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NEW SEQUENCE SPACES VIA ORLICZ FUNCTIONS

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ABSTRACT. In this article we define new sequence spaces and introduce the notion of λ-almost statistical convergence of weight $\hat{g} : [0,\infty) \to [0,\infty)$ where $\hat{g}(y_r) \to \infty$ for any sequence (y_r) in $[0,\infty)$ with $y_r \to \infty$. Additionally, we primarily investigate relationship between the spaces $\hat{S}^{\hat{g}}_{\widetilde{\lambda}}$ and $\left[\hat{V}^{\hat{g}}_{\tau}, \widetilde{\lambda}, \Lambda\right]$.

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1. INTRODUCTION

Fast [8] generated new insight of the idea of convergence by introducing the concept of statistical convergence, and also this concept had been independently expended by Buck [4] and Schoenberg [22] for real and complex sequences. Later this notion was considered by Šalát [20], Fridy [9], Connor [7] and some others. Afterward, the more general idea of $\widetilde{\lambda}$-statistical convergence was introduced by Mursaleen in [15]. Subsequently a lot of interesting studies have been done by many authors on several related notions of this convergence (see for example [5, 6, 14, 16, 21]).

On the other hand, in [3] and then in [5] a different direction was given to the study of these important summability methods where the notions of statistical convergence of order α and $\widetilde{\lambda}$-statistical convergence of order αwere introduced and studied. Very recently the notion of $\widetilde{\lambda}$-almost statistical convergence of order $\alpha$was studied in [6].

We first recall basic definitions and notations which will form the basis of our main results.

Throughout this paper, in line of [6], ω will stand for the class of all sequences of real numbers and ℓ_∞, c, c_0 will denote the Banach spaces of bounded, convergent and null sequences $y = (y_r)$ with the usual norm $||y|| = \sup_r |y_r|$.

theorem 1.1. [2] *A linear functional $\widehat{\mathrm{L}}$ on ℓ^∞ is said to be a Banach Limit if it satisfies the characteristics:*

i): $\widehat{\mathrm{L}}(y) \geq 0$ *if* $y \geq 0$ *(i.e* $y_r \geq 0$ *for all* r*),*
ii): $\widehat{\mathrm{L}}(\overline{f}) = 1$, *where* $\overline{f} = (1, 1, 1, ...)$,
iii): $\widehat{\mathrm{L}}(Dy) = \widehat{\mathrm{L}}(y)$, *where D is the shift operator defined by* $(Dy_r) = (y_{r+1})$.

Let B be the class of all Banach limits on ℓ_∞.

theorem 1.2. *([2], see also [11]) A bounded sequence y is said to be almost convergent to be a number ϖ if $\widehat{\mathrm{L}}(y) = \varpi$ for all $\widehat{\mathrm{L}} \in \mathrm{B}$. Equivalently y is almost convergent to be ϖ if*

$$\lim_{t\to\infty} \frac{1}{t} \sum_{r=1}^{t} y_{r+z} = \varpi$$

uniformly in z. We will denote the set of all almost convergent sequences by $\hat{c}$.

theorem 1.3. [13] *A sequence $y = (y_r) \in \ell^{\infty}$ is said to be strongly almost convergent to be a number ϖ if*

$$\lim_{t\to\infty} \frac{1}{t} \sum_{r=1}^{t} |y_{r+z} - \varpi| = 0$$

uniformly in z. We denote the space of all strongly almost convergent sequences by $[\hat{c}]$.

Also, it has been observed that $c \subset [\hat{c}] \subset \hat{c} \subset \ell_{\infty}$ and the inclusions are strict (see [6]).

theorem 1.4. [15] *Let $\widetilde{\lambda} = (\widetilde{\lambda}_t)$ be a non-decreasing sequence of positive numbers such that $\widetilde{\lambda}_{t+1} \leq \widetilde{\lambda}_t + 1$, $\widetilde{\lambda}_1 = 1$, $\widetilde{\lambda}_t \to \infty$ as $t \to \infty$. Let $J_t = \left[t - \widetilde{\lambda}_t + 1, t\right]$. The generalized de La Vallée-Poussin mean is defined by*

$$p_t(y) = \frac{1}{\widetilde{\lambda}_t} \sum_{r \in J_t} y_r.$$

A sequence $y = (y_r)$ is said to be $\left(V, \widetilde{\lambda}\right)$ − summable to a number ϖ provided that $p_t(y) \to \varpi$ as $t \to \infty$.

