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IDENTITIES OF SYMMETRY FOR EULER POLYNOMIALS AND ALTERNATING POWER SUMS

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ABSTRACT. It was a breakthrough of T. Kim that he introduced the p-adic Volkenborn integrals to the study of identities of symmetry in two variables for Bernoulli polynomials and power sums, which had been investigated by considering suitable symmetric identities. Very recently, this result was generalized to the case of arbitrary number of variables by using the p-adic Volkenborn integrals. The aim of this paper is to derive identities of symmetry in arbitrary number of variables for Euler polynomials and alternating power sums by using fermionic p-adic integrals and to illustrate the results with some examples, which is again initiated by T. Kim in the case of two variables.

1. INTRODUCTION AND PRELIMINARIES

Much work on identities of symmetry has been done for Bernoulli polynomials and power sums or Euler polynomials and alternating power sums or q-Bernoulli polynomials and q-power sums or q-Euler polynomials and alternating q-power sums. We let the reader refer to the papers [4-12,14-17] for some of the previous related works.

Suitable symmetric identities had been used to find identities of symmetry in two variables for Bernoulli polynomials and power sums (see [3,21,22]). T. Kim used a completely different tool, namely the p-adic Volkenborn integrals, in order to find identities of symmetry in two variables for such polynomials and sums (see [12-14]). It was observed that this p-adic approach can be generalized to the case of three variables in [11] and further to the case of arbitrary number of variables in [17].

T. Kim used fermionic p-adic integrals for continuous functions in order to find identities of symmetry in two variables for Euler polynomials and alternating power sums. Indeed, in [12] the following two identities were obtained:

$$(1)\qquad \sum_{i=0}^{n}\binom{n}{i}E_i(w_2x)T_{n-i}(w_1-1)w_1^iw_2^{n-i}=\sum_{i=0}^{n}\binom{n}{i}E_i(w_1x)T_{n-i}(w_2-1)w_2^iw_1^{n-i},$$

$$(2)\qquad w_1^n\sum_{i=0}^{w_1-1}(-1)^iE_n(w_2x+\frac{w_2}{w_1}i)=w_2^n\sum_{i=0}^{w_2-1}(-1)^iE_n(w_1x+\frac{w_1}{w_2}i),$$

where w_1,w_2 are any odd positive integers, n is a nonnegative integer, $E_n(x)$ are the Euler polynomials in (10), and $T_k(n)$ are the alternating power sums in (11).

Not much later, it is observed in [10] that many identities of symmetry for three variables can obtained by adopting the fermionic p-adic integral approach initiated in [12] (see Example 3.1). As was mentioned in Section 1 of [10], by specializing the variable w_3 as 1 in (c-1), (c-2) and (c-3) of Example 3.1, it was shown that (1) and (2) are all equal and that they are further equal to the following identities:

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Let w_1, w_2 be any odd positive integers. Then we have:

$$\sum_{k+l+m=n} \binom{n}{k,l,m} E_k(y_1) T_l(w_1-1) T_m(w_2-1) w_1^{k+m} w_2^{k+l} \tag{3}$$

$$= w_1^n \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{w_1-1} (-1)^i E_k(y_1 + \frac{i}{w_1}) T_{n-k}(w_2-1) w_2^k \tag{4}$$

$$= w_2^n \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{w_2-1} (-1)^i E_k(y_1 + \frac{i}{w_2}) T_{n-k}(w_1-1) w_1^k \tag{5}$$

$$= (w_1 w_2)^n \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} (-1)^{i+j} E_n(y_1 + \frac{i}{w_1} + \frac{j}{w_2}). \tag{6}$$

Those would not be discovered if more symmetries had not been available. Thus the abundance of symmetries in (c-1), (c-2) and (c-3) shed new light even on the existing identities in two variables. In addition, the identities of symmetry can be found also for q-Bernoulli polynomials [15] and q-Euler polynomials [7], respectively by using p-adic q-Volkenborn integrals and p-adic fermionic q-integrals. For the q-Bernoulli and q-Euler polynomials, we let the reader refer to the papers [1,2,18,20].

The aim of this paper is to generalize, by using the fermionic p-adic integrals, the results in three variables [10] to those in arbitrary number of variables in a suitable setting. The derivations of identities are based on the p-adic integral expression of the generating function for the Euler polynomials in (10) and the quotient of integrals in (13) and (14) that can be expressed as the exponential generating function for the alternating power sums. We indebted this idea to the paper [12]. Theorem 2.3 is the main result of this paper. In addition, we illustrate our results with examples in Section 3, which demonstrates that there are really abundant symmetries. In the rest of this section, we recall the facts that are needed throughout this paper.

Let p be a fixed odd prime. Throughout this paper, $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}_p$ will respectively denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. For a continuous function $f : \mathbb{Z}_p \to \mathbb{C}_p$, the p-adic fermionic integral of f is defined by

$$\int_{\mathbb{Z}_p} f(z) d\mu_{-1}(z) = \lim_{N\to\infty} \sum_{j=0}^{p^N-1} f(j)(-1)^j.$$

Then it is easy to see that

$$\int_{\mathbb{Z}_p} f(z+1) d\mu_{-1}(z) + \int_{\mathbb{Z}_p} f(z) d\mu_{-1}(z) = 2f(0). \tag{7}$$

Let $|\ |_p$ be the normalized absolute value of $\mathbb{C}_p$, such that $|p|_p = \frac{1}{p}$, and let

$$E = \{t \in \mathbb{C}_p \mid |t|_p < p^{-\frac{1}{p-1}}\}. \tag{8}$$

Then, for each fixed $t \in E$, the function $f(z) = e^{zt}$ is analytic on $\mathbb{Z}_p$ and by applying (7) to this f, we get the p-adic integral expression of the generating function for Euler numbers E_n:

$$\int_{\mathbb{Z}_p} e^{zt} d\mu_{-1}(z) = \frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}\ (t \in E). \tag{9}$$

So we have the following p-adic integral expression of the generating function for the Euler polynomials $E_n(x)$:

$$\int_{\mathbb{Z}_p} e^{(x+z)t} d\mu_{-1}(z) = \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \ (t \in E, x \in \mathbb{Z}_p). \tag{10}$$

Here and throughout this paper, we will have many instances to be able to interchange integral and infinite sum. That is justified by Proposition 55.4 in [19].

Let $T_k(n)$ denote the alternating k-th power sum of the first $(n+1)$ nonnegative integers, namely

$$T_k(n) = \sum_{i=0}^{n} (-1)^i i^k = (-1)^0 0^k + (-1)^1 1^k + (-1)^2 2^k + \cdots + (-1)^n n^k. \tag{11}$$

In particular,

$$T_0(n) = \begin{cases} 1, & \text{if } n \equiv 0 \ (mod\ 2), \\ 0, & \text{if } n \equiv 1 \ (mod\ 2), \end{cases} \quad T_k(0) = \begin{cases} 1, & \text{for } k = 0, \\ 0, & \text{for } k > 0. \end{cases} \tag{12}$$

From (9) and (11), one easily derives the following identities: for any odd positive integer w,

$$\frac{\int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} e^{wyt} d\mu_{-1}(y)} = \sum_{i=0}^{w-1} (-1)^i e^{it} \tag{13}$$

$$= \sum_{k=0}^{\infty} T_k(w-1)\frac{t^k}{k!} \ (t \in E). \tag{14}$$

In what follows, we will always assume that the p-adic fermionic integrals of the various exponential functions on $\mathbb{Z}_p$ are defined for $t \in E$ (see (9)), and therefore it will not be mentioned.

2. Main results

Let $n \geq 2$, and let $I_n = \{1, 2, \ldots, n\}$. Then the symmetric group S_n acts on I_n in a natural way as $(\sigma, j) \mapsto \sigma(j)$. For each integer j with $1 \leq j \leq n$, let Ω_j be the subset of 2^{I_n} consisting of all j-element subsets of I_n. The next result is easy to see and the proof is given in [17].

Proposition 2.1. *Let Ω be a nonempty subset of 2^{I_n} not containing the empty set ϕ. Then there is an action of S_n on Ω induced by the natural action of S_n on I_n if and only if $\Omega = \cup_{j \in J} \Omega_j$, for some nonempty subset J of I_n. Moreover, such an action of S_n on Ω is transitive if and only if $\Omega = \Omega_j$, for some $j\,(1 \leq j \leq n)$.*

Assume that n is any fixed integer ≥ 2. Then we will introduce notations that will be used throughout this paper.

- $I = I_n = \{1, 2, \ldots, n\}$.
- $\Omega_j = \Omega_j^{(n)}$=the set consisting of all j-element subsets of I, for any $j = 1, 2, \ldots, n-1$, $(|\Omega_j| = \binom{n}{j})$.

We give the reversed lexicographic ordering on each Ω_j and also on any nonempty subset Ω of Ω_j. For example, $\Omega_3 = \left\{\bar{1} = \{2,3,4\} < \bar{2} = \{1,3,4\} < \bar{3} = \{1,2,4\} < \bar{4} = \{1,2,3\}\right\}$, when $n = 4, j = 3$; $\Omega = \{\{2,3,4\} < \{1,2,4\}\}$ is a subset of Ω_j. We write $\Omega < \Omega'$, for disjoint nonempty subsets Ω, Ω' of Ω_j, if every member of Ω is smaller than that of Ω'. Also, we agree that $\phi < \Omega$, $\Omega < \phi$, for every (including the empty set) subset Ω of Ω_j.

Every permutation σ in S_n gives rise to a natural bijection of Ω_j onto itself given by $A \mapsto \sigma A$, where σA is the set obtained from A by applying σ to each member of A.

- $w_1, w_2, \ldots, w_n$ typical n odd positive integers.
- $w_A = \prod_{j\in A} w_j$, for any subset A of I, $(w_\phi = 1)$. For example, $w_{\{1,2,3\}} = w_1w_2w_3$. Also, we note that $\prod_{A\in\Omega_j} w_A = w_I^{\binom{n-1}{j-1}}$.
- $\hat{w}_i = \frac{w_I}{w_i}$.
- $\bar{A}$ = the complement of A in I, for any subset A of I, so that $w_A w_{\bar{A}} = w_I$, for any subset A of I.
- $\hat{w}_A = \prod_{i\in A} \hat{w}_i = (w_I)^{|A|-1} w_{\bar{A}}$, $(\hat{w}_I = (w_I)^{|A|-1})$.
- $x_A = x_{j_1 j_2 \cdots j_r}$, for any nonempty subset $A = \{j_1, j_2, \ldots, j_r\}$ of I, with $j_1 < j_2 < \cdots < j_r$. For example, $x_{\{1,2,3\}} = x_{123}$. These are typical variables of integration.
- $\phi_j : \Omega_j \to \Omega_{n-j}\, (A \mapsto \bar{A})$ is a bijection.
- $p_j(w;x) = p_j^{(n)}(w;x) = \sum_{A\in\Omega_j} w_A x_{\bar{A}}$, for $j = 1, 2, \ldots, n-1$. For example, with $n = 4$, we have

$$
\begin{aligned}
p_1(w;x) &= w_1x_{234} + w_2x_{134} + w_3x_{124} + w_4x_{123},\\
p_2(w;x) &= w_1w_2x_{34} + w_1w_3x_{24} + w_1w_4x_{23} + w_2w_3x_{14} + w_2w_4x_{13} + w_3w_4x_{12},\\
p_3(w;x) &= w_1w_2w_3x_4 + w_1w_2w_4x_3 + w_1w_3w_4x_2 + w_2w_3w_4x_1.
\end{aligned}
$$

