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PARTITIONS AND LABELED TREES

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Abstract. In this paper, we present a new representation of partitions of positive integers using labeled trees. Labeled trees representing a partition of an integer n is called a partition-tree of n. We discuss some results involving partition-trees, energy of an integer with respect to partition-trees, and gracefulness of the default labeling of partition trees.

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1. Introduction

For standard terminology and notion in graph theory, we follow the text-book of Deo [2]. The non-standard will be given in this paper as and when required.

Let $G = (V, E)$ be a graph (finite, simple, connected and undirected). We write $u \sim v$ to denote two vertices u and v are adjacent in G. The degree of a vertex v in G is denoted by $\deg(v)$.

A tree is simple connected graph without any cycle. Throughout this paper, by a labeled tree with k vertices, we means a tree whose k vertices are assigned unique numbers from 0 to $k-1$. Two labeled trees are isomorphic if their graphs are isomorphic and the corresponding points of the two trees have the same labels. The set of all internal(non-pendant) vertices of a tree T is denoted by $IV(T)$. Label value of a vertex v is denoted by $l(v)$.

A partition (integer partition) λ of a positive integer n, is a way of writing n as a sum of positive integers. Two sums that differ only in the order of their summands are considered the same partition (See [1]). There is a diagram representation of partitions of integers called Young's diagram. It is a finite collection of boxes, or cells, arranged in left-justified rows, with the row lengths in non-increasing order. Listing the number of boxes in each row gives a partition λ of a non-negative integer n, the total number of boxes of the diagram. This diagram is called a Ferrer's diagram, when partitions are represented using dots.

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The adjacency matrix of a graph G with n vertices $v_1, \ldots, v_n$, denoted by A_G, is an $n \times n$ matrix $(a_i j)$, whose (i, j)-th entry is given by

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \sim v_j; \\ 0, & \text{otherwise,} \end{cases}$$

The eigenvalues of A_G are said to be the eigenvalues of the graph G.

Definition 1.1. [4] *Let $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of a graph G. The energy of G is defined as*

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\mu_i|. \tag{1}$$

Let G be a graph with n vertices and m edges. The following inequalities are due to McClelland [8] and are obtained by Gutman [5, Theorem 5.1, Corollary 5.3.]:

$$\mathcal{E}(G) \leq \sqrt{2mn}, \tag{2}$$

$$2\sqrt{m} \leq \mathcal{E}(G) \leq 2m. \tag{3}$$

The following inequalities (4) and (5) are given by Koolen and Moulton [6, 7]:

$$\mathcal{E}(G) \leq \frac{n(\sqrt{n}+1)}{2}, \tag{4}$$

and if G is a bipartite graph with $n > 2$,

$$\mathcal{E}(G) \leq \frac{n(\sqrt{n}+\sqrt{2})}{\sqrt{8}}. \tag{5}$$

The following inequality gives a lower bound for the energy of G solely in terms of number of vertices [5, Theorem 5.4]:

$$\mathcal{E}(G) \geq 2\sqrt{n-1} \tag{6}$$

with equality if and only if G is the star $K_{1,n-1}$.

In [9], Rajendra et al. have discussed a ring and $\mathbb{Z}_n$-module structures on $\mathcal{T}_n$, the collection of all $\mathbb{Z}_n$-labeled trees with $n \geq 3$, Prüfer sequence construction method. This motivated to look at partitions of a positive integer as labeled trees. In this paper, we present a new representation of partitions of positive integers using labeled trees called partition-trees of n. We discuss some results involving partition-trees, energy of an integer with respect to partition-trees, and gracefulness of the default labeling of partition trees.

2. Correspondence between k vertex labeled trees and $(k-2)$-tuples

Let T be a k-vertex tree, whose vertices are labeled $0, 1, 2, \ldots, k-1$. We recall the following Prüfer sequence construction method (see [2, 3]) to define a $(k-2)$-tuple representing T: Remove the pendent vertex (and the edge incident on it) having the smallest label, which is, say, a_1. Suppose that b_1 was the vertex adjacent to a_1. Among the remaining $k-1$ vertices let a_2 be the pendant vertex with the smallest label and b_2 be the vertex adjacent to a_2. Remove the edge (a_2, b_2). This operation is repeated on the remaining $k-2$ vertices, and then on $k-3$ vertices, and so on. The process is terminated after $k-2$ steps, when only 2 vertices are left. The tree T defines the sequence ($(k-2)$-tuple)

$$(b_1, b_2, b_3, \ldots, b_{k-2}) \tag{7}$$

uniquely.

