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ON MODULE BUNDLES OVER ALGEBRA BUNDLES

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Abstract. We prove some results of module bundles over associative algebra bundles leading to the decomposition theorem for a semisimple module bundle. Using this theorem we show that all module bundles of an algebra bundle ξ are semisimple if and only if the cohomology modules of ξ of dimension greater than or equal to one are zero.

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1. Introduction

The notions of weak algebra bundle, algebra bundle, module bundle, module enlargements, representations and Hochschild cohomology are defined and studied extensively in [3, 7, 8].

The theory of module bundles over algebra bundle ξ is equivalent to that of representations of ξ. So the decomposition of a representation into direct sum of irreducible representations can be seen by decomposing a semisimple module bundle into direct sum of simple module bundles uniquely upto isomorphism.

Here we prove Nakayama lemma and Correspondence theorem for module bundles. Further we have exhibited the complete reducibility of a semisimple module bundle and uniqueness. The result mentioned in [3, Theorem 2.3] is interesting, so we prove this by proving necessary results.

All underlying vector spaces are real and finite dimensional and all algebras considered in the paper are finite dimensional associative algebras. All module bundles, algebra bundles, submodule bundles, subalgebra bundles and ideal bundles have same base space X which is compact Hausdorff. Throughout this paper $\xi = (\xi, p, X)$ denotes an associative algebra bundle. By a module bundle we mean a locally trivial module bundle unless otherwise specified.

2. Module Bundles and Representations

Here we recall a few definitions and further seen that the concept of module bundle and representation are equivalent.

A vector bundle $\xi = (\xi, p, X)$ is called a weak algebra bundle if there is a morphism $\theta : \xi \oplus \xi \to \xi$ which induces an algebra structure on each fiber $\xi_x, x \in X$ [3].

An algebra bundle is a vector bundle $\xi = (\xi, p, X)$ in which each fibre ξ_x is an algebra and for each x in X, there is an open neighbourhood U of x, an algebra A and a homeomorphism $\phi : U \times A \to p^{-1}(U)$ such that for each y in U, $\phi_y : A \to p^{-1}(y)$ is an algebra isomorphism [7].

A vector bundle $\mathscr{M} = (\mathscr{M}, q, X)$ is a right ξ-module bundle or simply a ξ-module bundle if there exists a morphism $\theta : \mathscr{M} \oplus \xi \to \mathscr{M}$ which induces a right ξ_x-module structure on $\mathscr{M}_x$ for each $x \in X$ [3].

A bimodule of an algebra A is a vector space M which is both right and left module of A and $(am)b = a(mb)$, for all $a, b \in A$, $m \in M$ [6].

A vector bundle $\mathscr{M} = (\mathscr{M}, q, X)$ is a ξ-bimodule bundle if there exist morphisms $\theta_1 : \mathscr{M} \oplus \xi \to \mathscr{M}$ and $\theta_2 : \xi \oplus \mathscr{M} \to \mathscr{M}$ which induce ξ_x-bimodule structure on $\mathscr{M}_x$ for each $x \in X$ [3].

A vector bundle morphism $\phi : \mathscr{M} \to \mathscr{N}$ of a ξ-bimodule bundle $\mathscr{M}$ into a ξ-bimodule bundle $\mathscr{N}$ is called a ξ-bimodule bundle morphism or simply ξ-morphism if for each $x \in X, \phi_x : \mathscr{M}_x \to \mathscr{N}_x$ is a ξ_x-bimodule homomorphism. If ϕ is a homeomorphism then it is called ξ-isomorphism. The notions of ξ-morphisms and isomorphisms on right ξ-module bundles are defined similarly [3].

A representation of an algebra bundle ξ on a vector bundle η is an algebra bundle morphism $\rho : \xi \to End(\eta)$, where $End(\eta) = \bigcup_{x \in X} End(\eta_x)$ and $End(\eta_x)$ is the algebra of all linear operators on η_x.

Let ξ be an algebra bundle. Then ξ itself a ξ-module bundle called a regular ξ-module bundle. Let A be an associative algebra and M be an A-bimodule. Let $\xi = X \times A$ and $\mathscr{M} = X \times M$. Then $\mathscr{M}$ is a ξ-bimodule bundle with the morphisms $\theta_1 : X \times (M \times A) \to X \times M$, $\theta_2 : X \times (A \times M) \to X \times M$ defined by $\theta_1(x, (m, a)) = (x, ma), \theta_2(x, (a, m)) = (x, am)$ $\forall$ $x \in X$, $a \in A$, $m \in M$. Such a module bundle is called a trivial bimodule bundle.

A ξ-bimodule bundle η is called locally trivial if for each $x \in X$, there is a neighborhood U of x such that $q^{-1}(U)$ is isomorphic to a trivial bimodule bundle. The notions of trivial ξ-module bundle and locally trivial ξ-module bundle are defined similarly [12].

Every locally trivial ξ-module bundle is a ξ-module bundle but converse is not true in general. Consider $I = [0, 1], \xi = I \times M_n(\mathbb{R})$ and $\mathscr{M} = I \times \mathbb{R}^n$. Then ξ is an algebra bundle and $\mathscr{M}$ is a vector bundle. Define $\theta : I \times (M_n(\mathbb{R}) \times \mathbb{R}^n) \to \mathbb{R}^n$ by $\theta(t, T, v) = (t, tTv)$, $t \in I, T \in M_n(\mathbb{R})$ and $v \in \mathbb{R}^n$.

The morphism θ induces an $M_n(\mathbb{R})$-module structure on $\mathbb{R}^n$. So $\mathscr{M}$ is a ξ-module bundle. But $\mathscr{M}$ is not locally trivial at 0.

Theorem 2.1. *Every ξ-module bundle induces a representation of ξ and conversely.*

Proof. Follows by standard methods in [12] as local triviality of underlying vector bundles implies the continuity. □

3. Bimodule bundle $Hom^R_\xi(\xi, \mathscr{M})$

Let $\mathscr{M}$ be a ξ-bimodule bundle and $Hom^R_\xi(\xi, \mathscr{M})$ be the set of all right ξ-morphisms of the ξ-module bundle ξ into the ξ-module bundle $\mathscr{M}$. Infact, $Hom^R_\xi(\xi, \mathscr{M}) = \bigcup_{x\in X} Hom^R_{\xi_x}(\xi_x, \mathscr{M}_x)$. Then $Hom^R_\xi(\xi, \mathscr{M})$ is a vector bundle with the induced topology from the vector bundle $Hom(\xi, \mathscr{M})$. Here we give ξ-bimodule bundle structure on $Hom^R_\xi(\xi, \mathscr{M})$ such that it is isomorphic to $\mathscr{M}$.