Recently, the notion of $\widetilde{\lambda}-$statistical convergent was introduced by Mursaleen [15] looks like this; a sequence $y = (y_r)$ is said to be $\widetilde{\lambda}-$ statistical convergent if there is a complex number ϖ in a manner that for every $\varepsilon > 0$

$$\lim_{t\to\infty} \frac{1}{\widetilde{\lambda}_t} |\{r \in J_t : |y_r - \varpi| \geq \varepsilon| = 0.$$

The class of all $\widetilde{\lambda}$-statistical convergent sequences is denoted by $S_{\widetilde{\lambda}}$. Additionally, the concept of almost $\widetilde{\lambda}-$statistical convergent was studied by Savaş [21].

The primary aim of this article is to introduce the idea of $\widetilde{\lambda}$-almost statistical convergent of weight $\widehat{g} : [0, \infty) \to [0, \infty)$where $\widehat{g}(y_r) \to \infty$ for any sequence (y_r) in $[0, \infty)$ with $y_r \to \infty$ (see [1]).

Further we study some new sequence spaces using Orlicz function and primarily examine its relation with certain other summability methods and some of its results. During this article we will consider functions $\widehat{g} : [0, \infty) \to [0, \infty)$ such that $\widehat{g}(y_r) \to \infty$ if $y_r \to \infty$. The class of all such functions will be denoted by $\mathbf{G}$.

2. Main Results

We first introduce our main definition.

theorem 2.1. *Let the sequence $\widetilde{\lambda} = (\widetilde{\lambda}_t)$ of real numbers be defined as above and let $\widehat{g} \in \mathbf{G}$. A sequence $y = (y_r) \in \omega$ is said to be $\widetilde{\lambda}-$ almost statistically convergent of weight $\widehat{g}$ if there is a complex number ϖ such that for every $\varepsilon > 0$*

$$\lim_{t\to\infty} \frac{1}{\widehat{g}(\widetilde{\lambda}_t)} \left|\{r \in J_t : |y_{r+z} - \varpi| \geq \varepsilon\right| = 0$$

uniformly in z where $J_t = [t-\widetilde{\lambda}_t+1, t]$. Whenever this limit appears, we write $\hat{S}^{\widehat{g}}_{\widetilde{\lambda}} - \lim y_r = \varpi$ and denote the set of all $\widetilde{\lambda}-$ almost statistically convergent sequences of weight $\widehat{g}$ by $\hat{S}^{\widehat{g}}_{\widetilde{\lambda}}$.

theorem 2.1. *When we take $\widehat{g}(t) = t^{\theta}$, $0 < \theta \leq 1$ we are granted the notion of $\widetilde{\lambda}-$ almost statistical convergence of order θ [6]. If $\widetilde{\lambda}_t = t$ and $\widehat{g}(t) = t^{\theta}$ then we obtain the notion of almost convergence of order θ. For $\widehat{g}(t) = t$ we also get the notion of almost $\widetilde{\lambda}-$ statistical convergence in* [21].

Recall in [10] that an Orlicz function $\Lambda : [0,\infty) \to [0,\infty)$ is continuous, convex, non-decreasing function such that $\Lambda(0) = 0$ and $\Delta(y) > 0$ for $y > 0$, and $\Lambda(y) \to \infty$ as $y \to \infty$. Subsequently Orlicz function was used to define sequence spaces by Parashar and Choudhary [17] and others ([19], [23], [24].

If convexity of Orlicz function Λ is replaced by $\Lambda(y+z) \leq \Lambda(y)+\Lambda(z)$, then this function is called Modulus function, which was presented and discussed by Ruckle [18] and Maddox [12].