- $d\mu(\Omega) = \prod_{A\in\Omega} d\mu(x_{\bar{A}})$, for any nonempty subset Ω of Ω_j

As before, assume that n is any fixed integer ≥ 2, j is an integer with $1 \leq j \leq n-1$, and that $w_1, w_2, \ldots, w_n$ are any odd positive integers. Then, in view of Proposition 1.1, for any subset Ω for Ω_j, we consider the following quotients of integrals given by

$$I_j(\Omega) = I_j^{(n)}(\Omega) = \frac{\int_{\mathbb{Z}_p^{|\Omega_j|}} e^{(p_j(w;x)+w_I(\sum_{A\in\Omega_j-\Omega} y_{\bar{A}}))t} d\mu_{-1}(\Omega_j)}{(\int_{\mathbb{Z}_p} e^{w_I z t} d\mu_{-1}(z))^{|\Omega|}} \tag{15}$$

$$= \frac{2^{\binom{n}{j}-|\Omega|} e^{w_I(\sum_{A\in\Omega_j-\Omega} y_{\bar{A}})t} (e^{w_I t}+1)^{|\Omega|}}{\prod_{A\in\Omega_j}(e^{w_A t}+1)}. \tag{16}$$

Here we have to observe that

$$
\begin{aligned}
\int_{\mathbb{Z}_p^{|\Omega_j|}} e^{p_j(w;x)t} d\mu_{-1}(\Omega_j) &= \prod_{A\in\Omega_j} \int_{\mathbb{Z}_p} e^{w_A x_{\bar{A}} t} d\mu_{-1}(x_{\bar{A}}) \\
&= \prod_{A\in\Omega_j} \frac{2}{e^{w_A t}+1} = \frac{2^{\binom{n}{j}}}{\prod_{A\in\Omega_j}(e^{w_A t}+1)}.
\end{aligned} \tag{17}
$$

It is important to observe here, either from (15) or from (16), that the integrals $I_j(\Omega)$ are invariant under any permutation of $w_1, w_2, \ldots, w_n$.

Now, we decompose Ω into a disjoint union $\Omega = \Omega^{(e)} \cup \Omega^{(s)}$, with $\Omega^{(e)} < \Omega^{(s)}$. As we allow either $\Omega^{(e)}$ or $\Omega^{(s)}$ to be the empty set, there are $|\Omega| + 1$ ways of doing this. Then, by invoking (13) and (14), we write the integral in (15) as:

$$I_j(\Omega) = \prod_{A\in\Omega_j-\Omega}\int_{\mathbb{Z}_p} e^{w_A(x_{\bar{A}}+w_{\bar{A}}y_{\bar{A}})t}d\mu_{-1}(x_{\bar{A}}) \times \prod_{A\in\Omega}\frac{\int_{\mathbb{Z}_p} e^{w_A x_{\bar{A}} t}d\mu_{-1}(x_{\bar{A}})}{\int_{\mathbb{Z}_p} e^{w_{\bar{A}}w_A z t}d\mu_{-1}(z)}$$
$$= \prod_{A\in\Omega_j-\Omega}\int_{\mathbb{Z}_p} e^{w_A(x_{\bar{A}}+w_{\bar{A}}y_{\bar{A}})t}d\mu_{-1}(x_{\bar{A}})$$
$$\times \prod_{A\in\Omega^{(e)}}\Big(\sum_{i_A=0}^{w_{\bar{A}}-1}(-1)^{i_A}e^{i_A w_A t}\Big) \times \prod_{A\in\Omega^{(s)}}\Big(\sum_{j_A=0}^{\infty} T_{j_A}(w_{\bar{A}}-1)\frac{(w_A t)^{j_A}}{j_A!}\Big). \tag{18}$$

Note here that we used the identity in (13) for all $A \in \Omega^{(e)}$, and that in (14) for all $A \in \Omega^{(s)}$.

Remark 2.2. *We observe that* (16) *depends essentially only on the size* $|\Omega|$ *of* Ω*. Namely, if* $\Omega^{(1)}$ *and* $\Omega^{(2)}$ *are two subsets of* Ω_j*, with* $|\Omega^{(1)}| = |\Omega^{(2)}|$*, and* $f : \Omega_j \to \Omega_j$ *is any bijective map sending* $\Omega^{(1)}$ *onto* $\Omega^{(2)}$*, then we have*

$$\frac{2^{\binom{n}{j}-|\Omega^{(2)}|}e^{w_I(\sum_{A\in\Omega_j-\Omega^{(2)}} y_{\bar{A}})t}(e^{w_I t}+1)^{|\Omega^{(2)}|}}{\prod_{A\in\Omega_j}(e^{w_A t}+1)}$$
$$= \frac{2^{\binom{n}{j}-|\Omega^{(1)}|}e^{w_I(\sum_{A\in\Omega_j-\Omega^{(1)}} y_{\overline{f(A)}})t}(e^{w_I t}+1)^{|\Omega^{(1)}|}}{\prod_{A\in\Omega_j}(e^{w_A t}+1)}. \tag{19}$$

Thus (15) *with* $\Omega^{(2)}$ *is the same as that with* $\Omega^{(1)}$*, with the 'y variables renamed.' Hence we only need to consider* (18) *for only one subset* Ω *of* Ω_j *with the given size. So, for each positive integer k with* $k \le |\Omega_j|$*, we only consider the subset* Ω *of* Ω_j *consisting of the first k (smaller) elements of* Ω_j*. We denote this subset by* Ω_{jk}*, and the empty subset of* Ω_j *by* Ω_{j0}*. From now on, we assume that* $\Omega = \Omega_{jk}$*, for some integer k* $(0 \le k \le |\Omega_j| = \binom{n}{j})$*. For example, when* $n = 3$*, we see that*

$$\Omega_{20} = \phi, \quad \Omega_{21} = \{\{2,3\}\}, \quad \Omega_{22} = \{\{2,3\},\{1,3\}\}, \quad \Omega_{23} = \{\{2,3\},\{1,3\},\{1,2\}\}.$$

Further, we assume that we have a decomposition of $\Omega^{(e)}$ as the disjoint union

$$\Omega^{(e)} = \cup_{A\in\Omega_j-\Omega}\Omega_A^{(e)} \tag{20}$$

satisfying the following conditions:

(i) $|\Omega_A^{(e)}| \le |\Omega_{A'}^{(e)}|$, for all $A, A' \in \Omega_j - \Omega$, with $A < A'$,

(ii) $\Omega_A^{(e)} < \Omega_{A'}^{(e)}$, for all $A, A' \in \Omega_j - \Omega$, with $A < A'$.

(Note: in view of (20), this requires in particular that we should choose $\Omega^{(e)} = \phi$, for $\Omega_j - \Omega = \phi$)

We assume that we are given such a decomposition as in (20) satisfying (i) and (ii). Then we write (18) as

$$
\begin{aligned}
(21)\qquad I_j(\Omega) &= \prod_{A\in\Omega_j-\Omega}\prod_{E\in\Omega_A^{(e)}}\sum_{i_E=0}^{w_{\bar{E}}-1}(-1)^{\sum_{E\in\Omega_A^{(e)}} i_E}\int_{\mathbb{Z}_p} e^{w_A(x_{\bar{A}}+w_{\bar{A}}y_{\bar{A}}+\sum_{E\in\Omega_A^{(e)}} i_E\frac{w_E}{w_A})t}\,d\mu_{-1}(x_{\bar{A}})\\
&\quad\times\prod_{A\in\Omega^{(s)}}\sum_{j_A=0}^{\infty}T_{j_A}(w_{\bar{A}}-1)\frac{(w_At)^{j_A}}{j_A!}\\
&= \prod_{A\in\Omega_j-\Omega}\sum_{l_A=0}^{\infty}\prod_{E\in\Omega_A^{(e)}}\sum_{i_E=0}^{w_{\bar{E}}-1}(-1)^{\sum_{E\in\Omega_A^{(e)}} i_E}E_{l_A}(w_{\bar{A}}y_{\bar{A}}+\sum_{E\in\Omega_A^{(e)}} i_E\frac{w_E}{w_A})\frac{(w_At)^{l_A}}{l_A!}\\
&\quad\times\prod_{A\in\Omega^{(s)}}\sum_{j_A=0}^{\infty}T_{j_A}(w_{\bar{A}}-1)\frac{(w_At)^{j_A}}{j_A!}.
\end{aligned}
$$

Further, by rearranging sums (21) can be written as

$$
\begin{aligned}
(22)\qquad I_j(\Omega) &= \sum_{n=0}^{\infty}\sum_{\sum_{A\in\Omega_j-\Omega} l_A+\sum_{A\in\Omega^{(s)}} j_A=n}\binom{n}{\ldots,l_A,\ldots,j_A,\ldots}\\
&\quad\times\prod_{A\in\Omega_j-\Omega}\prod_{E\in\Omega_A^{(e)}}\sum_{i_E=0}^{w_{\bar{E}}-1}(-1)^{\sum_{E\in\Omega_A^{(e)}} i_E}E_{l_A}(w_{\bar{A}}y_{\bar{A}}+\sum_{E\in\Omega_A^{(e)}} i_E\frac{w_E}{w_A})\\
&\quad\times\prod_{A\in\Omega^{(s)}}T_{j_A}(w_{\bar{A}}-1)\prod_{A\in\Omega_j-\Omega}w_A^{l_A}\prod_{A\in\Omega^{(s)}}w_A^{j_A}\frac{t^n}{n!}.
\end{aligned}
$$

Here $\binom{n}{\ldots,l_A,\ldots,j_A,\ldots}$ denotes the multinomial coefficient, where l_A and j_A are nonnegative integers varying respectively over the index sets $A\in\Omega_j-\Omega$ and $A\in\Omega^{(s)}$.

For the next theorem, we assume that $\Omega=\Omega_{jk}$, for some k, Ω has a decomposition into a disjoint union $\Omega=\Omega^{(e)}\cup\Omega^{(s)}$, with $\Omega^{(e)}<\Omega^{(s)}$, and that $\Omega^{(e)}$ has a decomposition $\Omega^{(e)}=\cup_{A\in\Omega_j-\Omega}\Omega_A^{(e)}$, satisfying the conditions (i) and (ii) in (20). As we noted earlier, the p-adic integrals in (15) are invariant under every permutation of $w_1,w_2,\ldots,w_n$, so that it gives the identities of symmetry with respect to $w_1,w_2,\ldots,w_n$, involving Bernoulli polynomials and power sums. Now, we have our main result from (22).