Let $End_\xi \mathscr{M} = \bigcup_{x\in X} End_{\xi_x} \mathscr{M}_x$, where $End_{\xi_x} \mathscr{M}_x$ is the set of all module endomorphisms of the ξ_x-module $\mathscr{M}_x$. Then $End_\xi \mathscr{M}$ is an algebra bundle over X.

Theorem 3.1. *Let $\mathscr{M}$ be a ξ-bimodule bundle. Then $Hom^R_\xi(\xi, \mathscr{M})$ and $\mathscr{M}$ are isomorphic as vector bundles.*

Proof. Let $p : \xi \to X, q : \mathscr{M} \to X$, $q' : Hom^R_\xi(\xi, \mathscr{M}) \to X$ be projections. Define $\phi : Hom^R_\xi(\xi, \mathscr{M}) \to \mathscr{M}$, $\phi_x(f_x) = f_x(1_x)$ where 1_x is the identity element of ξ_x. We show that ϕ is an isomorphism. Clearly $q \circ \phi = q'$ and each $\phi_x, x \in X$ is an isomorphism of the vector spaces $Hom^R_{\xi_x}(\xi_x, \mathscr{M}_x)$ and $\mathscr{M}_x$ [3, Theorem 1.7.1]. Now we show ϕ is continuous. Let

$h_1 : p^{-1}(U) \to U \times A, h_2 : q^{-1}(U) \to U \times M, h : q'^{-1}(U) \to U \times Hom^R_A(A, M)$

be local trivializations at x of the respective bundles, where A is an algebra and M is an A-module. Now we to show that $\phi^{-1}(W) \cap q'^{-1}(U)$ is open in $q'^{-1}(U)$ where W is an open set in $\mathscr{M}$. Define

$$g : U \times Hom^R_A(A, M) \to U \times M$$

by $(x, g) \mapsto (x, g(e))$, where e denote the identity element of A. Then g is continuous. So we have the following diagram:

$$\begin{array}{ccc} q'^{-1}(U) & \xrightarrow{\quad h \quad} & U \times Hom^R_A(A, M) \\ \Big\downarrow{\scriptstyle \phi|_{q'^{-1}(U)}} & & \Big\downarrow{\scriptstyle g} \\ q^{-1}(U) & \xrightarrow{\quad h_2 \quad} & U \times M \end{array}$$

Clearly this diagram is commutative, $g \circ h = h_2 \circ \phi\big|_{q'^{-1}(U)}$ implies $\phi\big|_{q'^{-1}(U)} = h_2^{-1} \circ g \circ h$ is continuous. Since W is open in $\mathscr{M}$, $W \cap q^{-1}(U)$

is open in $q^{-1}(U)$ and hence $\phi^{-1}(W \cap q^{-1}(U))$ is open in $q'^{-1}(U)$. Now $\phi^{-1}(W \cap q^{-1}(U) = \phi^{-1}(W) \cap (q \circ \phi)^{-1}(U) = \phi^{-1}(W) \cap q'^{-1}(U)$. Thus, ϕ is continuous. □

Corollary 3.2. *Let ξ be an algebra bundle. Then $End_{\xi}^{R}\ \xi$ and ξ are isomorphic algebra bundles.*

Proof. Consider $\phi : End_{\xi}^{R}\ \xi \to \xi,\ \phi_x(f_x) = f_x(1_x), x \in X$. Then for any $f_x, g_x \in End_{\xi_x}^{R} \xi_x$,

$$\begin{aligned} \phi_x(f_x \circ g_x) &= (f_x \circ g_x)(1_x) \\ &= f_x(g_x(1_x)) \\ &= f_x(1_x g_x(1_x)) \\ &= f_x(1_x) g_x(1_x) \quad (\because f_x \text{ is a right } \xi_x\text{-homomorphism}) \\ &= \phi_x(f_x)\phi_x(g_x). \end{aligned}$$

Hence ϕ is an isomorphism of algebra bundles. □

Now we shall make $Hom_{\xi}^{R}(\xi, \mathscr{M})$ into a ξ-bimodule bundle by defining,

$\theta_1(a, f)(b) = af(b)$ and $\theta_2(f, a)(b) = f(ab)$ for $a, b \in \xi, f \in Hom_{\xi}^{R}(\xi, \mathscr{M})$.

Clearly θ_1, θ_2 are morphisms which induce bimodule structures on fibers. The local triviality of $Hom_{\xi}^{R}(\xi, \mathscr{M})$ follows by the commutative diagram in the proof of the Theorem 3.1, where $Hom^{R}(A, M)$ has A-bimodule structure. Clearly each fiber map $\phi_x : Hom_{\xi_x}^{R}(\xi_x, \mathscr{M}_x) \to \mathscr{M}_x$ is ξ_x-bimodule isomorphism. Thus, we have the following result.

Corollary 3.3. *The ξ-bimodule bundle $Hom_{\xi}^{R}(\xi, \mathscr{M})$ is isomorphic to $\mathscr{M}$.*

4. Quotient module bundle and Radical

Let $\mathscr{N}$ be a submodule bundle of a ξ-module bundle $\mathscr{M} = (\mathscr{M}, q, X)$. Then $(\mathscr{M}/\mathscr{N}, \overline{q}, X)$ is a quotient ξ-module bundle. Let $\theta : \mathscr{M} \oplus \xi \to \mathscr{M}$ be a right ξ_x-module structure on $\mathscr{M}_x$ for each $x \in X$. We define $\overline{\theta} : (\mathscr{M}/\mathscr{N}) \oplus \xi \to \mathscr{M}/\mathscr{N}$ by $\overline{\theta}([m], r) = [\theta(m, r)]$, $m \in \mathscr{M}, r \in \xi$. Then $\overline{\theta}$ is continuous as the map $\pi_1 = (\pi \oplus Id) : \mathscr{M} \oplus \xi \to (\mathscr{M}/\mathscr{N}) \oplus \xi$ is continuous and open where $\pi : \mathscr{M} \to \mathscr{M}/\mathscr{N}$ is the quotient map.