We now define the following.

theorem 2.2. *Let Λ be an Orlicz function, $\tau = (\tau_r)$ be a sequence of strictly positive real numbers and let $\widehat{g} \in \mathbf{G}$. Uniformly in z, for some ϖ and $\rho > 0$, let us define*

$$\left[\hat{V}^{\widehat{g}}_{\tau}, \widetilde{\lambda}, \Lambda\right] = \{y = (y_r) : \lim_{t\to\infty} \frac{1}{\widehat{g}(\widetilde{\lambda}_t)} \sum_{r\in J_t} \left[\frac{\Lambda\left(|y_{r+z} - \varpi|\right)}{\rho}\right]^{\tau_r} = 0\}$$

If $y \in \left[\hat{V}^{\widehat{g}}_{\tau}, \widetilde{\lambda}, \Lambda\right]$ then we say that y is almost strongly $\widetilde{\lambda}-$ summable of weighted $\widehat{g}$ with respect to the Orlicz function Λ.

If we take various assignment of Λ, $\widetilde{\lambda}$, $\widehat{g}$ and τ in the above sequence space we can get the following:

(1) If $\Lambda(y) = y$, $\widetilde{\lambda}_t = t$, and $\tau_r = 1$ for all r then $\left[\hat{V}^{\widehat{g}}_{\tau}, \widetilde{\lambda}, \Lambda\right] = \left[\hat{V}^{\widehat{g}}\right]$.

(2) If $\tau_r = 1$ for all r, then $\left[\hat{V}^{\widehat{g}}_{\tau}, \widetilde{\lambda}, \Lambda\right] = \left[\hat{V}^{\widehat{g}}, \widetilde{\lambda}, \Lambda\right]$.

(3) If $\tau_r = 1$ for all r and $\widetilde{\lambda}_t = t$, then $\left[\hat{V}^{\widehat{g}}_{\tau}, \widetilde{\lambda}, \Lambda\right] = \left[\hat{V}^{\widehat{g}}, \Lambda\right]$.

(4) If $\widetilde{\lambda}_t = t$ then $\left[\hat{V}^{\widehat{g}}_{\tau}, \widetilde{\lambda}, \Lambda\right] = \left[\hat{V}^{\widehat{g}}_{\tau}, \Lambda\right]$.

(5) If $\widehat{g}(t) = t^{\theta}$, $0 < \theta \leq 1$, then $\left[\hat{V}^{\widehat{g}}_{\tau}, \widetilde{\lambda}, \Lambda\right] = \left[\hat{V}^{\theta}_{\tau}, \widetilde{\lambda}, \Lambda\right]$.

We now present the following theorem which can be proved easily.

Theorem 2.1. *If $\tau_r > 0$ and y is almost strongly $\widetilde{\lambda}-$ convergent of weight $\widehat{g}$ to ϖ_1, with respect to the Orlicz function Λ, that is $y_r \to \varpi_1(\left[\hat{V}_\tau^{\widehat{g}}, \widetilde{\lambda}, \right])$, then ϖ_1 is unique.*

For the following results we shall assume that the sequence $\tau = (\tau_r)$ is bounded and $0 < h = \inf_r \tau_r \leq \tau_r \leq \sup_r \tau_r = H < \infty$.

Theorem 2.2. *Let $\widehat{g}_1, \widehat{g}_2 \in \mathbf{G}$ be such that there exist $P > 0$ and a $u \in \mathbb{N}$ such that $\widehat{g}_1(\lambda_{tn})\widehat{g}_2(\lambda_t) \leq P$ for all $t \geq u$ and let Λ be an Orlicz function. Then $\left[\hat{V}_\tau^{\widehat{g}_1}, \widetilde{\lambda}, \Lambda\right] \subset \hat{S}_{\widetilde{\lambda}}^{\widehat{g}_2}$.*