Conversely, given a $(k-2)$-tuple i.e., sequence (7) of $k-2$ labels (from the set $\{0, 1, 2, \ldots, k-1\}$), a k-vertex tree can be constructed uniquely, as follows: Determine the first number in the sequence

$$0, 1, 2, \ldots, k-1 \tag{8}$$

that does not appear in the sequence (7). This number clearly is a_1. And thus the edge (a_1, b_1) is defined. Remove b_1 from the sequence (7) and a_1 from (8). In the remaining sequence of (8) find the first number that does not appear in the remainder of (7). This would be a_2, and thus the edge (a_2, b_2) is defined. The construction is continued till the sequence (7) has no element left. Finally, the last two vertices remaining in (8) are joined. So that we obtain T from (8).

Thus, there is a one-to-one correspondence between k-vertex labeled trees and $(k-2)$-tuples from the set $\{0, 1, 2, \ldots, k-1\}$. Hence, the number of labeled trees with k vertices is k^{k-2}.

3. Partition-trees

Definition 3.1. *An n-tuple $(x_1, x_2, \ldots, x_n)$ from the set $\{0, 1, 2, \ldots, n\}$ is called a (weak) composition n-tuple of n if*

$$x_1 \leq x_2 \leq \cdots \leq x_n \text{ and } x_1 + x_2 + \cdots + x_n = n.$$

There is a one-to-one correspondence between partitions of n and composition n-tuples of n. Hence, throughout this paper by a partition of n, we mean to consider a composition n-tuples of n.

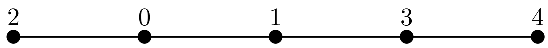
By the Prüfer sequence construction method discussed in section 2, given a partition λ of an in integer n, we can construct (uniquely) an $(n+2)$-vertex labeled tree (whose vertices are labeled $0, 1, 2, \ldots, n+1$), which we denote by T_λ. An $(n+2)$-vertex labeled tree corresponding to a partition λ of a positive integer n is called a partition-n-tree or a partition-tree of n with respect to λ. Partition-trees give a new representation of partitions of integers. Thus, we have

Theorem 3.2. *Let n be a positive integer, Λ_n be the set of all partitions of n and $\mathcal{T}_n$ be set of all the partition-trees of n. Then there is a one-to-one correspondence between Λ_n and $\mathcal{T}_n$ given by $\lambda \mapsto T_\lambda$.*

By Theorem 3.2 and the structure theorem for finite abelian groups, the following corollary is immediate:

Corollary 3.3. *Let p be a prime integer and n be a positive integer. Let $\mathcal{G}_n$ be the set of all equivalence classes of abelian groups of order p^n under isomorphism relation and $\mathcal{T}_n$ be set of all the partition-trees of n. Then there is a one-to-one correspondence between $\mathcal{G}_n$ and $\mathcal{T}_n$.*

Remark 3.4. *It should be noted that not every $(n+2)$-labeled tree is a partition-n-tree. For instance, consider the 5-vertex labeled tree T given in Figure 1.*



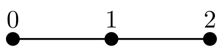
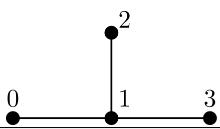
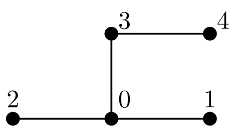
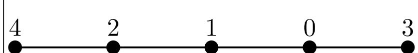
T

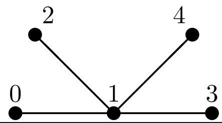
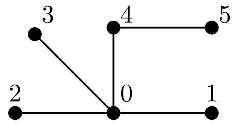
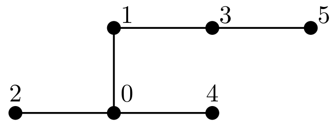
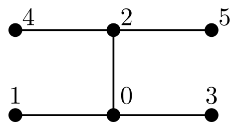
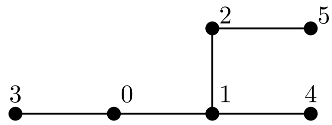
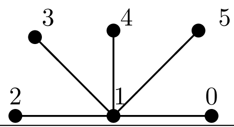
FIGURE 1. A labeled tree that is not a partition-3-tree

The tree T corresponds to the 3-tuple $(0, 1, 3)$ which is not a partition of 3. Hence the tree T is not a partition-3-tree. Hence, if $\mathcal{L}_n$ is the set of all labeled trees with $n+2$-vertices (whose vertices are labeled $0, 1, 2, \ldots, n+1$), then $\mathcal{T}_n \subsetneq \mathcal{L}_n$.