Definition 4.1. *Let $\mathscr{M}$ be a ξ-module bundle. Consider $R(\mathscr{M}) = \bigcup_{x \in X} R(\mathscr{M}_x)$, where $R(\mathscr{M}_x)$ is the Jacobson radical of the fiber $\mathscr{M}_x$. Then $R(\mathscr{M})$ is a submodule bundle of $\mathscr{M}$. For, let $h : \bigcup_{x \in U} \mathscr{M}_x \to U \times M$ be a local trivialization of $\mathscr{M}$ at x. Then $h_x(R(\mathscr{M}_x)) = R(M)$* [6, Proposition 3.1.3] *and hence $h|_{\bigcup_{x \in U} R(\mathscr{M}_x)}$ induces an isomorphism between $\bigcup_{x \in U} R(\mathscr{M}_x)$ and $U \times R(M)$. We call $R(\mathscr{M})$ as the radical bundle of $\mathscr{M}$.*

Remark 4.2. *Let $\mathscr{M} = (\mathscr{M}, q, X)$, $\mathscr{N} = (\mathscr{N}, q', X)$ be ξ-module bundles and $f : \mathscr{M} \to \mathscr{N}$ be a ξ-morphism. Then by* [6, Proposition 3.1.3], *$f(R(\mathscr{M})) \subseteq R(\mathscr{N})$. Hence f induces a map, $\bar{f} : \mathscr{M}/R(\mathscr{M}) \to \mathscr{N}/R(\mathscr{N})$ given by $\bar{f}_x([m]) = [f_x(m)]$ for each $m \in \mathscr{M}_x$, $x \in X$. If $[m_1] = [m_2]$ then $m_1, m_2 \in \mathscr{M}_x$ for some $x \in X$ and $m_1 - m_2 \in R(\mathscr{M}_x)$. Then*

$f_x(m_1 - m_2) \in f_x(R(\mathscr{M}_x)) \subseteq R(\mathscr{N}_x)$. So $f_x(m_1) - f_x(m_2) \in R(\mathscr{N}_x)$. Hence $[f_x(m_1)] = [f_x(m_2)]$. Thus, $\bar{f}$ is well defined.

Theorem 4.3 (Nakayama Lemma)**.** *A ξ-morphism $f : \mathscr{M} \to \mathscr{N}$ is an epimorphism iff $\bar{f} : \mathscr{M}/R(\mathscr{M}) \to \mathscr{N}/R(\mathscr{N})$ is an epimorphism.*

Proof. Sufficient to prove $\bar{f}$ is continuous as f_x is onto homomorphism if and only if $\bar{f}_x$ is onto homomorphism for each $x \in X$. Let $\pi : \mathscr{M} \to \mathscr{M}/R(\mathscr{M})$ and $\pi' : \mathscr{N} \to \mathscr{N}/R(\mathscr{N})$ be quotient maps. Let W be any open set in $\mathscr{N}/R(\mathscr{N})$. Then $\bar{f}^{-1}(W) = \pi(f^{-1}(\pi'^{-1}(W)))$ is open in $\mathscr{M}/R(\mathscr{M})$, since π', f are continuous and π is open. □

Proposition 4.4. *Let $\mathscr{M}, \mathscr{N}$ be ξ-module bundles and $\phi : \mathscr{M} \to \mathscr{N}$ be a ξ-morphism. Suppose ϕ is a monomorphism and* Im ϕ_x *is a complemented submodule of $\mathscr{N}_x$ for each $x \in X$ then* Im ϕ *is a submodule bundle of $\mathscr{N}$.*

Proof. Problem being local, assume $\xi = X \times A$ and $\mathscr{N} = X \times N$, trivial bundles, where A is an associative algebra and N is an A-module. Let $x \in X$. As Im ϕ_x is a complemented submodule of N there is a submodule L_x of N such that $N = L_x \oplus$ Im ϕ_x. Then $X \times L_x$ is a submodule bundle of $\mathscr{N}$. Hence Im ϕ is a submodule bundle of $\mathscr{N}$ from [1, Lemma 1.3.1]. □

Lemma 4.5. *Let $\mathscr{M}$ be a ξ-module bundle and $\mathscr{N}$ be a submodule bundle of $\mathscr{M}$. Then the following conditions are equivalent.*

(1) $\mathscr{N}$ is a direct summand;
(2) there is a ξ-morphism $\mathfrak{p} : \mathscr{M} \to \mathscr{N}$ such that $\mathfrak{p}(n) = n$ for every $n \in \mathscr{N}$;
(3) there is a ξ-morphism $\phi : \mathscr{M}/\mathscr{N} \to \mathscr{M}$ such that $\phi(\bar{m}) \in \bar{m}$ for every class $\bar{m} \in \mathscr{M}/\mathscr{N}$.

Proof. (1) $\Longrightarrow$ (2): Let $\mathscr{M} = \mathscr{N} \oplus \mathscr{L}$ for some submodule bundle $\mathscr{L}$ of $\mathscr{M}$. Then the projection map $\mathfrak{p} : \mathscr{M} \to \mathscr{N}$ is a morphism such that $\mathfrak{p}(n) = n$ for each $n \in \mathscr{N}$.

(2) $\Longrightarrow$ (3): Assume that there is a ξ-morphism $\mathfrak{p} : \mathscr{M} \to \mathscr{N}$ such that $\mathfrak{p}(n) = n$ for every $n \in \mathscr{N}$. It is sufficient to show that the map $\phi : \mathscr{M}/\mathscr{N} \to \mathscr{M}$ defined by $\phi(\bar{m}) = m - \mathfrak{p}(m)$ for each $\bar{m} \in \mathscr{M}/\mathscr{N}$ is continuous. Define $g : \mathscr{M} \to \mathscr{M}$ by $g(m) = m - \mathfrak{p}(m)\ \forall\ m \in \mathscr{M}$. Then g being sum of two continuous functions on vector bundles, is continuous. Let $\pi : \mathscr{M} \to \mathscr{M}/\mathscr{N}$ be the quotient map. Clearly $\phi \circ \pi = g$. So $\phi \circ \pi$ is continuous. Therefore, $\pi^{-1}(\phi^{-1}(U)) = (\phi \circ \pi)^{-1}(U)$ is open in $\mathscr{M}$ for every open set U in $\mathscr{M}$. This implies that $\phi^{-1}(U)$ is open in $\mathscr{M}/\mathscr{N}$ for every open set U in $\mathscr{M}$, by the definition of quotient topology. Thus, ϕ is continuous.