Theorem 2.3. *Let $w_1,w_2,\ldots,w_n$ be any odd positive integers. The following expression is invariant under every permutation of $w_1,w_2,\ldots,w_n$, so that it gives the identities of symmetry with respect to $w_1,w_2,\ldots,w_n$:*

$$
\begin{aligned}
(23)\qquad &\sum_{\sum_{A\in\Omega_j-\Omega} l_A+\sum_{A\in\Omega^{(s)}} j_A=n}\binom{n}{\ldots,l_A,\ldots,j_A,\ldots}\\
&\quad\times\prod_{A\in\Omega_j-\Omega}\prod_{E\in\Omega_A^{(e)}}\sum_{i_E=0}^{w_{\bar{E}}-1}(-1)^{\sum_{E\in\Omega_A^{(e)}} i_E}E_{l_A}(w_{\bar{A}}y_{\bar{A}}+\sum_{E\in\Omega_A^{(e)}} i_E\frac{w_E}{w_A})\\
&\quad\times\prod_{A\in\Omega^{(s)}}T_{j_A}(w_{\bar{A}}-1)\prod_{A\in\Omega_j-\Omega}w_A^{l_A}\prod_{A\in\Omega^{(s)}}w_A^{j_A}.
\end{aligned}
$$

In other words, for all permutations $\sigma \in S_n$, the following expressions are all the same:

$$\sum_{\sum_{A\in\Omega_j-\Omega} l_A+\sum_{A\in\Omega^{(s)}} j_A=n} \binom{n}{\ldots,l_A,\ldots,j_A,\ldots} \times \prod_{A\in\Omega_j-\Omega} \prod_{E\in\Omega_A^{(e)}} \sum_{i_E=0}^{w_{\overline{\sigma E}}-1} (-1)^{\sum_{E\in\Omega_A^{(e)}} i_E} E_{l_A}\Big(w_{\overline{\sigma A}} y_{\bar{A}} + \sum_{E\in\Omega_A^{(e)}} i_E \frac{w_{\sigma E}}{w_{\sigma A}}\Big) \times \prod_{A\in\Omega^{(s)}} T_{j_A}(w_{\overline{\sigma A}}-1) \prod_{A\in\Omega_j-\Omega} w_{\sigma A}^{l_A} \prod_{A\in\Omega^{(s)}} w_{\sigma A}^{j_A}.$$

Here $\binom{n}{\ldots,l_A,\ldots,j_A,\ldots}$ denotes the multinomial coefficient, where l_A and j_A are nonnegative integers varying respectively over the index sets $A \in \Omega_j - \Omega$ and $A \in \Omega^{(s)}$.

3. Examples

Here we would like to illustrate our Theorem 2.3.

Example 3.1. *Assume that $n = 3, j = 2$. Here $\Omega_2 = \Omega_2^{(3)} = \{\bar{1} = \{2,3\} < \bar{2} = \{1,3\} < \bar{3} = \{1,2\}\}$. In view of our discussion leading up to Theorem 2.3, we may consider only the subsets $\Omega = \Omega_{2i}$, $(i = 0,1,2,3)$ of Ω_2.*

(a) $\Omega_{20} = \phi \ \ (\Omega_2 - \Omega_{20} = \Omega_2)$
  (a-1) $\Omega_{20}^{(e)} = \phi \ \ (\Omega_{20,A}^{(e)} = \phi$, for each $A \in \Omega_2)$, $\Omega_{20}^{(s)} = \phi$.

(b) $\Omega_{21} = \{\bar{1}\} \ \ (\Omega_2 - \Omega_{21} = \{\bar{2},\bar{3}\})$
  (b-1) $\Omega_{21}^{(e)} = \phi \ \ (\Omega_{21,A}^{(e)} = \phi$, for each $A \in \Omega_2 - \Omega_{21})$, $\Omega_{21}^{(s)} = \{\bar{1}\}$,
  (b-2) $\Omega_{21}^{(e)} = \{\bar{1}\} \ \ (\Omega_{21,\bar{2}}^{(e)} = \phi, \Omega_{21,\bar{3}}^{(e)} = \{\bar{1}\}), \Omega_{21}^{(s)} = \phi$.

(c) $\Omega_{22} = \{\bar{1},\bar{2}\} \ \ (\Omega_2 - \Omega_{22} = \{\bar{3}\})$
  (c-1) $\Omega_{22}^{(e)} = \phi \ \ (\Omega_{22,\bar{3}}^{(e)} = \phi)$, $\Omega_{22}^{(s)} = \{\bar{1},\bar{2}\}$,
  (c-2) $\Omega_{22}^{(e)} = \{\bar{1}\} \ \ (\Omega_{22,\bar{3}}^{(e)} = \{\bar{1}\})$, $\Omega_{22}^{(s)} = \{\bar{2}\}$,
  (c-3) $\Omega_{22}^{(e)} = \{\bar{1},\bar{2}\} \ \ (\Omega_{22,\bar{3}}^{(e)} = \{\bar{1},\bar{2}\})$, $\Omega_{22}^{(s)} = \phi$.

(d) $\Omega_{23} = \{\bar{1},\bar{2},\bar{3}\} \ \ (\Omega_2 - \Omega_{23} = \phi)$
  (d-1) $\Omega_{23}^{(e)} = \phi$, $\Omega_{23}^{(s)} = \Omega_{23}$.

One checks now that the invariance of (15) under any permutation of w_1, w_2, w_3, applied to each of the cases (a-1), (b-1), (b-2), (c-1), (c-2), and (c-3), yield the results in Theorems 4.1, 4.2, 4.5, 4.8, 4.11, and 4.14 in [10]. As we noted in [10], not all of these give the full six identities of symmetry corresponding to the symmetric group S_3. The possible numbers of distinct identities of symmetry are 1, 2, 3, and 6 corresponding to the quotient $|S_3|/|H|$, where H is a subgroup of S_3, with the respective orders 6, 3, 2, and 1. In our case, (a-1), (b-1), (b-2) and (c-2) give the full six identities of symmetry, and (c-1) and (c-3) yield three identities of symmetry, while (d-1) gives no identities of symmetry. For convenience of the reader, we reproduce those results here with appropriate change of notations in (23). In the following, w_1, w_2, w_3 are all odd positive integers except for (a-1), where they are any positive integers.

(a-1)
$$\begin{aligned}
&\sum_{k+\ell+m=n}\binom{n}{k,\ell,m}E_k(w_1y_1)E_\ell(w_2y_2)E_m(w_3y_3)w_1^{\ell+m}w_2^{k+m}w_3^{k+\ell}\\
=&\sum_{k+\ell+m=n}\binom{n}{k,\ell,m}E_k(w_1y_1)E_\ell(w_3y_2)E_m(w_2y_3)w_1^{\ell+m}w_3^{k+m}w_2^{k+\ell}\\
=&\sum_{k+\ell+m=n}\binom{n}{k,\ell,m}E_k(w_2y_1)E_\ell(w_1y_2)E_m(w_3y_3)w_2^{\ell+m}w_1^{k+m}w_3^{k+\ell}\\
=&\sum_{k+\ell+m=n}\binom{n}{k,\ell,m}E_k(w_2y_1)E_\ell(w_3y_2)E_m(w_1y_3)w_2^{\ell+m}w_3^{k+m}w_1^{k+\ell}\\
=&\sum_{k+\ell+m=n}\binom{n}{k,\ell,m}E_k(w_3y_1)E_\ell(w_1y_2)E_m(w_2y_3)w_3^{\ell+m}w_1^{k+m}w_2^{k+\ell}\\
=&\sum_{k+\ell+m=n}\binom{n}{k,\ell,m}E_k(w_3y_1)E_\ell(w_2y_2)E_m(w_1y_3)w_3^{\ell+m}w_2^{k+m}w_1^{k+\ell}.
\end{aligned}$$

(b-1)
$$\begin{aligned}
&\sum_{k+l+m=n}\binom{n}{k,l,m}E_k(w_1y_2)E_l(w_2y_3)T_m(w_3-1)w_1^{l+m}w_2^{k+m}w_3^{k+l}\\
=&\sum_{k+l+m=n}\binom{n}{k,l,m}E_k(w_1y_2)E_l(w_3y_3)T_m(w_2-1)w_1^{l+m}w_3^{k+m}w_2^{k+l}\\
=&\sum_{k+l+m=n}\binom{n}{k,l,m}E_k(w_2y_2)E_l(w_1y_3)T_m(w_3-1)w_2^{l+m}w_1^{k+m}w_3^{k+l}\\
=&\sum_{k+l+m=n}\binom{n}{k,l,m}E_k(w_2y_2)E_l(w_3y_3)T_m(w_1-1)w_2^{l+m}w_3^{k+m}w_1^{k+l}\\
=&\sum_{k+l+m=n}\binom{n}{k,l,m}E_k(w_3y_2)E_l(w_2y_3)T_m(w_1-1)w_3^{l+m}w_2^{k+m}w_1^{k+l}\\
=&\sum_{k+l+m=n}\binom{n}{k,l,m}E_k(w_3y_2)E_l(w_1y_3)T_m(w_2-1)w_3^{l+m}w_1^{k+m}w_2^{k+l}.
\end{aligned}$$

(b-2)
$$\begin{aligned}
&w_1^n\sum_{k=0}^{n}\binom{n}{k}E_k(w_3y_2)\sum_{i=0}^{w_1-1}(-1)^iE_{n-k}(w_2y_3+\frac{w_2}{w_1}i)w_3^{n-k}w_2^k\\
=&w_1^n\sum_{k=0}^{n}\binom{n}{k}E_k(w_2y_2)\sum_{i=0}^{w_1-1}(-1)^iE_{n-k}(w_3y_3+\frac{w_3}{w_1}i)w_2^{n-k}w_3^k\\
=&w_2^n\sum_{k=0}^{n}\binom{n}{k}E_k(w_3y_2)\sum_{i=0}^{w_2-1}(-1)^iE_{n-k}(w_1y_3+\frac{w_1}{w_2}i)w_3^{n-k}w_1^k\\
=&w_2^n\sum_{k=0}^{n}\binom{n}{k}E_k(w_1y_2)\sum_{i=0}^{w_2-1}(-1)^iE_{n-k}(w_3y_3+\frac{w_3}{w_2}i)w_1^{n-k}w_3^k\\
=&w_3^n\sum_{k=0}^{n}\binom{n}{k}E_k(w_2y_2)\sum_{i=0}^{w_3-1}(-1)^iE_{n-k}(w_1y_3+\frac{w_1}{w_3}i)w_2^{n-k}w_1^k\\
=&w_3^n\sum_{k=0}^{n}\binom{n}{k}E_k(w_1y_2)\sum_{i=0}^{w_3-1}(-1)^iE_{n-k}(w_2y_3+\frac{w_2}{w_3}i)w_1^{n-k}w_2^k.
\end{aligned}$$

(c-1)
$$\begin{aligned}&\sum_{k+l+m=n}\binom{n}{k,l,m}E_k(w_1y_3)T_l(w_2-1)T_m(w_3-1)w_1^{l+m}w_2^{k+m}w_3^{k+l}\\=&\sum_{k+l+m=n}\binom{n}{k,l,m}E_k(w_2y_3)T_l(w_3-1)T_m(w_1-1)w_2^{l+m}w_3^{k+m}w_1^{k+l}\\=&\sum_{k+l+m=n}\binom{n}{k,l,m}E_k(w_3y_3)T_l(w_1-1)T_m(w_2-1)w_3^{l+m}w_1^{k+m}w_2^{k+l}.\end{aligned}$$

(c-2)
$$\begin{aligned}&w_1^n\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{w_1-1}(-1)^iE_k(w_2y_3+\frac{w_2}{w_1}i)T_{n-k}(w_3-1)w_2^{n-k}w_3^k\\=&w_1^n\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{w_1-1}(-1)^iE_k(w_3y_3+\frac{w_3}{w_1}i)T_{n-k}(w_2-1)w_3^{n-k}w_2^k\\=&w_2^n\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{w_2-1}(-1)^iE_k(w_1y_3+\frac{w_1}{w_2}i)T_{n-k}(w_3-1)w_1^{n-k}w_3^k\\=&w_2^n\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{w_2-1}(-1)^iE_k(w_3y_3+\frac{w_3}{w_2}i)T_{n-k}(w_1-1)w_3^{n-k}w_1^k\\=&w_3^n\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{w_3-1}(-1)^iE_k(w_1y_3+\frac{w_1}{w_3}i)T_{n-k}(w_2-1)w_1^{n-k}w_2^k\\=&w_3^n\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{w_3-1}(-1)^iE_k(w_2y_3+\frac{w_2}{w_3}i)T_{n-k}(w_1-1)w_2^{n-k}w_1^k.\end{aligned}$$