In Table 1, partition-trees corresponding to partitions of first four integers are listed.

Table 1: Partitions with corresponding composition n−tuples and labeled trees of few integers

n	Partitions	Composition n-tuples	partition-trees of n
1	1	(1)	
2	2	(0,2)	
	1+1	(1,1)	
3	3	(0,0,3)	
	1+2	(0,1,2)	

	1+1+1	(1,1,1)	
4	4	(0,0,0,4)	
	1+3	(0,0,1,3)	
	2+2	(0,0,2,2)	
	1+1+2	(0,1,1,2)	
	1+1+1+1	(1,1,1,1)	

For a partition-tree T of n, the partition of n corresponding to T is denoted by λ_T.

Proposition 3.5. *Let n be a positive integer and T be a partition-tree of n. Let $IV(T) = \{v_1, \ldots, v_k\}$. We have,*

(i) If $l(v_1) \leq l(v_2) \leq \cdots \leq l(v_k)$, then

$$\lambda_T = \left(\underbrace{l(v_1), \ldots, l(v_1)}_{\deg(v_1)-1 \text{ terms}}, \underbrace{l(v_2), \ldots, l(v_2)}_{\deg(v_2)-1 \text{ terms}}, \ldots, \underbrace{l(v_k), \ldots, l(v_k)}_{\deg(v_k)-1 \text{ terms}} \right).$$

(ii) $\displaystyle\sum_{v_i \in IV(T_\lambda)} (\deg(v_i) - 1) \cdot l(v_i) = n.$

Proof. (i) Follows by the Prüfer sequence construction method discussed in section 2.

(ii) Follows from (i). □

Corollary 3.6. *If n be a positive integer ≥ 4, then no partition-tree of n is isomorphic to P_{n+2}.*

Proof. Suppose that n is a positive integer ≥ 4 and assume that λ is a partition of n such that $\lambda_T \cong P_{n+2}$. By Proposition 3.5(ii),

$$\sum_{v_i \in IV(T_\lambda)} l(v_i) = n \tag{9}$$

But we have,

$$\sum_{v_i \in IV(T_\lambda)} l(v_i) = \sum_{v_i \in V(T_\lambda)} l(v_i) - (l(u) + l(w)), \tag{10}$$

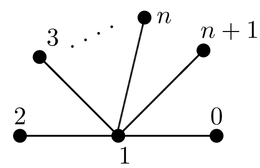
where u and w are pendant vertices of λ_T. Since λ_T is a partition-tree of n, it is clear that one of u and w has labeling $n+1$, say $l(w) = n+1$. Also, we have $0 \leq l(w) \leq n$. Since $n \geq 4$, from (10), we have,

$$\sum_{v_i \in IV(T_\lambda)} l(v_i) = 0 + 1 + 2 + \cdots + (n+1) - l(u) - (n+1) > n,$$

which is a contradiction to the fact (9). □

Lemma 3.7. *Let n be a positive integer and λ be a partition of n. Then $\lambda = (1, 1, \ldots, 1)$ if and only if $T_\lambda \cong K_{1,n+1}$.*

Proof. Suppose that $\lambda = (1, 1, \ldots, 1)$ (which is an $(n+2)$-tuple). Then by the method of construction of a labeled tree from an $(n+2)$-tuple discussed in section 2, it follows that, T_λ is the labeled tree given below:



T_λ

Hence, $T_\lambda \cong K_{1,n+1}$.

Conversely, suppose that $T_\lambda \cong K_{1,n+1}$. Then by the Prüfer sequence construction method λ is an n-tuple of the form $(t, t, \ldots, t)$. Since it is a (weak) composition n-tuple of n, by Definition 3.1, it follows that $t = 1$ and consequently, $\lambda = (1, 1, \ldots, 1)$. □

Theorem 3.8. *Let n be a positive integer. For any partition λ of n, the vertices labeled 0 and 1 are adjacent in T_λ.*

Proof. If $\lambda = (1, 1, \ldots, 1)$, by Lemma 3.7, it follows that the vertices labeled 0 and 1 are adjacent in T_λ. Assume that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \neq (1, 1, \ldots, 1)$. Then clearly, $\lambda_1 = 0$ and the vertex labeled 0 is an internal vertex in T_λ. We observe that, in the partition-tree corresponding $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, either the vertices labeled λ_i and λ_{i+1} are adjacent or $\lambda_i = \lambda_{i+1}$. So, if 1 is a part in the partition λ, then the vertices labeled 0 and 1 are adjacent. If 1 is not a part in λ, then the vertex labeled 1 is a pendant vertex and from the method of construction of the partition-tree T_λ from the partition λ, it follows that the vertices labeled 0 and 1 are adjacent. □