(3) $\Longrightarrow$ (1): Let $\phi : \mathscr{M}/\mathscr{N} \to \mathscr{M}$ be a ξ-morphism such that $\phi(\bar{m}) \in \bar{m}$ for every class $\bar{m} \in \mathscr{M}/\mathscr{N}$. Let $x \in X$. Then $\phi_x : (\mathscr{M}/\mathscr{N})_x \to (\mathscr{M})_x$ is a ξ_x module homomorphism such that $\phi_x(\bar{m}) \in \bar{m}$ for every class $\bar{m} \in (\mathscr{M})_x/(\mathscr{N})_x$. If $\phi_x(\bar{m}) = 0$ then $0 \in \bar{m}$ so that $\bar{m} = 0$. So Ker $\phi_x = 0$ and hence ϕ is a monomorphism. For each $x \in X$ we have $\mathscr{N}_x \oplus$ Im $\phi_x = \mathscr{M}_x$. Hence Im ϕ_x is a complemented submodule of $\mathscr{M}_x$ for each $x \in X$. By the Proposition 4.4, $\mathscr{L} =$ Im ϕ is a submodule bundle of $\mathscr{M}$. Since $\mathscr{N} \oplus \mathscr{L} = \mathscr{M}$, $\mathscr{N}$ is a direct summand. □

5. Semisimple Module Bundles

Definition 5.1. *A ξ-module bundle $\mathscr{M}$ is said to be semisimple if each fiber $\mathscr{M}_x$ is a semisimple ξ_x-module. An algebra bundle ξ is called semisimple if the regular module bundle ξ is semisimple.*

Definition 5.2. *A ξ-module bundle $\mathscr{M}$ is said to be simple if $\mathscr{M} \neq 0$ and has no submodule bundle other than 0 and $\mathscr{M}$ itself.*

By using the methods in [5], we prove the following result.

Proposition 5.3. *Let $\mathscr{M}, \mathscr{N}$ be ξ-module bundles and $\phi : \mathscr{M} \to \mathscr{N}$ be a ξ-morphism. Suppose $\mathscr{M}$ is semisimple and ϕ is an epimorphism then Ker ϕ is a submodule bundle of $\mathscr{M}$.*

Proof. Since the problem is local, we may assume that $\xi = X \times A$, $\mathscr{M} = X \times M$ and $\mathscr{N} = X \times N$, A is an associative algebra and M, N are A-modules with M semisimple. Then $\phi : X \times M \to X \times N$ has the form $\phi(x, m) = (x, \phi_x(m))$, $x \mapsto \phi_x$ is a continuous map of X into $Hom_A(M, N)$ where $Hom_A(M, N)$ is the space of all A-module homomorphisms of M into N. Let $x_0 \in X$. Consider $\phi_{x_0} : M \to N$ and $M_2 = \text{Ker } \phi_{x_0}$. Then M_2 is a submodule of M. Since M is semisimple, there is a submodule M_1 of M such that $M = M_1 \oplus M_2$. Let $L = N \oplus M_2$. Then $\phi_{x_0}|_{M_1} : M_1 \to N$ is an isomorphism as ϕ is onto. For each $x \in X$, define A-module homomorphism $\psi_x : M = M_1 \oplus M_2 \to L = N \oplus M_2$ by the conditions, $\psi_x(m_1) = \phi_x(m_1), \psi_x(m_2) = \phi_x(m_2) + m_2$. Since $\phi_{x_0} : M_1 \to N$ is an isomorphism, $\psi_{x_0} : M \to L$ is also an isomorphism. Since the map $x \mapsto \psi_x$ is continuous from X into $Hom_A(M, L)$, there is an open set U in X such that ψ_x is an isomorphism for each $x \in U$. Now define $h : U \times M_2 \to \text{Ker } \phi|_U$ by $h(x, m_2) = (x, (\psi_x)^{-1}(m_2))$. For any $x \in U$, we have $(m_1, m_2) \in \text{Ker } \phi_x$ if and only if $\phi_x(m_1, m_2) = 0$ if and only if $\psi_x(m_1, m_2) = m_2$ or equivalently, $(m_1, m_2) \in (\psi_x)^{-1}(M_2)$. This shows that Ker $\phi_x = (\psi_x)^{-1}(M_2)$ for each $x \in U$. Hence the map h is an isomorphism whose inverse is given by $(x, m) \mapsto (x, \psi_x(m))$. This proves the local triviality of Ker ϕ. Hence Ker ϕ is a ξ-submodule bundle of $\mathscr{M}$. □

Proposition 5.4. *Let $\mathscr{M}, \mathscr{N}$ be ξ-module bundles and $\phi : \mathscr{M} \to \mathscr{N}$ be a ξ-morphism. Suppose ϕ is a monomorphism and $\mathscr{N}$ is semisimple then* Im ϕ *is a submodule bundle of $\mathscr{N}$.*

Proof. Follows by the Proposition 4.4, since $\mathscr{N}$ being semisimple, each Im ϕ_x is complemented in $\mathscr{N}_x$ for $x \in X$. □

Theorem 5.5 (First Isomorphism Theorem)**.** *Let $\mathscr{M}, \mathscr{N}$ be ξ-module bundles and $\phi : \mathscr{M} \to \mathscr{N}$ be a ξ-morphism. Suppose $\mathscr{M}$ is semisimple and ϕ is an epimorphism then the quotient module bundle $\mathscr{M}$/Ker ϕ is isomorphic to $\mathscr{N}$*

Proof. By the Proposition 5.3, Ker ϕ is a submodule bundle of $\mathscr{M}$. Let $\pi : \mathscr{M} \to \mathscr{M}/\text{Ker } \phi$ be the quotient map. Define $\bar{\phi} : \mathscr{M}/\text{Ker } \phi \to \mathscr{N}$ by $\bar{\phi}([m]) = \phi(m)$. Clearly $\bar{\phi} \circ \pi = \phi$. Then for any open set U in $\mathscr{N}$ we have $\pi^{-1}(\bar{\phi}^{-1}(U)) = \phi^{-1}(U)$, which is open in $\mathscr{M}$. Hence $\bar{\phi}$ is continuous since $(\bar{\phi})^{-1}(U)$ is open in the quotient topology of $\mathscr{M}/\text{Ker } \phi$. Hence the proposition. □

Theorem 5.6 (Correspondence Theorem)**.** *Let $\mathscr{M}$ be a semisimple ξ-module bundle $\mathscr{N}$ be a submodule bundle of $\mathscr{M}$. Then there is a one-one correspondence between the submodule bundles of $\mathscr{M}/\mathscr{N}$ and the submodule bundles of $\mathscr{M}$ containing $\mathscr{N}$.*