(c-3)
$$\begin{aligned}&(w_1w_2)^n\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}(-1)^{i+j}E_n(w_3y_3+\frac{w_3}{w_1}i+\frac{w_3}{w_2}j)\\=&(w_2w_3)^n\sum_{i=0}^{w_2-1}\sum_{j=0}^{w_3-1}(-1)^{i+j}E_n(w_1y_3+\frac{w_1}{w_2}i+\frac{w_1}{w_3}j)\\=&(w_3w_1)^n\sum_{i=0}^{w_3-1}\sum_{j=0}^{w_1-1}(-1)^{i+j}E_n(w_2y_3+\frac{w_2}{w_3}i+\frac{w_2}{w_1}j).\end{aligned}$$

Example 3.2. *Assume that $n=4, j=3$. Here $\Omega_3=\Omega_3^{(4)}=\{\bar{1}=\{2,3,4\}<\bar{2}=\{1,3,4\}<\bar{3}=\{1,2,4\}<\bar{4}=\{1,2,3\}\}$. In view of our discussion leading up to Theorem 2.3, we may consider only the subsets $\Omega=\Omega_{3i}$, $(i=0,1,2,3,4)$ of Ω_3. The details on all possible cases can be found in [17]. We let the interested reader write out the identities of symmetry for each of those cases by using Theorem 2.3.*

Example 3.3. *Assume that $n=4, j=2$. Here $\Omega_2=\Omega_2^{(4)}=\{34=\{3,4\}<24=\{2,4\}<23=\{2,3\}<14=\{1,4\}<13=\{1,3\}<12=\{1,2\}\}$. In view of our discussion leading up to Theorem 2.3, we may consider only the subsets $\Omega=\Omega_{2i}$, $(i=0,1,2,3,4,5,6)$ of Ω_2. Again, all the possible cases are detailed in [17]. We let the interested reader write out the identities of symmetry for each of those cases by using Theorem 2.3.*

Example 3.4. *Assume that $n=5, j=2$. Here $\Omega_2=\Omega_2^{(5)}=\{45=\{4,5\}<35=\{3,4\}<34=\{3,4\}<25=\{2,5\}<24=\{2,4\}<23=\{2,3\}<15=\{1,5\}<14=\{1,4\}<13=\{1,3\}<12=\{1,2\}\}$.*

In view of our discussion leading up to Theorem 2.3, we may consider only the following subsets $\Omega = \Omega_{2i}$, $(i = 0,1,2,3,4,5,6,7,8,9,10)$ *of* Ω_2. *We let the interested reader write out the identities of symmetry for each of the following cases by using Theorem 2.3.*

(a) $\Omega_{20} = \phi$ $(\Omega_2 - \Omega_{20} = \Omega_2)$

(a-1) $\Omega_{20}^{(e)} = \phi(\Omega_{20,A}^{(e)} = \phi$ for each $A \in \Omega_2)$, $\Omega_{20}^{(s)} = \phi$.

(b) $\Omega_{21} = \{45\}$ $(\Omega_2 - \Omega_{21} = \{35,34,25,24,23,15,14,13,12\})$

(b-1) $\Omega_{21}^{(e)} = \phi$ $(\Omega_{21,A}^{(e)} = \phi$ for each $A \in \Omega_2 - \Omega_{21})$, $\Omega_{21}^{(s)} = \{45\}$,

(b-2) $\Omega_{21}^{(e)} = \{45\}$ $(\Omega_{21,A}^{(e)} = \phi$ for each $A \in \{35,34,25,24,23,15,14,13\})$, $\Omega_{21,12}^{(e)} = \{45\})$, $\Omega_{21}^{(s)} = \phi$.

(c) $\Omega_{22} = \{45,35\}$ $(\Omega_2 - \Omega_{22} = \{34,25,24,23,15,14,13,12\})$

(c-1) $\Omega_{22}^{(e)} = \phi$ $(\Omega_{22,A}^{(e)} = \phi$ for each $A \in \Omega_2 - \Omega_{22})$, $\Omega_{22}^{(s)} = \{45,35\}$,

(c-2) $\Omega_{22}^{(e)} = \{45\}$ $(\Omega_{22,A}^{(e)} = \phi$ for each $A \in \{34,25,24,23,15,14,13\})$, $\Omega_{22,12}^{(e)} = \{45\})$, $\Omega_{22}^{(s)} = \{35\}$,

(c-3-1) $\Omega_{22}^{(e)} = \{45,35\}$ $(\Omega_{22,A}^{(e)} = \phi$ for each $A \in \{34,25,24,23,15,14,13\})$, $\Omega_{22,12}^{(e)} = \{45,35\})$, $\Omega_{22}^{(s)} = \phi$,

(c-3-2) $\Omega_{22}^{(e)} = \{45,35\}$ $(\Omega_{22,A}^{(e)} = \phi$ for each $A \in \{34,25,24,23,15,14\})$, $\Omega_{22,13}^{(e)} = \{45\}$, $\Omega_{22,12}^{(e)} = \{35\})$, $\Omega_{22}^{(s)} = \phi$.

(d) $\Omega_{23} = \{45,35,34\}$ $(\Omega_2 - \Omega_{23} = \{25,24,23,15,14,13,12\})$

(d-1) $\Omega_{23}^{(e)} = \phi$ $(\Omega_{23,A}^{(e)} = \phi$ for each $A \in \Omega_2 - \Omega_{23})$, $\Omega_{22}^{(s)} = \{45,35,34\}$,

(d-2) $\Omega_{23}^{(e)} = \{45\}$ $(\Omega_{23,A}^{(e)} = \phi$ for each $A \in \{25,24,23,15,14,13\})$, $\Omega_{23,12}^{(e)} = \{45\})$, $\Omega_{22}^{(s)} = \{35,34\}$,

(d-3-1) $\Omega_{23}^{(e)} = \{45,35\}$ $(\Omega_{23,A}^{(e)} = \phi$ for each $A \in \{25,24,23,15,14,13\})$, $\Omega_{23,12}^{(e)} = \{45,35\})$, $\Omega_{23}^{(s)} = \{34\}$,

(d-3-2) $\Omega_{23}^{(e)} = \{45,35\}$ $(\Omega_{23,A}^{(e)} = \phi$ for each $A \in \{25,24,23,15,14\})$, $\Omega_{23,13}^{(e)} = \{45\}$, $\Omega_{23,12}^{(e)} = \{35\})$, $\Omega_{23}^{(s)} = \{34\}$.

(d-4-1) $\Omega_{23}^{(e)} = \{45,35,34\}$ $(\Omega_{23,A}^{(e)} = \phi$ for each $A \in \{25,24,23,15,14,13\})$, $\Omega_{23,12}^{(e)} = \{45,35,34\})$, $\Omega_{23}^{(s)} = \phi$,

(d-4-2) $\Omega_{23}^{(e)} = \{45,35,34\}$ $(\Omega_{23,A}^{(e)} = \phi$ for each $A \in \{25,24,23,15,14\})$, $\Omega_{23,13}^{(e)} = \{45\}$, $\Omega_{23,12}^{(e)} = \{35,34\})$, $\Omega_{23}^{(s)} = \phi$,

(d-4-3) $\Omega_{23}^{(e)} = \{45,35,34\}$ $(\Omega_{23,A}^{(e)} = \phi$ for each $A \in \{25,24,23,15\})$, $\Omega_{23,14}^{(e)} = \{45\}$, $\Omega_{23,13}^{(e)} = \{35\})$, $\Omega_{23,12}^{(e)} = \{34\})$, $\Omega_{23}^{(s)} = \phi$,

(e) $\Omega_{24} = \{45,35,34,25\}$ $(\Omega_2 - \Omega_{24} = \{24,23,15,14,13,12\})$

(e-1) $\Omega_{24}^{(e)} = \phi$ $(\Omega_{24,A}^{(e)} = \phi$ for each $A \in \Omega_2 - \Omega_{24})$, $\Omega_{24}^{(s)} = \{45,35,34,25\}$,

(e-2) $\Omega_{24}^{(e)} = \{45\}$ $(\Omega_{24,A}^{(e)} = \phi$ for each $A \in \{24,23,15,14,13\})$, $\Omega_{24,12}^{(e)} = \{45\})$, $\Omega_{24}^{(s)} = \{35,34,25\}$,

(e-3-1) $\Omega_{24}^{(e)} = \{45,35\}$ $(\Omega_{24,A}^{(e)} = \phi$ for each $A \in \{24,23,15,14,13\})$, $\Omega_{24,12}^{(e)} = \{45,35\})$, $\Omega_{24}^{(s)} = \{34,25\}$,

(e-3-2) $\Omega_{24}^{(e)} = \{45,35\}$ $(\Omega_{24,A}^{(e)} = \phi$ for each $A \in \{24,23,15,14\})$, $\Omega_{24,13}^{(e)} = \{45\}$,

$\Omega^{(e)}_{24,12} = \{35\}), \Omega^{(s)}_{24} = \{34, 25\}$,

(e-4-1) $\Omega^{(e)}_{24} = \{45, 35, 34\}$ $(\Omega^{(e)}_{24,A} = \phi$ for each $A \in \{24, 23, 15, 14, 13\}), \Omega^{(e)}_{24,12}$
$= \{45, 35, 34\}), \Omega^{(s)}_{24} = \{25\}$,

(e-4-2) $\Omega^{(e)}_{24} = \{45, 35, 34\}$ $(\Omega^{(e)}_{24,A} = \phi$ for each $A \in \{24, 23, 15, 14\}), \Omega^{(e)}_{24,13}$
$= \{45\}, \Omega^{(e)}_{24,12} = \{35, 34\}), \Omega^{(s)}_{24} = \{25\}$,

(e-4-3) $\Omega^{(e)}_{24} = \{45, 35, 34\}$ $(\Omega^{(e)}_{24,A} = \phi$ for each $A \in \{24, 23, 15\}), \Omega^{(e)}_{24,14}$
$= \{45\}, \Omega^{(e)}_{24,13} = \{35\}, \Omega^{(e)}_{24,12} = \{34\}), \Omega^{(s)}_{24} = \{25\}$,

(e-5-1) $\Omega^{(e)}_{24} = \{45, 35, 34, 25\}$ $(\Omega^{(e)}_{24,A} = \phi$ for each $A \in \{24, 23, 15, 14, 13\}), \Omega^{(e)}_{24,12}$
$= \{45, 35, 34, 25\}), \Omega^{(s)}_{24} = \phi$,

(e-5-2) $\Omega^{(e)}_{24} = \{45, 35, 34, 25\}$ $(\Omega^{(e)}_{24,A} = \phi$ for each $A \in \{24, 23, 15, 14\}), \Omega^{(e)}_{24,13}$
$= \{45\}, \Omega^{(e)}_{24,12} = \{35, 34, 25\}), \Omega^{(s)}_{24} = \phi$,

(e-5-3) $\Omega^{(e)}_{24} = \{45, 35, 34, 25\}$ $(\Omega^{(e)}_{24,A} = \phi$ for each $A \in \{24, 23, 15, 14\}), \Omega^{(e)}_{24,13}$
$= \{45, 35\}, \Omega^{(e)}_{24,12} = \{34, 25\}), \Omega^{(s)}_{24} = \phi$,

(e-5-4) $\Omega^{(e)}_{24} = \{45, 35, 34, 25\}$ $(\Omega^{(e)}_{24,A} = \phi$ for each $A \in \{24, 23, 15\}), \Omega^{(e)}_{24,14}$
$= \{45\}, \Omega^{(e)}_{24,13} = \{35\}, \Omega^{(e)}_{24,12} = \{34, 25\}), \Omega^{(s)}_{24} = \phi$,

(e-5-5) $\Omega^{(e)}_{24} = \{45, 35, 34, 25\}$ $(\Omega^{(e)}_{24,A} = \phi$ for each $A \in \{24, 23\}), \Omega^{(e)}_{24,15}$
$= \{45\}, \Omega^{(e)}_{24,14} = \{35\}, \Omega^{(e)}_{24,13} = \{34\}, \Omega^{(e)}_{24,12} = \{25\}), \Omega^{(s)}_{24} = \phi$.