Proof. Let $y = (y_r) \in \left[\hat{V}_\tau^{\widehat{g}_1}, \widetilde{\lambda}, \Lambda\right]$ and let $\varepsilon > 0$ be given. Now see that

$$
\begin{aligned}
& \frac{1}{\widehat{g}_1(\widetilde{\lambda}_t)} \sum_{r \in J_t} \left[\frac{\Lambda\left(|y_{r+z} - \varpi|\right)}{\rho}\right]^{\tau_r} \\
= \quad & \frac{1}{\widehat{g}_1(\widetilde{\lambda}_t)} \left[\sum_{\substack{r \in J_t \\ |y_{r+z}-\varpi| \geq \varepsilon}} \left[\frac{\Lambda\left(|y_{r+z} - \varpi|\right)}{\rho}\right]^{\tau_r} + \sum_{\substack{r \in J_t \\ |y_{r+z}-\varpi| < \varepsilon}} \left[\frac{\Lambda\left(|y_{r+z} - \varpi|\right)}{\rho}\right]^{\tau_r} \right] \\
\geq \quad & \frac{1}{P \cdot \widehat{g}_2(\widetilde{\lambda}_t)} \left[\sum_{\substack{r \in J_t \\ |y_{r+z}-\varpi| \geq \varepsilon}} \left[\frac{\Lambda\left(|y_{r+z} - \varpi|\right)}{\rho}\right]^{\tau_r} + \sum_{\substack{r \in J_t \\ |y_{r+z}-\varpi| < \varepsilon}} \left[\frac{\Lambda\left(|y_{r+z} - \varpi|\right)}{\rho}\right]^{\tau_r} \right] \\
\geq \quad & \frac{1}{P \cdot \widehat{g}_2(\widetilde{\lambda}_t)} \sum_{\substack{r \in J_t \\ |y_{r+z}-\varpi| \geq \varepsilon}} [\Lambda(\varepsilon)]^{\tau_r} \\
\geq \quad & \frac{1}{P \cdot \widehat{g}_2(\widetilde{\lambda}_t)} \sum_{\substack{r \in J_t \\ |y_{r+z}-\varpi| \geq \varepsilon}} \min\left(\left[\Lambda\left(\epsilon_1\right)\right]^h, \left[\Lambda\left(\epsilon_1\right)\right]^H\right), \ \epsilon_1 = \frac{\epsilon}{\rho} \\
\geq \quad & \frac{1}{P \cdot \widehat{g}_2(\widetilde{\lambda}_t)} \left|\{r \in J_t : |y_{r+z} - \varpi| \geq \varepsilon\right| . \min\left(\left[\Lambda\left(\epsilon_1\right)\right]^h, \left[\Lambda\left(\epsilon_1\right)\right]^H\right)
\end{aligned}
$$

for all $t \geq u$.

Observe that the left hand side of this inequality tends to zero since $y \in \left[\hat{V}_\tau^{\widehat{g}_1}, \widetilde{\lambda}, \Lambda\right]$. As by our assumption $\min\left(\left[\Lambda\left(\epsilon_1\right)\right]^h, \left[\Lambda\left(\epsilon_1\right)\right]^H\right) P > 0$, so the right hand side also tends to zero uniformly in z as $t \to \infty$. Therefore, $y \in \hat{S}_{\widetilde{\lambda}}^{\widehat{g}_2}$ also $\left[\hat{V}_\tau^{\widehat{g}_1}, \widetilde{\lambda}, \Lambda\right] \subset \hat{S}_{\widetilde{\lambda}}^{\widehat{g}_2}$. □

Corollary 2.3. *Let $\widehat{g} \in \mathbf{G}$ and Λ be an Orlicz function, then $\left[\hat{V}_\tau^{\widehat{g}}, \widetilde{\lambda}, \Lambda\right] \subset \hat{S}_{\widetilde{\lambda}}^{\widehat{g}}$.*

Theorem 2.4. *Let $\widehat{g}_1, \widehat{g}_2 \in \mathbf{G}$ be such that there exist $P > 0$ and a $u \in \mathbb{N}$ such that $\widehat{g}_1(\widetilde{\lambda}_t)/\widehat{g}_2(\widetilde{\lambda}_t) \leq P$ for all $t \geq u$, let Λ be an Orlicz function and*

$y = (y_r)$ be a bounded sequence, Then $\hat{S}_{\widetilde{\lambda}}^{\widehat{g}_1} \subset \left[\hat{V}_{\tau}^{\widehat{g}_2}, \widetilde{\lambda}, \Lambda\right]$ provided $\widehat{g}_1 \in \mathbf{G}$ satisfies that $\limsup_t \dfrac{\widetilde{\lambda}_t}{\widehat{g}_1(\widetilde{\lambda}_t)} < \infty$.