Proof. Let $\pi : \mathscr{M} \to \mathscr{M}/\mathscr{N}$ be the quotient map. Then π is an epimorphism of module bundles. Let $\mathscr{M}_1$ be a submodule bundle of $\mathscr{M}$ containing $\mathscr{N}$. Let $\overline{\mathscr{M}_1} = \pi(\mathscr{M}_1)$. Then $\overline{\mathscr{M}_1} = \{[m] : m \in \mathscr{M}_1\}$. This can be identified with the module bundle $\mathscr{M}_1/\mathscr{N}$. Hence $\overline{\mathscr{M}_1}$ is a submodule bundle of $\mathscr{M}/\mathscr{N}$ which is uniquely determined by $\mathscr{M}_1$. Conversely, let $\overline{\mathscr{M}_1}$ be a submodule bundle of $\mathscr{M}/\mathscr{N}$. Take $\mathscr{M}_1 = \pi^{-1}(\overline{\mathscr{M}_1})$. Let $\pi_1 : \mathscr{M}/\mathscr{N} \to (\mathscr{M}/\mathscr{N})/\overline{\mathscr{M}_1}$ be the corresponding quotient map. Then $\psi = \pi_1 \circ \pi : \mathscr{M} \to (\mathscr{M}/\mathscr{N})/\overline{\mathscr{M}_1}$ is an epimorphism of module bundles. It can be seen that Ker $\psi = \mathscr{M}_1$. Since $\mathscr{M}$ is semisimple, by the Proposition 5.3, it follows that $\mathscr{M}_1$ is submodule bundle of $\mathscr{M}$. Clearly $\mathscr{M}_1$ contains $\mathscr{N}$. □

Corollary 5.7. *A submodule bundle $\mathscr{N}$ of a semisimple module bundle $\mathscr{M}$ is maximal if and only if $\mathscr{M}/\mathscr{N}$ is simple.*

Proof. Follows from the above theorem. □

Proof of complete reducibility of a semisimple module bundle needs the following definitions and results.
Pseudo Algebraic Subroup[2]: Let H be a subgroup of $GL(n, \mathbb{C})$. Denote by $x_{ij}(\sigma), (1 \leq i, j \leq n)$, the coefficients of a matrix $\sigma \in GL(n, \mathbb{C})$. We shall say that H is an algebraic subgroup of $GL(n, \mathbb{C})$ if there exists a set of polynomials $P_\alpha(\cdots, x_{ij}, \cdots)$ in n^2 arguments such that the conditions $\sigma \in H$ and $P_\alpha(\cdots, x_{ij}(\sigma), \cdots) = 0 \ \forall\ \alpha$ are equivalent. Denote by $x'_{ij}(\sigma)$, $x''_{ij}(\sigma)$ the real and imaginary parts of $x_{ij}(\sigma)$. If there exists set of polynomials $P_\beta(\cdots, x'_{ij}, x''_{ij}, \cdots)$ in $2n^2$ arguments such that the conditions $\sigma \in H$ and $P_\beta(\cdots, x'_{ij}(\sigma), x''_{ij}(\sigma), \cdots) = 0, \ \forall \beta$ are equivalent then H is called as pseudo algebraic subgroup of $GL(n, \mathbb{C})$.

Proposition 5.8. *Let M be a module over an associative algebra A and $Aut(M)$ be the Lie group of all linear automorphisms of M. Let $\hat{G}$ be the collection of all A-module automorphisms of M. Then $\hat{G}$ forms a pseudo algebraic subgroup of Aut(M) and hence it is a Lie subgroup of Aut(M).*

Proof. Let $g = (a_{ij}) \in Aut(M)$. Let $\{m_1, m_2, \dots, m_n\}$ be a basis of M such that $g(m_i) = \sum_j a_{ij} m_j$. Let $\{a_1, a_2, \dots, a_m\}$ be a basis of A and A-module structure on M be given by

$$a_i m_j = \sum_k b_{ij}^{(k)} m_k, \ 1 \leq i \leq m, \ 1 \leq j \leq n.$$

Then

$$\begin{aligned} g \in \hat{G} &\iff g(a_i m_j) = a_i g(m_j) \\ &\iff g(\sum_k b_{ij}^{(k)} m_k) = a_i \sum_k a_{jk} v_k \\ &\iff \sum_l b_{ij}^{(k)} a_{kl} m_l = \sum_l a_{jk} b_{ik}^{(l)} m_l \\ &\iff \sum_l b_{ij}^{k} a_{kl} = \sum_l a_{jk} b_{ik}^{l}. \end{aligned}$$

This implies $g \in \hat{G}$ is equivalent to saying that a_{ij}'s satisfy a polynomial relation. Hence $\hat{G}$ is a pseudo algebraic subgroup of $Aut(M)$. By [2, p. 136], it follows that $\hat{G}$ is a Lie subgroup of $Aut(M)$. □

Remark 5.9. *Let M be a module over an associative algebra A. Let $G_r(M)$ be the Grassmann manifold of all r-dimensional subspaces of M and $\Gamma_r(M)$ be the set of all r-dimensional submodules of M, where $r < \dim M$. Then $G_r(M)$ is a compact Hausdorff, second countable space* [9]. *Further, $\Gamma_r(M)$ is closed in $G_r(M)$. For this, let*

$$V_r(M) = \{T : \mathbb{R}^r \to M : T \text{ is a linear transformation and } \operatorname{rank} T = r\}$$

be the Stiefel manifold of all r-frames in M. Then $V_r(M)$ has subspace topology as a subspace of $L(\mathbb{R}^r, M)$, the space of all linear transformations of $\mathbb{R}^r$ to M. The topology on $G_r(M)$ is the quotient topology induced by $q : V_r(M) \to G_r(M), q(T) = T(\mathbb{R}^r)$. We give A-module structure on $\mathbb{R}^r$ by identifying it as linear subspace of M. Let

$$W = \{T \in V_r(M) : T \text{ is an } A\text{-module homomorphism}\}.$$

Then W is closed in $V_r(M)$, as every convergent sequence in W converges in W. Since $q^{-1}(\Gamma_r(M)) = W$, it follows that $\Gamma_r(M)$ is closed in $G_r(M)$.

We recall the definition of rigidity. Let A be an associative algebra, M be an A-module, N be an r-dimensional submodule of M, $r < \dim M$. Let $\Gamma_r(M)$ denote the space of all r-dimensional submodules of M. Let $\hat{G}$ be the collection of all A-module automorphisms, which is a Lie subgroup of $\mathrm{Aut}(M)$. Then $\hat{G}$ acts on $\Gamma_r(M)$ by the action $g.L = g(L)$. We say that N is rigid if $\hat{G}.N$ is open in $\Gamma_r(M)$.

Richardson [11, Proposition 15.3] has given the rigidity of submodules over an algebraically closed field. Here we need the rigidity when the field is real.

Theorem 5.10 (Rigidity Theorem for Submodules)**.** *Any submodule of a semisimple module over a real associative algebra is rigid with respect to the Lie group of all module automorphisms defined on the given module.*

Proof. Follows from the methods in [11, 12]. □

Theorem 5.11 (Decomposition Theorem)**.** *Every semisimple $\xi = (\xi, p, X)$-module bundle $\mathscr{M} = (\mathscr{M}, q, X)$ is isomorphic to a direct sum of simple ξ-module bundles.*

Proof. It is sufficient to show that every proper submodule bundle $\mathscr{N}$ of $\mathscr{M}$ is a direct summand. By the Lemma 4.5, we need to construct a continuous ξ-morphism $f : \mathscr{M} \to \mathscr{N}$ such that f is identity on $\mathscr{N}$.