(f) $\Omega_{25} = \{45, 35, 34, 25, 24\}$ $(\Omega_2 - \Omega_{25} = \{23, 15, 14, 13, 12\})$

(f-1) $\Omega^{(e)}_{25} = \phi$ $(\Omega^{(e)}_{25,A} = \phi$ for each $A \in \Omega_2 - \Omega_{25}), \Omega^{(s)}_{25} = \{45, 35, 34, 25, 24\}$,

(f-2) $\Omega^{(e)}_{25} = \{45\}$ $(\Omega^{(e)}_{25,A} = \phi$ for each $A \in \{23, 15, 14, 13\}), \Omega^{(e)}_{25,12} = \{45\})$,
$\Omega^{(s)}_{25} = \{35, 34, 25, 24\}$,

(f-3-1) $\Omega^{(e)}_{25} = \{45, 35\}$ $(\Omega^{(e)}_{25,A} = \phi$ for each $A \in \{23, 15, 14, 13\}), \Omega^{(e)}_{25,12}$
$= \{45, 35\}), \Omega^{(s)}_{25} = \{34, 25, 24\}$,

(f-3-2) $\Omega^{(e)}_{25} = \{45, 35\}$ $(\Omega^{(e)}_{25,A} = \phi$ for each $A \in \{23, 15, 14\}), \Omega^{(e)}_{25,13} = \{45\}$,
$\Omega^{(e)}_{25,12} = \{35\}), \Omega^{(s)}_{25} = \{34, 25, 24\}$,

(f-4-1) $\Omega^{(e)}_{25} = \{45, 35, 34\}$ $(\Omega^{(e)}_{25,A} = \phi$ for each $A \in \{23, 15, 14, 13\}), \Omega^{(e)}_{25,12}$
$= \{45, 35, 34\}), \Omega^{(s)}_{24} = \{25, 24\}$,

(f-4-2) $\Omega^{(e)}_{25} = \{45, 35, 34\}$ $(\Omega^{(e)}_{25,A} = \phi$ for each $A \in \{23, 15, 14\}), \Omega^{(e)}_{25,13}$
$= \{45\}, \Omega^{(e)}_{25,12} = \{35, 34\}), \Omega^{(s)}_{25} = \{25, 24\}$,

(f-4-3) $\Omega^{(e)}_{25} = \{45, 35, 34\}$ $(\Omega^{(e)}_{25,A} = \phi$ for each $A \in \{23, 15\}), \Omega^{(e)}_{25,14}$
$= \{45\}, \Omega^{(e)}_{25,13} = \{35\}, \Omega^{(e)}_{25,12} = \{34\}), \Omega^{(s)}_{25} = \{25, 24\}$,

(f-5-1) $\Omega^{(e)}_{25} = \{45, 35, 34, 25\}$ $(\Omega^{(e)}_{25,A} = \phi$ for each $A \in \{23, 15, 14, 13\}), \Omega^{(e)}_{25,12}$
$= \{45, 35, 34, 25\}), \Omega^{(s)}_{25} = \{24\}$,

(f-5-2) $\Omega^{(e)}_{25} = \{45, 35, 34, 25\}$ $(\Omega^{(e)}_{25,A} = \phi$ for each $A \in \{23, 15, 14\}), \Omega^{(e)}_{25,13}$
$= \{45\}, \Omega^{(e)}_{25,12} = \{35, 34, 25\}), \Omega^{(s)}_{25} = \{24\}$,

(f-5-3) $\Omega^{(e)}_{25} = \{45, 35, 34, 25\}$ $(\Omega^{(e)}_{25,A} = \phi$ for each $A \in \{23, 15, 14\}), \Omega^{(e)}_{25,13}$

$= \{45,35\}, \Omega^{(e)}_{25,12} = \{34,25\}), \Omega^{(s)}_{25} = \{24\}$,

(f-5-4) $\Omega^{(e)}_{25} = \{45,35,34,25\}$ $(\Omega^{(e)}_{25,A} = \phi$ for each $A \in \{23,15\}), \Omega^{(e)}_{25,14}$
$= \{45\}, \Omega^{(e)}_{25,13} = \{35\}, \Omega^{(e)}_{25,12} = \{34,25\}), \Omega^{(s)}_{25} = \{24\}$,

(f-5-5) $\Omega^{(e)}_{25} = \{45,35,34,25\}$ $(\Omega^{(e)}_{25,23} = \phi, \Omega^{(e)}_{25,15}$
$= \{45\}, \Omega^{(e)}_{25,14} = \{35\}, \Omega^{(e)}_{25,13} = \{34\}, \Omega^{(e)}_{25,12} = \{25\}), \Omega^{(s)}_{25} = \{24\}$,

(f-6-1) $\Omega^{(e)}_{25} = \{45,35,34,25,24\}$ $(\Omega^{(e)}_{25,A} = \phi$ for each $A \in \{23,15,14,13\}), \Omega^{(e)}_{25,12}$
$= \{45,35,34,25,24\}), \Omega^{(s)}_{25} = \phi$,

(f-6-2) $\Omega^{(e)}_{25} = \{45,35,34,25,24\}$ $(\Omega^{(e)}_{25,A} = \phi$ for each $A \in \{23,15,14\}), \Omega^{(e)}_{25,13}$
$= \{45\}, \Omega^{(e)}_{25,12} = \{35,34,25,24\}), \Omega^{(s)}_{25} = \phi$,

(f-6-3) $\Omega^{(e)}_{25} = \{45,35,34,25,24\}$ $(\Omega^{(e)}_{25,A} = \phi$ for each $A \in \{23,15,14\}), \Omega^{(e)}_{25,13}$
$= \{45,35\}, \Omega^{(e)}_{25,12} = \{34,25,24\}), \Omega^{(s)}_{25} = \phi$,

(f-6-4) $\Omega^{(e)}_{25} = \{45,35,34,25,24\}$ $(\Omega^{(e)}_{25,A} = \phi$ for each $A \in \{23,15\}), \Omega^{(e)}_{25,14}$
$= \{45\}, \Omega^{(e)}_{25,13} = \{35\}, \Omega^{(e)}_{24,12} = \{34,25,24\}), \Omega^{(s)}_{25} = \phi$,

(f-6-5) $\Omega^{(e)}_{25} = \{45,35,34,25,24\}$ $(\Omega^{(e)}_{25,A} = \phi$ for each $A \in \{23,15\}), \Omega^{(e)}_{25,14}$
$= \{45\}, \Omega^{(e)}_{25,13} = \{35,34\}, \Omega^{(e)}_{24,12} = \{25,24\}), \Omega^{(s)}_{25} = \phi$,

(f-6-6) $\Omega^{(e)}_{25} = \{45,35,34,25,24\}$ $(\Omega^{(e)}_{25,23} = \phi, \Omega^{(e)}_{25,15}$
$= \{45\}, \Omega^{(e)}_{25,14} = \{35\}, \Omega^{(e)}_{25,13} = \{34\}, \Omega^{(e)}_{25,12} = \{25,24\}), \Omega^{(s)}_{25} = \phi$,

(f-6-7) $\Omega^{(e)}_{25} = \{45,35,34,25,24\}$ $(\Omega^{(e)}_{25,23} = \{45\}, \Omega^{(e)}_{25,15}$
$= \{35\}, \Omega^{(e)}_{25,14} = \{34\}, \Omega^{(e)}_{25,13} = \{25\}, \Omega^{(e)}_{25,12} = \{24\}), \Omega^{(s)}_{25} = \phi$.

(g) $\Omega_{26} = \{45,35,34,25,24,23\}$ $(\Omega_2 - \Omega_{26} = \{15,14,13,12\})$

(g-1) $\Omega^{(e)}_{26} = \phi$ $(\Omega^{(e)}_{26,A} = \phi$ for each $A \in \Omega_2 - \Omega_{26}), \Omega^{(s)}_{26} = \{45,35,34,25,24,23\}$,

(g-2) $\Omega^{(e)}_{26} = \{45\}$ $(\Omega^{(e)}_{26,A} = \phi$ for each $A \in \{15,14,13\}, \Omega^{(e)}_{26,12} = \{45\})$,
$\Omega^{(s)}_{26} = \{35,34,25,24,23\}$,

(g-3-1) $\Omega^{(e)}_{26} = \{45,35\}$ $(\Omega^{(e)}_{26,A} = \phi$ for each $A \in \{15,14,13\}, \Omega^{(e)}_{26,12}$
$= \{45,35\}), \Omega^{(s)}_{26} = \{34,25,24,23\}$,

(g-3-2) $\Omega^{(e)}_{26} = \{45,35\}$ $(\Omega^{(e)}_{26,A} = \phi$ for each $A \in \{15,14\}, \Omega^{(e)}_{26,13} = \{45\}$,
$\Omega^{(e)}_{26,12} = \{35\}), \Omega^{(s)}_{26} = \{34,25,24,23\}$,

(g-4-1) $\Omega^{(e)}_{26} = \{45,35,34\}$ $(\Omega^{(e)}_{26,A} = \phi$ for each $A \in \{23,15,14,13\}), \Omega^{(e)}_{26,12}$
$= \{45,35,34\}), \Omega^{(s)}_{26} = \{25,24,23\}$,

(g-4-2) $\Omega^{(e)}_{26} = \{45,35,34\}$ $(\Omega^{(e)}_{26,A} = \phi$ for each $A \in \{23,15,14\}), \Omega^{(e)}_{26,13}$
$= \{45\}, \Omega^{(e)}_{26,12} = \{35,34\}), \Omega^{(s)}_{26} = \{25,24,23\}$,

(g-4-3) $\Omega^{(e)}_{26} = \{45,35,34\}$ $(\Omega^{(e)}_{26,A} = \phi$ for each $A \in \{23,15\}), \Omega^{(e)}_{26,14}$
$= \{45\}, \Omega^{(e)}_{26,13} = \{35\}, \Omega^{(e)}_{26,12} = \{34\}), \Omega^{(s)}_{26} = \{25,24,23\}$,

(g-5-1) $\Omega^{(e)}_{26} = \{45,35,34,25\}$ $(\Omega^{(e)}_{26,A} = \phi$ for each $A \in \{15,14,13\}), \Omega^{(e)}_{26,12}$
$= \{45,35,34,25\}), \Omega^{(s)}_{26} = \{24,23\}$,

