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New Relations for the Normal Subgroups of Hecke Groups

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Abstract

For an integer $q \geq 3$, Hecke groups $H(\lambda_q)$ are an important class of discrete groups with the most important member being the famous modular group obtained in the case of $q = 3$. They were defined by E. Hecke in 1936 when he was studying with Dirichlet series. There are a number of research papers on the properties of Hecke groups, their normal subgroups and the relation with regular maps. Here we add some recent results to state new relations between the parameters of the normal subgroups of Hecke groups and the corresponding regular maps which are also graphs in combinatorial sense by means of a new graph invariant called omega which was recently defined in 2018.

1 Introduction

[1 2 3] The general Hecke groups $H(\lambda)$ is the group generated by two linear fractional transformations

$$R(z) = \frac{-1}{z} \quad \text{and} \quad T(z) = z + \lambda$$

where λ is a fixed positive real number. R and T have matrix representations $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$. Let $S = R \cdot T$. These groups are useful in the study of Dirichlet series and it was shown that this is possible only when the group is properly discontinuous, in other words, Fuchsian. For the groups $H(\lambda)$, Hecke showed that $H(\lambda)$ is Fuchsian when $\lambda \geq 2$ and real or

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when $\lambda = \lambda_q = 2cos(\pi/q)$ where q is an integer ≥ 3 when $\lambda < 2$ in [9]. Here we deal with the latter case where $\lambda = \lambda_q = 2cos(\pi/q)$ and denote the corresponding Hecke group by $H(\lambda_q)$.

The most famous Hecke group is the modular group obtained for $q = 3$. It is usually denoted by Γ. For the modular group, $\lambda_3 = 1$ and therefore the underlying field is the rationals. That is, all coefficients of the elements of the modular group are rational integers. Apart from the modular group, the most studied ones are obtained for $q = 4, 5$ and 6 as the underlying fields are the quadratic extensions of the field of rationals.

Several works related to the study of Hecke groups and their normal subgroups date back to 1990s, see e.g. [1, 2, 3, 4, 8, 9, 10, 11].

Recall that permutation method given by Jones and Singerman in [10] can be applied to Hecke groups to obtain the normal subgroups as follows: for every $m \in \mathbb{N}$ so that $m \mid q$, we have an epimorphism

$$\theta : H(\lambda_q) \longrightarrow (2, m, n)$$

where R goes to $2-$cycles, S goes to $m-$cycles and $T = R \cdot S$ goes to $n-$cycles. If $|G| = \mu$, then the number of cycles that $\overline{R}$ has is $\frac{\mu}{2}$; the number of cycles that $\overline{S}$ has is $\frac{\mu}{m}$ and the number of cycles that $\overline{T}$ has is $\frac{\mu}{n}$. Each cycle in $\overline{R}$ in the quotient group $H(\lambda_q)/N$ corresponds to an edge in the regular map corresponding to the normal subgroup N. That is $e = \frac{\mu}{2}$. Each cycle in $\overline{S}$ similarly corresponds to a face of the regular map corresponds to N and finally, each cycle in $\overline{T}$ corresponds to a vertex in the regular map. Hence $f = \frac{\mu}{m}$ and $v = \frac{\mu}{n}$. Here

$$N \cong \left(g; \frac{q}{m}^{(\frac{\mu}{m})}, \infty^{(\frac{\mu}{n})}\right) = \left(g; \frac{q}{m}^{(f)}, \infty^{(v)}\right)$$

where g is the genus of the underlying surface of the regular map corresponding to N. g can be calculated by means of the Riemann-Hurwitz formula.

Theorem 1.1. *If* $N \cong \left(g; d^{(f)}, \infty^{(v)}\right)$ *is a normal subgroup of* $H(\lambda_q)$*, then*

$$N \cong F_{1+e-f} * \prod_{i=1}^{f} C_d.$$

Proof. It was proven in [3] by Cangul that the rank of the free part of the normal subgroup N is $r(N) = 2g + v - 1$. Also the Euler formula for a graph on an orientable surface gives that

$$v - e + f = 2 - 2g.$$

Hence

$$\begin{aligned} r(N) &= 2g + v - 1 \\ &= 2 - v + e - f + v - 1 \\ &= e - f + 1. \end{aligned}$$

Finally for each $d = \frac{q}{m}$, we obtain a cycle C_d of length d and their number in the free product representation is f. Then the result follows. □

The following four results are easy consequences of Theorem 1.1:

Corollary 1.1. *If $N \cong \left(g; d^{(f)}, \infty^{(v)}\right)$ is a normal subgroup of $H(\lambda_q)$, then*

$$N \cong F_{1+\frac{2-m}{2m}\mu} * \prod_{i=1}^{f} C_d,$$

where $m = q/d$.

Proof. If follows from the fact that $e = \frac{\mu}{2}$ and that $f = \frac{\mu}{m}$. □

Corollary 1.2. *If $N \cong \left(0; d^{(f)}, \infty^{(v)}\right)$ is a normal subgroup of $H(\lambda_q)$ on the sphere, then N has rank $r(N) = v - 1$.*

Corollary 1.3. *If $N \cong \left(1; d^{(f)}, \infty^{(v)}\right)$ is a normal subgroup of $H(\lambda_q)$ on a torus, then N has rank $r(N) = v + 1$.*

Corollary 1.4. *If $N \cong \left(2; d^{(f)}, \infty^{(v)}\right)$ is a normal subgroup of $H(\lambda_q)$ on a double torus, then the rank of N is $r(N) = v + 3$.*

In [5], a new topological graph invariant denoted by Ω was defined and studied. See [5, 6, 7, 12] for some fundamental properties of Ω invariant. In [5], it was shown that $\Omega(G) = 2(e - v)$ for any graph G. So as every regular map is a graph embedded on a surface, we can use the same notation for regular maps. Now we have the following result:

Theorem 1.2. *For a regular map G, the number f of faces of G is*

$$f = \frac{\Omega(G)}{2} + \chi(G)$$

where $\chi(G)$ denotes the Euler characteristic of G.

Proof. By [5], we have $\Omega(G) = 2(e - v)$. As $v - e + f = 2 - 2g$, we obtain

$$\Omega(G) = 2(f - 2 + 2g).$$

Hence $f = \frac{\Omega(G)}{2} + 2 - 2g$ implying the result. □

The following is an immediate result of Theorem 1.2:

Corollary 1.5. *The rank of a normal subgroup N of $H(\lambda_q)$ is*

$$r(N) = 1 + v - \chi(G)$$

where v is the number of vertices of G.

Proof. By Theorem 1.1, we have $r = 1 + e - f$. By Theorem 1.2, we get

$$\begin{aligned} r(N) &= 1 + e - (\tfrac{\Omega(G)}{2} + \chi(G)) \\ &= 1 + e - (e - v + \chi(G)) \\ &= 1 + v - \chi(G). \end{aligned}$$

□

2 Conclusion

Using a recently introduced graph invariant called omega invariant, we have given several new properties of the normal subgroups of Hecke groups. As each normal subgroup of a Hecke group corresponds in a one to one way to a regular map which is an embedding of a connected graph into a compact orientable surface, we are able to use this new graph invariant defined for graphs in the study of normal subgroups, so connecting two interesting branches of algebra.

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