Proposition 3.9. *Let n be a positive integer n, $\mathcal{T}_n^{(0)}$ be the set of all partition-trees of n having an internal vertex labeled 0, and $\mathcal{L}_n^{(0)}$ the set of all $(n+2)$-vertex labeled trees having an internal vertex labeled 0. Then*

$$p(n) = |\mathcal{T}_n^{(0)}| + 1 \leq |\mathcal{L}_n^{(0)}|.$$

Proof. For $n = 1$, $\mathcal{T}_1^{(0)} = \emptyset$ and the result follows in this case. For any partition $\lambda \neq (1, 1, \dots, 1)$ of $n \geq 2$, 0 is an internal vertex of the partition-tree λ_T. Also, $\mathcal{T}_n^{(0)} \subsetneq \mathcal{L}_n^{(0)}$. Therefore, $p(n) = |\mathcal{T}_n^{(0)}| + 1 \leq |\mathcal{L}_n^{(0)}|$. □

4. Energy of an integer corresponding to a partition

Definition 4.1. *Let n be a positive integer, λ be a partition of n and T_λ be the partition-tree corresponding to λ. Let A_λ be the adjacency matrix of T_λ. We call A_λ as the matrix of the partition λ. The energy of the integer n corresponding to the partition λ, denoted by $\mathcal{E}(n, \lambda)$ or $\mathcal{E}_\lambda(n)$, is defined to be the energy of the tree T_λ.*

We note that A_λ is an $(n+2) \times (n+2)$ real symmetric matrix. If $\mu_1, \mu_2, \dots, \mu_{n+2}$ are the eigenvalues of A_λ, then the energy of n corresponding to λ is

$$\mathcal{E}_\lambda(n) = \sum_{i=1}^{n+2} |\mu_i|.$$

Theorem 4.2. *Let n be a positive integer and λ be a partition of n. Then*

$$\mathcal{E}_\lambda(n) \leq \sqrt{2(n+1)(n+2)}, \tag{11}$$

$$2\sqrt{n+1} \leq \mathcal{E}_\lambda(n) \leq 2(n+1) \tag{12}$$

$$\mathcal{E}_\lambda(n) \leq \frac{(n+2)(\sqrt{(n+2)}+1)}{2} \tag{13}$$

$$\mathcal{E}_\lambda(n) \leq \frac{(n+2)(\sqrt{(n+2)}+\sqrt{2})}{\sqrt{8}} \tag{14}$$

$$\mathcal{E}_\lambda(n) \geq 2\sqrt{n+1} \tag{15}$$

with equality in (15) *if and only if* $\lambda = (1, 1, \dots, 1)$.

Proof. Since T_λ is a tree with $n+2$ vertices (it has $n+1$ edges), the proof of the inequalities (11) to (15) follows from the inequalities (2) to (6). By the Lemma 3.7, $\lambda = (1, 1, \dots, 1)$ if and only if $T_\lambda \cong K_{1,n+1}$. Hence from the equality case in (6), it follows that,

$$\mathcal{E}_\lambda(n) = 2\sqrt{n+1} \iff \lambda = (1, 1, \dots, 1).$$ □

5. Gracefulness of default labeling of partition-trees

Let n be a positive integer, λ be a partition of n and T_λ be the partition-tree corresponding to λ. Since T_λ is a labeled tree, a natural question arises that - the default labeling of T_λ is graceful or not? We attempt to answer this question.

Proposition 5.1. *Let n be a positive integer and $\lambda = (1, 1, \dots, 1)$. Then default labeling of T_λ is not graceful.*

Proof. Since T_λ is a star (by Lemma 3.7) for $\lambda = (1, 1, \ldots, 1)$ where 1 is labeled at the center, the two edges from the center adjacent to the vertices labeled 0 and 2 will have the same labeling 1. So T_λ is not graceful. □

Proposition 5.2. *Let n be a positive integer and λ be a partition of n. If 1 and 2 are parts of λ, then default labeling of T_λ is not graceful and converse is not true.*

Proof. First part of the proposition follows by Theorem 3.8. A counter example to second part is the partition-4-tree of $(0, 0, 0, 4)$. □

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