Proof. Suppose that $y \in \ell_\infty$ and $\hat{S}_{\widetilde{\lambda}}^{\widehat{g}} - \lim y_r = L$. Since $y \in \ell_\infty$, then there is a constant $T > 0$ such that $|y_{r+z} - \varpi| \leq T$. Given $\epsilon > 0$, for all z we observe that

$$\begin{aligned}
& \frac{1}{\widehat{g}_2(\widetilde{\lambda}_t)} \sum_{r \in J_t} \left[\frac{\Lambda\left(|y_{r+z} - \varpi|\right)}{\rho}\right]^{\tau_r} \\
\leq \ & \frac{P}{\widehat{g}_1(\widetilde{\lambda}_t)} \sum_{r \in J_t} \left[\frac{\Lambda\left(|y_{r+z} - \varpi|\right)}{\rho}\right]^{\tau_r} \\
= \ & \frac{P}{\widehat{g}_1(\widetilde{\lambda}_t)} \left(\sum_{\substack{r \in J_t \\ |y_{r+z} - \varpi| \geq \varepsilon}} \left[\frac{\Lambda\left(|y_{r+z} - \varpi|\right)}{\rho}\right]^{\tau_r} + \sum_{\substack{r \in J_t \\ |y_{r+z} - \varpi| < \varepsilon}} \left[\frac{\Lambda\left(|y_{r+z} - \varpi|\right)}{\rho}\right]^{\tau_r} \right) \\
\leq \ & \frac{P}{\widehat{g}_1(\widetilde{\lambda}_t)} \sum_{\substack{r \in J_t \\ |y_{r+z} - \varpi| \geq \varepsilon}} \max\left\{ \left[\Lambda\left(\frac{T}{\rho}\right)\right]^h, \left[\Lambda\left(\frac{T}{\rho}\right)\right]^H \right\} + \frac{P}{\widehat{g}_1(\widetilde{\lambda}_t)} \sum_{\substack{r \in J_t \\ |y_{r+z} - \varpi| < \varepsilon}} \left[\Lambda\left(\frac{\epsilon}{\rho}\right)\right]^{\tau_r} \\
\leq \ & \max\left\{ \left[\Lambda\left(K\right)\right]^h, \left[\Lambda\left(K\right)\right]^H \right\} \cdot P \cdot \frac{1}{\widehat{g}_1(\widetilde{\lambda}_t} \left|\{r \in J_t : |y_{r+z} - \varpi| \geq \varepsilon\}\right| \\
+ \ & P \cdot \limsup_t \frac{\widetilde{\lambda}_t}{\widehat{g}_1(\widetilde{\lambda}_t)} \cdot \max\left\{ \left[\Lambda\left(\epsilon_1\right)\right]^h, \left[\Lambda\left(\epsilon_1\right)\right]^H \right\}, \ \frac{T}{\rho} = K, \ \frac{\epsilon}{\rho} = \epsilon_1.
\end{aligned}$$

for all $t \geq u$.

Since $\limsup_t \dfrac{\widetilde{\lambda}_t}{\widehat{g}_1(\widetilde{\lambda}_t)} < \infty$ so the second term an the right hand side can be made arbitrarily small as we take $\varepsilon > 0$ arbitrarily small. Again $y \in \hat{S}_{\widetilde{\lambda}}^{\widehat{g}_1}$ implies that the first term on the the right hand side tends to zero as $t \to \infty$ uniformly in z. Consequently, the left hand side of above inequality also tends to zero as $t \to \infty$ uniformly in z. Thus, $y \in \left[\hat{V}_{\tau}^{\widehat{g}_2}, \widetilde{\lambda}, \Lambda\right]$. □

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