(g-5-2) $\Omega^{(e)}_{26} = \{45,35,34,25\}$ $(\Omega^{(e)}_{26,A} = \phi$ for each $A \in \{15,14\}), \Omega^{(e)}_{26,13}$

$= \{45\}, \Omega^{(e)}_{26,12} = \{35,34,25\}), \Omega^{(s)}_{26} = \{24,23\}$,

(g-5-3) $\Omega^{(e)}_{26} = \{45,35,34,25\} \quad (\Omega^{(e)}_{26,A} = \phi \text{ for each } A \in \{15,14\}), \Omega^{(e)}_{26,13}$
$= \{45,35\}, \Omega^{(e)}_{26,12} = \{34,25\}), \Omega^{(s)}_{26} = \{24,23\}$,

(g-5-4) $\Omega^{(e)}_{26} = \{45,35,34,25\} \quad (\Omega^{(e)}_{26,A} = \phi \text{ for each } A \in \{15\}), \Omega^{(e)}_{26,14}$
$= \{45\}, \Omega^{(e)}_{26,13} = \{35\}, \Omega^{(e)}_{26,12} = \{34,25\}), \Omega^{(s)}_{26} = \{24,23\}$,

(g-5-5) $\Omega^{(e)}_{26} = \{45,35,34,25\} \quad (\Omega^{(e)}_{26,15} = \{45\}, \Omega^{(e)}_{26,14} = \{35\}, \Omega^{(e)}_{26,13} = \{34\}$,
$\Omega^{(e)}_{26,12} = \{25\}), \Omega^{(s)}_{26} = \{24,23\}$,

(g-6-1) $\Omega^{(e)}_{26} = \{45,35,34,25,24\} \quad (\Omega^{(e)}_{26,A} = \phi \text{ for each } A \in \{15,14,13\}), \Omega^{(e)}_{26,12}$
$= \{45,35,34,25,24\}), \Omega^{(s)}_{26} = \{23\}$,

(g-6-2) $\Omega^{(e)}_{26} = \{45,35,34,25,24\} \quad (\Omega^{(e)}_{26,A} = \phi \text{ for each } A \in \{15,14\}), \Omega^{(e)}_{26,13}$
$= \{45\}, \Omega^{(e)}_{26,12} = \{35,34,25,24\}), \Omega^{(s)}_{26} = \{23\}$,

(g-6-3) $\Omega^{(e)}_{26} = \{45,35,34,25,24\} \quad (\Omega^{(e)}_{26,A} = \phi \text{ for each } A \in \{15,14\}), \Omega^{(e)}_{26,13}$
$= \{45,35\}, \Omega^{(e)}_{26,12} = \{34,25,24\}), \Omega^{(s)}_{26} = \{23\}$,

(g-6-4) $\Omega^{(e)}_{26} = \{45,35,34,25,24\} \quad (\Omega^{(e)}_{26,15} = \phi, \Omega^{(e)}_{26,14} = \{45\}, \Omega^{(e)}_{26,13} = \{35\}$,
$\Omega^{(e)}_{26,12} = \{34,25,24\}), \Omega^{(s)}_{26} = \{23\}$,

(g-6-5) $\Omega^{(e)}_{26} = \{45,35,34,25,24\} \quad (\Omega^{(e)}_{26,15} = \phi, \Omega^{(e)}_{26,14} = \{45\}, \Omega^{(e)}_{26,13} = \{35,34\}$,
$\Omega^{(e)}_{26,12} = \{25,24\}), \Omega^{(s)}_{26} = \{23\}$,

(g-6-6) $\Omega^{(e)}_{26} = \{45,35,34,25,24\} \quad (\Omega^{(e)}_{26,15} = \{45\}, \Omega^{(e)}_{26,14} = \{35\}, \Omega^{(e)}_{26,13} = \{34\}$,
$\Omega^{(e)}_{26,12} = \{25,24\}), \Omega^{(s)}_{26} = \{23\}$,

(g-7-1) $\Omega^{(e)}_{26} = \{45,35,34,25,24,23\} \quad (\Omega^{(e)}_{26,A} = \phi \text{ for each } A \in \{15,14,13\}), \Omega^{(e)}_{26,12}$
$= \{45,35,34,25,24,23\}), \Omega^{(s)}_{26} = \phi$,

(g-7-2) $\Omega^{(e)}_{26} = \{45,35,34,25,24,23\} \quad (\Omega^{(e)}_{26,A} = \phi \text{ for each } A \in \{15,14\}), \Omega^{(e)}_{26,13}$
$= \{45\}, \Omega^{(e)}_{26,12} = \{35,34,25,24,23\}), \Omega^{(s)}_{26} = \phi$,

(g-7-3) $\Omega^{(e)}_{26} = \{45,35,34,25,24,23\} \quad (\Omega^{(e)}_{26,A} = \phi \text{ for each } A \in \{15,14\}), \Omega^{(e)}_{26,13}$
$= \{45,35\}, \Omega^{(e)}_{26,12} = \{34,25,24,23\}), \Omega^{(s)}_{26} = \phi$,

(g-7-4) $\Omega^{(e)}_{26} = \{45,35,34,25,24,23\} \quad (\Omega^{(e)}_{26,A} = \phi \text{ for each } A \in \{15,14\}), \Omega^{(e)}_{26,13}$
$= \{45,35,34\}, \Omega^{(e)}_{26,12} = \{25,24,23\}), \Omega^{(s)}_{26} = \phi$,

(g-7-5) $\Omega^{(e)}_{26} = \{45,35,34,25,24,23\} \quad (\Omega^{(e)}_{26,15} = \phi, \Omega^{(e)}_{26,14} = \{45\}, \Omega^{(e)}_{26,13} = \{35\}$,
$\Omega^{(e)}_{26,12} = \{34,25,24,23\}), \Omega^{(s)}_{26} = \phi$,

(g-7-6) $\Omega^{(e)}_{26} = \{45,35,34,25,24,23\} \quad (\Omega^{(e)}_{26,15} = \phi, \Omega^{(e)}_{26,14} = \{45\}, \Omega^{(e)}_{26,13} = \{35,34\}$,
$\Omega^{(e)}_{26,12} = \{25,24,23\}), \Omega^{(s)}_{26} = \phi$,

(g-7-7) $\Omega^{(e)}_{26} = \{45,35,34,25,24,23\} \quad (\Omega^{(e)}_{26,15} = \phi, \Omega^{(e)}_{26,14} = \{45,35\}, \Omega^{(e)}_{26,13} = \{34,25\}$,
$\Omega^{(e)}_{26,12} = \{24,23\}), \Omega^{(s)}_{26} = \phi$,

(g-7-8) $\Omega^{(e)}_{26} = \{45,35,34,25,24,23\} \quad (\Omega^{(e)}_{26,15} = \{45\}, \Omega^{(e)}_{26,14} = \{35\}, \Omega^{(e)}_{26,13} = \{34\}$,
$\Omega^{(e)}_{26,12} = \{25,24,23\}), \Omega^{(s)}_{26} = \phi$,

(g-7-9) $\Omega^{(e)}_{26} = \{45,35,34,25,24,23\} \quad (\Omega^{(e)}_{26,15} = \{45\}, \Omega^{(e)}_{26,14} = \{35\}, \Omega^{(e)}_{26,13} = \{34,25\}$,

$\Omega^{(e)}_{26,12} = \{24,23\}), \Omega^{(s)}_{26} = \phi$.

(h) $\Omega_{27} = \{45,35,34,25,24,23,15\}$ $(\Omega_2 - \Omega_{27} = \{14,13,12\})$

(h-1) $\Omega^{(e)}_{27} = \phi$ $(\Omega^{(e)}_{27,A} = \phi$ for each $A \in \Omega_2 - \Omega_{27}), \Omega^{(s)}_{27} = \{45,35,34,25,24,23,15\}$,

(h-2) $\Omega^{(e)}_{27} = \{45\}$ $(\Omega^{(e)}_{27,A} = \phi$ for each $A \in \{14,13\}), \Omega^{(e)}_{27,12} = \{45\})$,
$\Omega^{(s)}_{27} = \{35,34,25,24,23,15\}$,

(h-3-1) $\Omega^{(e)}_{27} = \{45,35\}$ $(\Omega^{(e)}_{27,A} = \phi$ for each $A \in \{14,13\}), \Omega^{(e)}_{27,12}$
$= \{45,35\}), \Omega^{(s)}_{27} = \{34,25,24,23,15\}$,

(h-3-2) $\Omega^{(e)}_{27} = \{45,35\}$ $(\Omega^{(e)}_{27,14} = \phi$, $\Omega^{(e)}_{27,13} = \{45\}, \Omega^{(e)}_{27,12} = \{35\})$,
$\Omega^{(s)}_{27} = \{34,25,24,23,15\}$,

(h-4-1) $\Omega^{(e)}_{27} = \{45,35,34\}$ $(\Omega^{(e)}_{27,A} = \phi$ for each $A \in \{14,13\}), \Omega^{(e)}_{27,12}$
$= \{45,35,34\}), \Omega^{(s)}_{27} = \{25,24,23,15\}$,

(h-4-2) $\Omega^{(e)}_{27} = \{45,35,34\}$ $(\Omega^{(e)}_{27,14} = \phi, \Omega^{(e)}_{27,13} = \{45\}, \Omega^{(e)}_{27,12} = \{35,34\})$,
$\Omega^{(s)}_{27} = \{25,24,23,15\}$,

(h-4-3) $\Omega^{(e)}_{27} = \{45,35,34\}$ $(\Omega^{(e)}_{27,14} = \{45\}, \Omega^{(e)}_{27,13} = \{35\}, \Omega^{(e)}_{27,12} = \{34\})$,
$\Omega^{(s)}_{27} = \{25,24,23\}$,

(h-5-1) $\Omega^{(e)}_{27} = \{45,35,34,25\}$ $(\Omega^{(e)}_{27,A} = \phi$ for each $A \in \{14,13\}), \Omega^{(e)}_{27,12}$
$= \{45,35,34,25\}), \Omega^{(s)}_{27} = \{24,23,15\}$,

(h-5-2) $\Omega^{(e)}_{27} = \{45,35,34,25\}$ $(\Omega^{(e)}_{27,14} = \phi, \Omega^{(e)}_{27,13} = \{45\}, \Omega^{(e)}_{27,12} = \{35,34,25\})$,
$\Omega^{(s)}_{27} = \{24,23,15\}$,

(h-5-3) $\Omega^{(e)}_{27} = \{45,35,34,25\}$ $(\Omega^{(e)}_{27,14} = \phi, \Omega^{(e)}_{27,13} = \{45,35\}, \Omega^{(e)}_{27,12} = \{34,25\})$,
$\Omega^{(s)}_{27} = \{24,23,15\}$,

(h-5-4) $\Omega^{(e)}_{27} = \{45,35,34,25\}$ $(\Omega^{(e)}_{27,14} = \{45\}, \Omega^{(e)}_{27,13} = \{35\}, \Omega^{(e)}_{27,12} = \{34,25\})$,
$\Omega^{(s)}_{27} = \{24,23,15\}$,

(h-6-1) $\Omega^{(e)}_{27} = \{45,35,34,25,24\}$ $(\Omega^{(e)}_{27,A} = \phi$ for each $A \in \{14,13\}), \Omega^{(e)}_{27,12}$
$= \{45,35,34,25,24\}), \Omega^{(s)}_{27} = \{23,15\}$,