Let $x_0 \in X$ and let

$$\phi : U \times A \to \bigcup_{x \in U} \xi_x, \;\; \alpha : U \times M \to \bigcup_{x \in U} \mathscr{M}_x, \text{ and } \beta : U \times N \to \bigcup_{x \in U} \mathscr{N}_x$$

be local trivializations at x_0 of $\xi, \mathscr{M}$ and $\mathscr{N}$ respectively, where A is an algebra, M is semisimple A-module and N is a simple A-module. Now we show the existence of a submodule L of M and a module bundle isomorphism $\psi : U \times M \to \bigcup_{x \in U} \mathscr{M}_x$ such that $\psi|_{U \times L}$ maps $U \times L$ onto $\bigcup_{x \in U} \mathscr{N}_x$.

The map $\gamma = \alpha^{-1}\beta : U \times N \to U \times M$, is continuous being composition of continuous maps. Let $r = \dim N$. Then $U \to \Gamma_r(M)$, $x \mapsto \gamma_x(N)$ is continuous since $U \to V_r(M)$, $x \mapsto \gamma_x$ is continuous. Further, by the Remark 5.9, $\Gamma_r(M)$ is compact being a closed subspace of compact space $G_r(M)$.

By the Rigidity Theorem 5.10, the orbit

$$\hat{G}(x_0) = \{g(L) : g \in \hat{G}\}$$

of $L = \gamma_{x_0}(N)$ is open in compact Hausdorff space $\Gamma_r(M)$ and hence it is locally compact. Also, $\hat{G}(x_0)$ is second countable being a subspace of second countable space $G_r(M)$. Hence $\hat{G}(x_0)$ is a $\hat{G}$-space, which satisfy the hypothesis of [14, Lemma 2.9.1]. So $\hat{G}(x_0)$ is homeomorphic to $\hat{G}/\hat{G}_0$ by the map $g.L \mapsto g\hat{G}_0$, where $\hat{G}_0 = \{g \in \hat{G} : g.L = L\}$ is the stability subgroup corresponding to the element L of $\Gamma_r(M)$. Since $x \mapsto \gamma_x(N)$ is continuous map of U into $\Gamma_r(M)$ and $\hat{G}(x_0)$ is open in $\Gamma_r(M)$ it follows that $U' = \{x \in U : \gamma_x(N) \in \hat{G}(x_0)\}$ is open in X. Further, $x_0 \in U'$ and for each $x \in U'$, there exists $g_x \in \hat{G}$ such that $\gamma_x(N) = g_x(L)$. Since $\hat{G}$ is a Lie group and $\hat{G}_0$ is a closed subgroup $\hat{G}$, $\hat{G}_0$ has a local cross section given by $g.\hat{G}_0 \mapsto g$ [4, Theorem 6.5.2]. Hence $\hat{G} \to \hat{G}/\hat{G}_0$ is a locally trivial principal bundle. Hence by shrinking we get a neighborhood of x_0 on which the map $x \mapsto g_x$ is continuous being the composition of following continuous maps:

$$U' \to \Gamma_r(M) \to \hat{G}/\hat{G}_0 \to \hat{G}$$

$$x \mapsto \gamma_x(N) = g_x(L) \mapsto g_x\hat{G}_0 \mapsto g_x.$$

Without loss of generality we may assume that $x \mapsto g_x$ is continuous on U to $\hat{G}$. Then the map $\psi : U \times M \to \bigcup_{x \in U} \mathscr{M}_x$, $\psi(x, m) = \alpha(x, g_x(m))$ is a module bundle isomorphism, since α_x and g_x are module isomorphisms for each $x \in U$. Also $\psi|_{U \times L}$ maps $U \times L$ onto $\bigcup_{x \in U} \mathscr{N}_x$. For, let $x \in X, l \in L$. As $g_x(N) = \gamma_x(L)$, $g_x(l) = \gamma_x(n)$ for some $n \in N$ and

$$\psi_x(l) = \alpha_x(g_x(l)) = \alpha_x(\gamma_x(n)) = \alpha_x(\alpha_x^{-1}\beta_x(n)) = \beta_x(n) \in \mathscr{N}_x.$$

Now, as M is semisimple, there is an A-submodule L' of M such that $M = L \oplus L'$ [6, Proposition 2.2.1]. Let $\hat{f} : U \times M \to U \times L$ be the map given by the projection onto first factor. Clearly $\hat{f}$ is a module bundle homomorphism which is identity on $U \times L$. Let $f : \bigcup_{x \in U} \mathscr{M}_x \to \bigcup_{x \in U} \mathscr{N}_x$ be given by $f = \psi \hat{f} \psi^{-1}$. Then f is a $p^{-1}(U)$-morphism which is identity on $\mathscr{N}_x$ for each $x \in X$. Since X is compact Hausdorff, we get a finite open covering $\{U_n\}$ of X together with the partition of unity $p_n : X \to \mathbb{R}$ and $p^{-1}(U_n)$-module bundle homomorphisms $f_n : \bigcup_{x \in U_n} \mathscr{M}_x \to \bigcup_{x \in U_n} \mathscr{N}_x$ such that f_n is identity on N_x for each $x \in U_n$. Now let us define $g_n : \mathscr{M} \to \mathscr{N}$ by,

$$g_n(m) = \begin{cases} p_n(x) f_n(m), & \text{if } m \in \mathscr{M}_x, x \in U_n; \\ 0, & \text{otherwise.} \end{cases}$$

Let $g = \sum_n g_n$. Then $g : \mathscr{M} \to \mathscr{N}$ is a ξ- morphism which is identity on $\mathscr{N}$. Thus, $\mathscr{M} \cong \mathscr{N} \oplus \mathscr{N}'$ for some submodule bundle $\mathscr{N}'$ of $\mathscr{M}$. By continuing this process we get a decomposition of $\mathscr{M}$ into direct sum of simple ξ-module bundles. □

Lemma 5.12. *Let $(X \times V, p, X)$ be a trivial vector bundle and $(X \times M, q, X)$ be a trivial semisimple ξ-module bundle where $\xi = (X \times A, p_A, X)$ is trivial algebra bundle, M is an A-module. Suppose $\phi : X \times V \to X \times M$ is a vector bundle monomorphism such that $\phi_x(V)$ is a submodule of M for each $x \in X$. Then there is a finite open partition $\cup U_i = X$ such that $\phi_x(V) = \phi_y(V)\ \forall\ x, y \in U_i,\ \forall i$.*

