i=0}^{\infty}E^*_{i,\lambda}\frac{t^i}{i!}\sum_{l=0}^{\infty}\bigg(\sum_{m=1}^{l+1}\frac{m!p(m)(-1)^l\lambda^{m-1}(1)_{m,1/\lambda}2^lS_2(l+1,m)}{m^k(l+1)}\bigg)\frac{t^l}{l!}\\&=\sum_{n=0}^{\infty}\bigg(\sum_{l=0}^{n}\sum_{m=1}^{l+1}\binom{n}{l}\frac{m!p(m)(-1)^l\lambda^{m-1}(1)_{m,1/\lambda}2^l}{m^k(l+1)}S_{2,\lambda}(l+1,m)E^*_{n-l,\lambda}\bigg)\frac{t^n}{n!}.\end{aligned}$$

Thus, by comparing the coefficients on both sides of (45), we have the desired result. □

Theorem 3.3. *For $n \geq 0$, $k \in \mathbb{Z}$, we have*

$$\mathcal{E}^{*(k)}_{n,\lambda,p}(x) = \sum_{\alpha=0}^{n}\sum_{l=0}^{\alpha}\sum_{m=1}^{l+1}\sum_{\upsilon=0}^{[\frac{\alpha-l}{2}]}\binom{n}{\alpha}\binom{\alpha}{l}\frac{(-1)^l(1)_{m,1/\lambda}\lambda^{\alpha-l-2\upsilon+m-1}2^{n-\alpha+l+1}m!p(m)}{(\alpha-l+1)(l+1)m^k} \times S_1(\alpha-l+1,2\upsilon+1)S_{2,\lambda}(l+1,m)B^*_{n-\alpha,\frac{\lambda}{2}}\left(\frac{x}{2}\right).$$

Proof. From (3), (5), (17), (36) and Lemma 14, we get

(46)

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{E}^{*(k)}_{n,\lambda,p}(x)\frac{t^n}{n!} = \frac{u_{k,\lambda}(1-e_\lambda(-2t)|p)}{t(e_\lambda(t)+e_\lambda^{-1}(t))}e_\lambda^x(t)\\
&= \frac{e_\lambda^x(t)}{e_\lambda(t)+e_\lambda^{-1}(t)}\frac{e_\lambda(t)-e_\lambda^{-1}(t)}{e_\lambda(t)-e_\lambda^{-1}(t)}\\
&\qquad\times\sum_{l=0}^{\infty}\left(\sum_{m=1}^{l+1}\frac{p(m)(-1)^l\lambda^{m-1}(1)_{m,1/\lambda}m!2^{l+1}}{m^k}S_{2,\lambda}(l+1,m)\right)\frac{t^l}{(l+1)!}\\
&= \frac{2t}{e_{\frac{\lambda}{2}}(2t)-e_{\frac{\lambda}{2}}^{-1}(2t)}e_{\frac{\lambda}{2}}^{\frac{x}{2}}(2t)\frac{1}{2t}\sum_{j=1}^{\infty}\left(\sum_{\upsilon=0}^{[\frac{j-1}{2}]}2\lambda^{j-2\upsilon-1}S_1(j,2\upsilon+1)\right)\frac{t^j}{j!}\\
&\qquad\times\sum_{l=0}^{\infty}\left(\sum_{m=1}^{l+1}\frac{p(m)(-1)^l\lambda^{m-1}(1)_{m,1/\lambda}m!2^{l+1}}{(l+1)m^k}S_{2,\lambda}(l+1,m)\right)\frac{t^l}{l!}\\
&= \sum_{i=0}^{\infty}B^*_{i,\frac{\lambda}{2}}(\frac{x}{2})\frac{2^it^i}{i!}\sum_{j=0}^{\infty}\left(\sum_{\upsilon=0}^{[\frac{j}{2}]}\lambda^{j-2\upsilon}S_1(j+1,2\upsilon+1)\right)\frac{t^j}{(j+1)!}\\
&\qquad\times\sum_{l=0}^{\infty}\left(\sum_{m=1}^{l+1}\frac{p(m)(-1)^l\lambda^{m-1}(1)_{m,1/\lambda}m!2^{l+1}}{(l+1)m^k}S_{2,\lambda}(l+1,m)\right)\frac{t^l}{l!}\\
&= \sum_{i=0}^{\infty}B^*_{i,\frac{\lambda}{2}}\left(\frac{x}{2}\right)\frac{2^it^i}{i!}\sum_{\alpha=0}^{\infty}\left(\sum_{l=0}^{\alpha}\sum_{m=1}^{l+1}\sum_{\upsilon=0}^{[\frac{\alpha-l}{2}]}\binom{\alpha}{l}\frac{(-1)^l(1)_{m,1/\lambda}\lambda^{\alpha-l-2\upsilon+m-1}2^{l+1}m!p(m)}{(\alpha-l+1)(l+1)m^k}\right.\\
&\qquad\left.\times S_1(\alpha-l+1,2\upsilon+1)S_{2,\lambda}(l+1,m)\right)\frac{t^\alpha}{\alpha!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{\alpha=0}^{n}\sum_{l=0}^{\alpha}\sum_{m=1}^{l+1}\sum_{\upsilon=0}^{[\frac{\alpha-l}{2}]}\binom{n}{\alpha}\binom{\alpha}{l}\frac{(-1)^l(1)_{m,1/\lambda}\lambda^{\alpha-l-2\upsilon+m-1}2^{n-\alpha+l+1}m!p(m)}{(\alpha-l+1)(l+1)m^k}\right.\\
&\qquad\left.\times S_1(\alpha-l+1,2\upsilon+1)S_{2,\lambda}(l+1,m)B^*_{n-\alpha,\frac{\lambda}{2}}\left(\frac{x}{2}\right)\right)\frac{t^n}{n!}.
\end{aligned}$$

Thus, by comparing the coefficients on both sides of (46), we obtain the desired theorem. □

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