(h-6-2) $\Omega^{(e)}_{27} = \{45,35,34,25,24\}$ $(\Omega^{(e)}_{27,14} = \phi, \Omega^{(e)}_{27,13} = \{45\}, \Omega^{(e)}_{27,12} = \{35,34,25,24\})$,
$\Omega^{(s)}_{27} = \{23,15\}$,

(h-6-3) $\Omega^{(e)}_{27} = \{45,35,34,25,24\}$ $(\Omega^{(e)}_{27,14} = \phi, \Omega^{(e)}_{27,13} = \{45,35\}, \Omega^{(e)}_{27,12} = \{34,25,24\})$,
$\Omega^{(s)}_{27} = \{23,15\}$,

(h-6-4) $\Omega^{(e)}_{27} = \{45,35,34,25,24\}$ $(\Omega^{(e)}_{27,14} = \{45\}, \Omega^{(e)}_{27,13} = \{35\}, \Omega^{(e)}_{27,12} = \{34,25,24\})$,
$\Omega^{(s)}_{27} = \{23,15\}$,

(h-6-5) $\Omega^{(e)}_{27} = \{45,35,34,25,24\}$ $(\Omega^{(e)}_{27,14} = \{45\}, \Omega^{(e)}_{27,13} = \{35,34\}, \Omega^{(e)}_{27,12} = \{25,24\})$,
$\Omega^{(s)}_{27} = \{23,15\}$,

(h-7-1) $\Omega^{(e)}_{27} = \{45,35,34,25,24,23\}$ $(\Omega^{(e)}_{27,A} = \phi$ for each $A \in \{14,13\}), \Omega^{(e)}_{27,12}$
$= \{45,35,34,25,24,23\}), \Omega^{(s)}_{27} = \{15\}$,

(h-7-2) $\Omega^{(e)}_{27} = \{45,35,34,25,24,23\}$ $(\Omega^{(e)}_{27,14} = \phi, \Omega^{(e)}_{27,13} = \{45\}, \Omega^{(e)}_{27,12} = \{35,34,25,24,23\})$,
$\Omega^{(s)}_{27} = \{15\}$,

(h-7-3) $\Omega_{27}^{(e)}=\{45,35,34,25,24,23\}$ $(\Omega_{27,14}^{(e)}=\phi,\Omega_{27,13}^{(e)}=\{45,35\},\Omega_{27,12}^{(e)}=\{34,25,24,23\})$, $\Omega_{27}^{(s)}=\{15\}$,

(h-7-4) $\Omega_{27}^{(e)}=\{45,35,34,25,24,23\}$ $(\Omega_{27,14}^{(e)}=\phi,\Omega_{27,13}^{(e)}=\{45,35,34\},\Omega_{27,12}^{(e)}=\{25,24,23\})$, $\Omega_{27}^{(s)}=\{15\}$,

(h-7-5) $\Omega_{27}^{(e)}=\{45,35,34,25,24,23\}$ $(\Omega_{27,14}^{(e)}=\{45\},\Omega_{27,13}^{(e)}=\{35\},\Omega_{27,12}^{(e)}=\{34,25,24,23\})$, $\Omega_{27}^{(s)}=\{15\}$,

(h-7-6) $\Omega_{27}^{(e)}=\{45,35,34,25,24,23\}$ $(\Omega_{27,14}^{(e)}=\{45\},\Omega_{27,13}^{(e)}=\{35,34\},\Omega_{27,12}^{(e)}=\{25,24,23\})$, $\Omega_{27}^{(s)}=\{15\}$,

(h-7-7) $\Omega_{27}^{(e)}=\{45,35,34,25,24,23\}$ $(\Omega_{27,14}^{(e)}=\{45,35\},\Omega_{27,13}^{(e)}=\{34,25\},\Omega_{27,12}^{(e)}=\{24,23\})$, $\Omega_{27}^{(s)}=\{15\}$,

(h-8-1) $\Omega_{27}^{(e)}=\{45,35,34,25,24,23,15\}$ $(\Omega_{27,A}^{(e)}=\phi$ for each $A\in\{14,13\}),\Omega_{27,12}^{(e)}$ $=\{45,35,34,25,24,23,15\}),\Omega_{27}^{(s)}=\phi$,

(h-8-2) $\Omega_{27}^{(e)}=\{45,35,34,25,24,23,15\}$ $(\Omega_{27,14}^{(e)}=\phi,\Omega_{27,13}^{(e)}=\{45\},\Omega_{27,12}^{(e)}=\{35,34,25,24,23,15\})$, $\Omega_{27}^{(s)}=\phi$,

(h-8-3) $\Omega_{27}^{(e)}=\{45,35,34,25,24,23,15\}$ $(\Omega_{27,14}^{(e)}=\phi,\Omega_{27,13}^{(e)}=\{45,35\},\Omega_{27,12}^{(e)}=\{34,25,24,23,15\})$, $\Omega_{27}^{(s)}=\phi$,

(h-8-4) $\Omega_{27}^{(e)}=\{45,35,34,25,24,23,15\}$ $(\Omega_{27,14}^{(e)}=\phi,\Omega_{27,13}^{(e)}=\{45,35,34\},\Omega_{27,12}^{(e)}=\{25,24,23,15\})$, $\Omega_{27}^{(s)}=\phi$,

(h-8-5) $\Omega_{27}^{(e)}=\{45,35,34,25,24,23,15\}$ $(\Omega_{27,14}^{(e)}=\{45\},\Omega_{27,13}^{(e)}=\{35\},\Omega_{27,12}^{(e)}=\{34,25,24,23,15\})$, $\Omega_{27}^{(s)}=\phi$,

(h-8-6) $\Omega_{27}^{(e)}=\{45,35,34,25,24,23,15\}$ $(\Omega_{27,14}^{(e)}=\{45\},\Omega_{27,13}^{(e)}=\{35,34\},\Omega_{27,12}^{(e)}=\{25,24,23,15\})$, $\Omega_{27}^{(s)}=\phi$,

(h-8-7) $\Omega_{27}^{(e)}=\{45,35,34,25,24,23,15\}$ $(\Omega_{27,14}^{(e)}=\{45\},\Omega_{27,13}^{(e)}=\{35,34,24\}$, $\Omega_{27,12}^{(e)}=\{24,23,15\}),\Omega_{27}^{(s)}=\phi$,

(h-8-8) $\Omega_{27}^{(e)}=\{45,35,34,25,24,23,15\}$ $(\Omega_{27,14}^{(e)}=\{45,35\},\Omega_{27,13}^{(e)}=\{34,24\}$, $\Omega_{27,12}^{(e)}=\{24,23,15\}),\Omega_{27}^{(s)}=\phi$.

(i) $\Omega_{28}=\{45,35,34,25,24,23,15,14\}$ $(\Omega_2-\Omega_{28}=\{13,12\})$

(i-1) $\Omega_{28}^{(e)}=\phi$ $(\Omega_{28,A}^{(e)}=\phi$ for each $A\in\Omega_2-\Omega_{28}),\Omega_{28}^{(s)}=\{45,35,34,25,24,23,15,14\}$,

(i-2) $\Omega_{28}^{(e)}=\{45\}$ $(\Omega_{28,13}^{(e)}=\phi,\Omega_{28,12}^{(e)}=\{45\}),\Omega_{28}^{(s)}=\{35,34,25,24,23,15,14\}$,

(i-3-1) $\Omega_{28}^{(e)}=\{45,35\}$ $(\Omega_{28,13}^{(e)}=\phi,\Omega_{28,12}^{(e)}=\{45,35\}),\Omega_{28}^{(s)}=\{34,25,24,23,15,14\}$,

(i-3-2) $\Omega_{28}^{(e)}=\{45,35\}$ $(\Omega_{28,13}^{(e)}=\{45\},\Omega_{28,12}^{(e)}=\{35\}),\Omega_{28}^{(s)}=\{34,25,24,23,15,14\}$,

(i-4-1) $\Omega_{28}^{(e)}=\{45,35,34\}$ $(\Omega_{28,13}^{(e)}=\phi,\Omega_{28,12}^{(e)}=\{45,35,34\}),\Omega_{28}^{(s)}=\{25,24,23,15,14\}$,

(i-4-2) $\Omega_{28}^{(e)}=\{45,35,34\}$ $(\Omega_{28,13}^{(e)}=\{45\},\Omega_{28,12}^{(e)}=\{35,34\}),\Omega_{28}^{(s)}=\{25,24,23,15,14\}$,

(i-5-1) $\Omega_{28}^{(e)}=\{45,35,34,25\}$ $(\Omega_{28,13}^{(e)}=\phi,\Omega_{28,12}^{(e)}=\{45,35,34,25\}),\Omega_{28}^{(s)}=\{24,23,15,14\}$,

(i-5-2) $\Omega_{28}^{(e)}=\{45,35,34,25\}$ $(\Omega_{28,13}^{(e)}=\{45\},\Omega_{28,12}^{(e)}=\{35,34,25\}),\Omega_{28}^{(s)}=\{24,23,15,14\}$,

(i-5-3) $\Omega_{28}^{(e)}=\{45,35,34,25\}$ $(\Omega_{28,13}^{(e)}=\{45,35\},\Omega_{28,12}^{(e)}=\{34,25\}),\Omega_{28}^{(s)}=\{24,23,15,14\}$,

(i-6-1) $\Omega_{28}^{(e)}=\{45,35,34,25,24\}$ $(\Omega_{28,13}^{(e)}=\phi,\Omega_{28,12}^{(e)}=\{45,35,34,25,24\}),\Omega_{28}^{(s)}=\{23,15,14\}$, (i-6-2) $\Omega_{28}^{(e)} = \{45,35,34,25,24\}$ $(\Omega_{28,13}^{(e)} = \{45\}, \Omega_{28,12}^{(e)} = \{35,34,25,24\}), \Omega_{28}^{(s)} = \{23,15,14\}$,

(i-6-3) $\Omega_{28}^{(e)} = \{45,35,34,25,24\}$ $(\Omega_{28,13}^{(e)} = \{45,35\}, \Omega_{28,12}^{(e)} = \{34,25,24\}), \Omega_{28}^{(s)} = \{23,15,14\}$,

(i-7-1) $\Omega_{28}^{(e)} = \{45,35,34,25,24,23\}$ $(\Omega_{28,13}^{(e)} = \phi, \Omega_{28,12}^{(e)} = \{45,35,34,25,24,23\})$,
$\Omega_{28}^{(s)} = \{15,14\}$,

(i-7-2) $\Omega_{28}^{(e)} = \{45,35,34,25,24,23\}$ $(\Omega_{28,13}^{(e)} = \{45\}, \Omega_{28,12}^{(e)} = \{35,34,25,24,23\})$,
$\Omega_{28}^{(s)} = \{15,14\}$,

(i-7-3) $\Omega_{28}^{(e)} = \{45,35,34,25,24,23\}$ $(\Omega_{28,13}^{(e)} = \{45,35\}, \Omega_{28,12}^{(e)} = \{34,25,24,23\})$,
$\Omega_{28}^{(s)} = \{15,14\}$,

(i-7-4) $\Omega_{28}^{(e)} = \{45,35,34,25,24,23\}$ $(\Omega_{28,13}^{(e)} = \{45,35,34\}, \Omega_{28,12}^{(e)} = \{25,24,23\})$,
$\Omega_{28}^{(s)} = \{15,14\}$,