Proof. The morphism $\phi : X \times V \to X \times M$ induces a continuous map

$$\Phi : X \to Hom(V, M),$$

where $Hom(V, M)$ is the space of all linear maps of V into M. A module M being semisimple, it has finite number of (non isomorphic) submodules [6]. Let $\{M_i\}_{i=1}^n$ be the set of all submodules of M whose dimension equals that of V and let $U_i = \{x \in X : \phi_x(V) = M_i\}$ for each i. Then $X = \bigcup_i U_i$. Now the proof follows by applying the methods of [10, Lemma 2.4]. □

Lemma 5.13. *Let $\mathscr{M}$ be a semisimple ξ-module bundle and $\mathscr{M}_1, \mathscr{M}_2$ be submodule bundles of $\mathscr{M}$. Then $\mathscr{M}_1 \cap \mathscr{M}_2$ is a submodule bundle of $\mathscr{M}$.*

Proof. Follows by Lemma 5.12 and by the methods of [8, Lemma 2.5]. □

Remark 5.14. *The Lemma 5.13 need not be true in case of non semisimple module bundles. A counter example is given below.*

Example 5.15. *Let $A = \{(a, b, b) : a, b \in \mathbb{R}\}$. Clearly A is an algebra with respect to component wise multiplication and $M = \mathbb{R}^3$ is an A-module. Let $I = [0, 1]$ and $\xi = I \times A$ and $\mathscr{M} = I \times M$. Then $\mathscr{M}$ is a ξ-module bundle. Let $\mathscr{N}_1 = I \times \{(x, y, 0) : x, y \in \mathbb{R}\}$ and $\mathscr{N}_2 = \{(t, x, y, ty) : t \in [0, 1], x, y \in \mathbb{R}\}$. Note that $\mathbb{R}^2$ is also an A-module with respect to natural multiplication and $\mathscr{N}_2$ is isomorphic to $I \times \mathbb{R}^2$. So $\mathscr{N}_1, \mathscr{N}_2$ are submodule bundles of $\mathscr{M}$. Note that the fiber of $\mathscr{N}_1 \cap \mathscr{N}_2$ has dimension 2 at $t = 0$ and 1 elsewhere. So $\mathscr{N}_1 \cap \mathscr{N}_2$ is not a subbundle of $\mathscr{M}$.*

Theorem 5.16 (Uniqueness of decomposition)**.** *Let*

$$\mathscr{M} = \mathscr{M}_1 \oplus \mathscr{M}_2 \oplus \cdots \oplus \mathscr{M}_n = \mathscr{N}_1 \oplus \mathscr{N}_2 \oplus \cdots \oplus \mathscr{N}_k$$

be decompositions of a semisimple ξ-module bundle $\mathscr{M}$ into direct Whitney sums of simple submodule bundles. Then $n = k$ and with suitable permutation, $\mathscr{M}_i$ is isomorphic to $\mathscr{N}_i$ for each i.

Proof. For each j, let $P_j = \mathscr{N}_1 \oplus \mathscr{N}_2 \oplus \cdots \oplus \widehat{\mathscr{N}_j} \oplus \cdots \oplus \mathscr{N}_k$. Then P_j is a submodule bundle of $\mathscr{M}$ and $\mathscr{M}/P_j$ is isomorphic to $\mathscr{N}_j$. Since N_j is simple, by the Corollary 5.7, it follows that P_j is a maximal submodule bundle of $\mathscr{M}$. By the Lemma 5.13, $\mathscr{M}_n \cap P_j$ is a submodule bundle of $\mathscr{M}$. Note that $\bigcap_{j=1}^{k} P_j = 0$. Since $\mathscr{M}_n \neq 0$, there is j such that $\mathscr{M}_n \not\subseteq P_j$. As $\mathscr{M}_n$ is simple, we must have $\mathscr{M}_n \cap P_j = 0$. Since P_j is maximal we conclude that $\mathscr{M} = \mathscr{M}_n \oplus P_j$. So, we have

$$\mathscr{M}/\mathscr{M}_n \cong \mathscr{M}_1 \oplus \mathscr{M}_2 \oplus \cdots \oplus \mathscr{M}_{n-1} \cong P_j = \mathscr{N}_1 \oplus \mathscr{N}_2 \oplus \cdots \oplus \widehat{\mathscr{N}_j} \oplus \cdots \oplus \mathscr{N}_k.$$

Hence the theorem follows by induction on n. □

6. Hochschild Cohomology and Semisimple Module Bundles

We recall the definition of Hochschild cohomology of an algebra bundle [3, 8]. Let η be a ξ-bimodule bundle. Let $\rho_1 : \xi \oplus \eta \to \eta$ and $\rho_2 : \eta \oplus \xi \to \eta$ be left and right module bundle structures of η. Let $C^n(\xi, \eta)$ denotes a finitely generated projective $C(X)$-module of all multilinear morphisms from $\xi^n = \xi \oplus \cdots \oplus \xi$ (n times) to η, where $C(X)$ is the ring of all continuous real valued functions on X. We identify $C^0(\xi, \eta)$ with $\Gamma(\eta)$. We define n^{th} coboundray $C(X)$-module homomorphism from $C^n(\xi, \eta)$ to $C^{n+1}(\xi, \eta)$ as follows: given $s \in C^0(\xi, \eta)$, consider a function

$$f_s : \xi \to \eta$$

given by

$$f_s(v) = \rho_1\big(v, s(x)\big) - \rho_2\big(s(x), v\big), \quad \text{for all} \quad v \in \xi_x.$$

Then f is a morphism being a combination of following continuous maps

$$\xi \to \xi \times X \to \xi \oplus \eta \to \eta, \quad v \mapsto \big(v, p(v) = x\big) \mapsto \big(v, s(x)\big) \mapsto \rho_1\big(v, s(x)\big)$$

and

$$\xi \to X \times \xi \to \eta \oplus \xi \to \eta, \quad v \mapsto \big(p(v) = x, v\big) \mapsto \big(s(x), v\big) \mapsto \rho_2\big(s(x), v\big).$$

Hence $f_s \in C^1(\xi, \eta)$. Thus we define

$$\delta^0 : C^0(\xi, \eta) \to C^1(\xi, \eta)$$

by

$$\delta^0(s) = f_s.$$

Given $f \in C^n(\xi, \eta)$, we define $g : \xi^{n+1} \to \eta$ by

$$\begin{aligned} g(x_1, \cdots, x_{n+1}) &= \rho_1\big(x_1, f(x_2, \cdots, x_{n+1})\big) \\ &\quad + \sum_{i=1}^{n} (-1)^i f(x_1, \cdots, x_i x_{i+1}, \cdots, x_{n+1}) \\ &\quad + (-1)^{n+1} \rho_2\big(f(x_1, \cdots, x_n), x_{n+1}\big). \end{aligned}$$

Then $g \in C^{n+1}(\xi, \eta)$. So we define

$$\delta^n : C^n(\xi, \eta) \to C^{n+1}(\xi, \eta)$$

by

$$\delta^n(f) = g.$$

Then $\{C^n(\xi,\eta),\delta^n\}$ forms a cochain complex.