(i-8-1) $\Omega_{28}^{(e)} = \{45,35,34,25,24,23,15\}$ $(\Omega_{28,13}^{(e)} = \phi, \Omega_{28,12}^{(e)} = \{45,35,34,25,24,23,15\})$,
$\Omega_{28}^{(s)} = \{14\}$,

(i-8-2) $\Omega_{28}^{(e)} = \{45,35,34,25,24,23,15\}$ $(\Omega_{28,13}^{(e)} = \{45\}, \Omega_{28,12}^{(e)} = \{35,34,25,24,23,15\})$,
$\Omega_{28}^{(s)} = \{14\}$,

(i-8-3) $\Omega_{28}^{(e)} = \{45,35,34,25,24,23,15\}$ $(\Omega_{28,13}^{(e)} = \{45,35\}, \Omega_{28,12}^{(e)} = \{34,25,24,23,15\})$,
$\Omega_{28}^{(s)} = \{14\}$,

(i-8-4) $\Omega_{28}^{(e)} = \{45,35,34,25,24,23,15\}$ $(\Omega_{28,13}^{(e)} = \{45,35,34\}, \Omega_{28,12}^{(e)} = \{25,24,23,15\})$,
$\Omega_{28}^{(s)} = \{14\}$,

(i-9-1) $\Omega_{28}^{(e)} = \{45,35,34,25,24,23,15,14\}$ $(\Omega_{28,13}^{(e)} = \phi, \Omega_{28,12}^{(e)} = \{45,35,34,25,24,23,15,14\})$,
$\Omega_{28}^{(s)} = \phi$,

(i-9-2) $\Omega_{28}^{(e)} = \{45,35,34,25,24,23,15,14\}$ $(\Omega_{28,13}^{(e)} = \{45\}, \Omega_{28,12}^{(e)} = \{35,34,25,24,23,15,14\})$,
$\Omega_{28}^{(s)} = \phi$,

(i-9-3) $\Omega_{28}^{(e)} = \{45,35,34,25,24,23,15,14\}$ $(\Omega_{28,13}^{(e)} = \{45,35\}, \Omega_{28,12}^{(e)} = \{34,25,24,23,15,14\})$,
$\Omega_{28}^{(s)} = \phi$,

(i-9-4) $\Omega_{28}^{(e)} = \{45,35,34,25,24,23,15,14\}$ $(\Omega_{28,13}^{(e)} = \{45,35,34\}, \Omega_{28,12}^{(e)} = \{25,24,23,15,14\})$,
$\Omega_{28}^{(s)} = \phi$,

(i-9-5) $\Omega_{28}^{(e)} = \{45,35,34,25,24,23,15,14\}$ $(\Omega_{28,13}^{(e)} = \{45,35,34,25\}, \Omega_{28,12}^{(e)} = \{24,23,15,14\})$,
$\Omega_{28}^{(s)} = \phi$.

(j) $\Omega_{29} = \{45,35,34,25,24,23,15,14,13\}$ $(\Omega_2 - \Omega_{29} = \{12\})$

(j-1) $\Omega_{29}^{(e)} = \phi$ $(\Omega_{29,12}^{(e)} = \phi), \Omega_{29}^{(s)} = \{45,35,34,25,24,23,15,14,13\}$,

(j-2) $\Omega_{29}^{(e)} = \{45\}$ $(\Omega_{29,12}^{(e)} = \{45\}), \Omega_{29}^{(s)} = \{35,34,25,24,23,15,14,13\}$,

(j-3) $\Omega_{29}^{(e)} = \{45,35\}$ $(\Omega_{29,12}^{(e)} = \{45,35\}), \Omega_{29}^{(s)} = \{34,25,24,23,15,14,13\}$,

(j-4) $\Omega_{29}^{(e)} = \{45,35,34\}$ $(\Omega_{29,12}^{(e)} = \{45,35,34\}), \Omega_{29}^{(s)} = \{25,24,23,15,14,13\}$,

(j-5) $\Omega_{29}^{(e)} = \{45,35,34,25\}$ $(\Omega_{29,12}^{(e)} = \{45,35,34,25\}), \Omega_{29}^{(s)} = \{24,23,15,14,13\}$,

(j-6) $\Omega_{29}^{(e)} = \{45,35,34,25,24\}$ $(\Omega_{29,12}^{(e)} = \{45,35,34,25,24\}), \Omega_{29}^{(s)} = \{23,15,14,13\}$,

(j-7) $\Omega_{29}^{(e)} = \{45,35,34,25,24,23\}$ $(\Omega_{29,12}^{(e)} = \{45,35,34,25,24,23\}), \Omega_{29}^{(s)} = \{15,14,13\}$,

(j-8) $\Omega_{29}^{(e)} = \{45,35,34,25,24,23,15\}$ $(\Omega_{29,12}^{(e)} = \{45,35,34,25,24,23,15\}), \Omega_{29}^{(s)} = \{14,13\}$,

(j-9) $\Omega_{29}^{(e)} = \{45, 35, 34, 25, 24, 23, 15, 14\}$ $(\Omega_{29,12}^{(e)} = \{45, 35, 34, 25, 24, 23, 15, 14\})$,
$\Omega_{29}^{(s)} = \{13\}$,
(j-10) $\Omega_{29}^{(e)} = \{45, 35, 34, 25, 24, 23, 15, 14, 13\}$ $(\Omega_{29,12}^{(e)} = \{45, 35, 34, 25, 24, 23, 15, 14, 13\})$,
$\Omega_{29}^{(s)} = \phi$,
(k) $\Omega_{210} = \{45, 35, 34, 25, 24, 23, 15, 14, 13, 12\}$ $(\Omega_2 - \Omega_{210} = \phi)$
(k-1) $\Omega_{210}^{(e)} = \phi, \Omega_{210}^{(s)} = \Omega_{210}$.

4. Conclusion

In [12], identities of symmetry in two variables for Bernoulli polynomials and power sums and those for Euler polynomials and alternating power sums, which had been shown by manipulating suitable symmetric identities, were derived respectively by employing the p-adic Volkenborn integrals and the fermionic p-adic integrals.

Not much later, it was observed in [10,11] that the identities of symmetry in two variables can be extended to those in three variables again by using those two kinds of p-adic integrals. We mentioned that the abundant identities of symmetry in three variables, by specializing one of the three variables as 1, shed new light even on the existing identities in two variables.

In this paper, we generalized the identities of symmetry in three variables for Euler polynomials and alternating power sums to those in arbitrary number of variables in a suitable setting. We proved our main result, Theorem 2.3, and illustrated our result with some examples. Introducing p-adic integrals in search for symmetries was a breakthrough in the sense that this approach allows one to find truly adundant symmetries, which seem like impossible with the 'classical' method of manipulating suitable symmetric identities.

As was noted in [10] and recalled in Example 3.1, the number of distinct identities of symmetry in three variables is not always 6=$|S_3|$, but it is 1, 2, 3, or 6. This is because the identities of symmetry in each case are obtained by permuting w_1, w_2, w_3 in any one equation and hence there is a transitive action of S_3 on those set of symmetric identities. Thus it is equal to the quotient $|S_3|/|H|$, where H is a subgroup of S_3.

We think that it is an interesting probelm to determine the possible numbers of distinct symmetries. We leave this as a challenging problem for the interested reader. Similar results for other special polynomials together with the corresponding suitable power sums will appear in forthcoming papers.

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References

1. S. Araci, U. Duran, M. Acikgoz, *A study on a class of q -Euler polynomials under the symmetric group of degree n,* J. Nonlinear Sci. Appl. **9** (2016), no. 8, 5196-5201.
2. J. Choi, P. J. Anderson, H. M. Srivastava, *Carlitz's q-Bernoulli and q-Euler numbers and polynomials and a class of q-Hurwitz zeta functions,* Appl. Math. Comput. **215** (2009), no. 3, 1185-1208.
3. E. Deeba, D. Rodriguez, *Stirling's and Bernoulli numbers,* Amer. Math. Monthly **98** (1991), 423–426.
4. U. Duran, M. Acikgoz, S. Araci, *Symmetric identities involving weighted q-Genocchi polynomials under S_4,* Proc. Jangjeon Math. Soc. **18** (2015), no. 4, 455-465.
5. Y. He, *Symmetric identities for Calitzs q-Bernoulli numbers and polynomials,* Adv. Differ. Equ. **2013**, 2013:246, 10pp.
6. Y. He, W. Zhang, *A convolution formula for Bernoulli polynomials,* Ars Combin. 108 (2013), 97-104.

7. D. S. Kim, T. Kim, *Three variable symmetric identities involving Carlitz-type q-Euler polynomials,* Math. Sci. **8** (2014), no. 4, 147-152.
8. D. S. Kim, N. Lee, J. Na, K. H. Park, *Abundant symmetry for higher-order Bernoulli polynomials (I),* Adv. Stud. Contemp. Math. (Kyungshang) **23** (2013), no. 3, 461–482.
9. D. S. Kim, N. Lee, J. Na, K. H. Park, *Abundant symmetry for higher-order Bernoulli polynomials (II),* Proc. Jangjeon Math. Soc. **16** (2013), no. 3, 359–378.
10. D. S. Kim, K. H. Park, *Identities of symmetry for Euler polynomials arising from quotients of fermionic integrals invariant under* S_3*,* J. Inequal. Appl. **2010**, Art. ID 851521, 16 pp.
11. D. S. Kim, K. H. Park, *Identities of symmetry for Bernoulli polynomials arising from quotients of Volkenborn integrals invariant under* S_3*,* Appl. Math. Comput. **219** (2013), no. 10, 5096-5104.
12. T. Kim, *Symmetry p-adic invariant integral on* $\mathbb{Z}_p$ *for Bernoulli and Euler polynomials,* J. Difference Equ. Appl. **14** (2008), 1267-1277.
13. T. Kim, *q-Volkenborn integration,* Russ. J. Math. Phys. **9** (2002), no. 3, 288–299.
14. T. Kim, *Symmetry of power sum polynomials and multivariate fermionic p-adic invariant integral on* $\mathbb{Z}_p$*,* Russ. J. Math. Phys. **16** (2009), no. 1, 93-96.
15. T. Kim, D. V. Dolgy, D. S. Kim, *Some identities of q-Bernoulli polynomials under symmetry group* S_3*,* J. Nonlinear Convex Anal. **16** (2015), no. 9, 1869–1880.
16. T. Kim, D. S. Kim, *Identities of symmetry for degenerate Euler polynomials and alternating generalized falling factorial sums,* Iran. J. Sci. Technol. Trans. A Sci. **41** (2017), no. 4, 939-949.
17. T. Kim, D. S. Kim, H. Y. Kim, J. Kwon, *Identities of symmetry for Bernoulli polynomials and power sums,* J. Inequal. Appl. **2020**, 2020:245, 12 pp.
18. H. Ozden and Y. Simsek, *A new extension of q-Euler numbers and polynomials related to their interpolation functions,* Appl. Math. Lett. **21** (2008), no. 9, 934–939.
19. W. H. Schikhof, *Ultrametric calculus: An introduction to p-adic analysis,* Cambridge University Press, 2006.
20. Y. Simsek, *Complete sum of products of* (h,q)*-extension of Euler polynomials and numbers,* J. Difference Equ. Appl. **16** (2010), no. 11, 1331–1348.
21. H. Tuenter, *A symmetry of power sum polynomials and Bernoulli numbers,* Amer. Math. Monthly **108** (2001), 258–261.
22. S. Yang, *An identity of symmetry for the Bernoulli polynomials,* Discrete Math. **308** (2008), 550–554.

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