Denote $Z^n(\xi,\eta) = \text{Ker }\delta^n$ and $B^n(\xi,\eta) = \text{Im }\delta^{n-1}$. The n-th Hochschild cohomology module of ξ over $C(X)$ with coefficients in η is a factor module

$$H^n(\xi,\eta) = \text{Ker }\delta^n/\text{Im }\delta^{n-1} = Z^n(\xi,\eta)/B^n(\xi,\eta).$$

Enlargement of a module bundle [3]: Let η,η' be ξ-bimodule bundles. By an enlargement of η' by η we mean a pair (E,π), where E is a ξ-bimodule bundle containing η' as a submodule bundle and π is a module bundle morphism of E onto η whose kernel is η'. We say that (E,π) is right inessential if there is a right module bundle isomorphism π^{-1} of η into E such that $\pi\pi^{-1}$ is the identity map on η.

Lemma 6.1. *Let η be a ξ-module bundle and η' be a submodule bundle of η. Let $\pi:\eta\to\eta/\eta'$ be the natural morphism. Then the enlargement (η,π) of η' by η/η' is inessential if and only if η' is a complemented submodule bundle of η.*

Proof. Suppose (η,π) is inessential. Then there is a ξ-module bundle isomorphism γ of η/η' into η such that $\pi\gamma$ is the identity map on η/η'. Let $\eta'' = \text{Im }\gamma$. Since each Im γ_x is a complemented submodule of η_x, by the Proposition 4.4, η'' is a submodule bundle of η. Clearly $\eta = \eta'\oplus\eta''$. Conversely suppose η' is a complemented submodule bundle of η. Then by the Lemma 4.5, there is a ξ-morphism $\gamma:\eta/\eta'\to\eta$ such that $\gamma(\bar m)\in\bar m$ for each $\bar m\in\eta/\eta'$. But then it is easy to check that $\pi\gamma$ is identity. □

We recall the construction of ξ-module bundles P_n [3]. Consider $\xi^* = \bigcup_{x\in X}\xi^*_x$, where ξ^*_x is the algebra obtained by adjoining an identity element[13]. Let $\phi: U\times A\to\bigcup_{x\in U}\xi_x$ be a local trivialization of ξ. Define $\phi^*: U\times A^*\to\bigcup_{x\in U}\xi^*_x$ by $\phi^*(x,(a,\alpha)) = (\phi(x,a),\alpha)$. where α is in the ground field of A. Define a topology on $\bigcup_{x\in U}\xi^*_x$ such that ϕ^* is a homeomorphism. Then ξ^* is an algebra bundle.

Consider the vector bundle $P_n = \xi^{\otimes n}\otimes\xi^*$, where $\xi^{\otimes n} = \xi\otimes\cdots\otimes\xi$ (n times). Define

$$\theta:\xi\oplus P_n\to P_n$$

by

$$\begin{aligned}\theta_1(a,a_1\otimes\cdots\otimes a_n\otimes a^*_{n+1}) = {} & a.a_1\otimes\cdots\otimes a^*_{n+1} + \\ & \sum_{i=1}^{n}(-1)^i a\otimes a_1\otimes\cdots\otimes a_ia_{i+1}\otimes\cdots\otimes a_n\otimes a^*_{n+1} \\ & + (-1)^n a\otimes a_1\otimes\cdots\otimes a_na^*_{n+1}\end{aligned}$$

and

$$\theta_2: P_n\oplus\xi\to P_n$$

by $\theta_2(a_1\otimes\cdots\otimes a_n\otimes a^*_{n+1},a) = a_1\otimes\cdots\otimes a_n\otimes a^*_{n+1}.a$. Then P_n is a locally trivial ξ-bimodule bundle.

We prove the following result mentioned in [3, Theorem 2.3] by using the Theorem 5.11 and Lemma 6.1.

Theorem 6.2. *A necessary and sufficient condition for all ξ-module bundles to be semisimple is that all the cohomology modules of dimension ≥ 1 are zero.*

Proof. Suppose all the ξ-module bundles are semisimple. Let η be any ξ-module bundle and (E, π) be a right inessential enlargement of η by P_n. Module bundle E is semisimple being ξ-module bundle and hence $\eta = \text{Ker } \pi$ is a submodule bundle of E. Then by Theorem 5.11, there is a submodule bundle η' of E such that $E = \eta \oplus \eta'$. Clearly $\pi|_{\eta'} : \eta' \to P_n$ is an isomorphism. Consider the enlargement (E_0, π_0), where $E_0 = \eta \oplus P_n$ and π_0 is the usual projection of E_0 onto P_n. By [3, Theorem 1.2], the set of all equivalence classes of the right inessential enlargements of η by P_n is a $C(X)$ module and the equivalence class containing the enlargement $(E_0 = \eta \oplus P_n, \pi_0)$ is the zero element of $m(P_n, \eta)$. Define $I : E \to E_0$ by, $I(e, e') = (e, \pi(e'))$. Then I is an isomorphism such that $\pi_0 I = \pi$. Hence (E, π) is equivalent to (E_0, π_0). Hence $m(P_n, \eta) = 0$. By [3, Lemma 1.2], $m(P_n, \eta)$ is isomorphic to $H^{n+1}(\xi, \eta)$ for $n \geq 0$.

Conversely, suppose all the cohomology modules of dimension ≥ 1 are zero. Let η be any ξ-module bundle and η' be a submodule bundle of η. Let π be the natural morphism of η onto η/η'. Consider η as right ξ-module bundle η^* where the morphism that induces the right module structure on the fibers of η^* is same as that of η and the morphism which induce the left module structures is defined to be zero. Let π^* denote the corresponding morphism of η^* onto η^*/η'^*. Clearly the enlargement (η^*, π^*) is left inessential. Since the cohomology classes are zero, by [3, Proposition 1.1], we get $m(\eta/\eta', \eta') = 0$. Hence this enlargement is equivalent to the trivial one which is also right inessential. Hence (η^*, π^*) is right inessential. But then corresponding enlargement (η, π) is right inessential. By applying the same Proposition again we can conclude that (η, π) is also left inessential. Hence by the Lemma 6.1, η' is a complemented submodule bundle of η. Thus η is semisimple. □

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