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FRACTIONAL CALCULUS OPERATORS APPLIED TO THE FUNCTION INVOLVING THE PRODUCT OF SRIVASTAVA POLYNOMIALS AND INCOMPLETE I-FUNCTIONS

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Abstract. In this study, we investigate the Marichev-Saigo-Maeda fractional order differentiation and integral for the function pertaining the product of Srivastava polynomials and incomplete I-functions. Moreover, their special cases are also depicted in terms of the corollaries associated with Saigo, Riemann-Liouville, and Erdelyi-Kober fractional operators.

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1. Introduction and Preliminary

These days, the theory of incomplete functions are evidently used in the various field of science and engineering for solving many applied problems ([2], [19]). In this article, we use incomplete I-function which is the generalized form of I-function described by Rathie [20], which are the expansion formulae of well recognized Fox's H-function [5] and many related functions. In the theory of special functions, fractional calculus is widely used in the field of science and engineering due to its application in wireless communication, mathematical modeling, statistical distribution etc., see [1]-[4],[6]-[8],[11],[12],[14]-[15],[17]-[18],[21],[24],[26].
Recently, new classes of incomplete I-functions have been investigated by Jangid et al. [9]. The incomplete I-functions ${}^{\gamma}I_{p,q}^{m,n}(z)$ and ${}^{\Gamma}I_{p,q}^{m,n}(z)$ are defined by Jangid et al. [10] as follows:

$$(1)\qquad {}^{\gamma}I_{p,q}^{m,n}(z) = {}^{\gamma}I_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (c_1,\zeta_1;C_1:x),(c_2,\zeta_2;C_2),\cdots,(c_p,\zeta_p;C_p) \\ (d_1,\eta_1;D_1),\cdots,(d_q,\eta_q;D_q) \end{matrix}\right]$$
$$= {}^{\gamma}I_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (c_1,\zeta_1;C_1:x),(c_j,\zeta_j;C_j)_{2,p} \\ (d_j,\eta_j;D_j)_{1,q} \end{matrix}\right]$$
$$= \frac{1}{2\pi i}\int_L \phi(s,x) z^s ds,$$

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and

(2)
$$\begin{aligned}
{}^{\Gamma}I_{p,q}^{m,n}(z) &= {}^{\Gamma}I_{p,q}^{m,n}\left[z \left| \begin{array}{c} (c_1,\zeta_1;C_1:x),(c_2,\zeta_2;C_2),\cdots,(c_p,\zeta_p;C_p) \\ (d_1,\eta_1;D_1),\cdots,(d_q,\eta_q;D_q) \end{array} \right.\right] \\
&= {}^{\Gamma}I_{p,q}^{m,n}\left[z \left| \begin{array}{c} (c_1,\zeta_1;C_1:x),(c_j,\zeta_j;C_j)_{2,p} \\ (d_j,\eta_j;D_j)_{1,q} \end{array} \right.\right] \\
&= \frac{1}{2\pi i}\int_L \Phi(s,x) z^s ds,
\end{aligned}$$

for all $z \neq 0$ where

(3)
$$\phi(s,x) = \frac{\{\gamma(1-c_1+\zeta_1 s,x)\}^{C_1}\prod_{j=1}^{m}\{\Gamma(d_j-\eta_j s)\}^{D_j}\prod_{j=2}^{n}\{\Gamma(1-c_j+\zeta_j s)\}^{C_j}}{\prod_{j=m+1}^{q}\{\Gamma(1-d_j+\eta_j s)\}^{D_j}\prod_{j=n+1}^{p}\{\Gamma(c_j-\zeta_j s)\}^{C_j}},$$

and

(4)
$$\Phi(s,x) = \frac{\{\Gamma(1-c_1+\zeta_1 s,x)\}^{C_1}\prod_{j=1}^{m}\{\Gamma(d_j-\eta_j s)\}^{D_j}\prod_{j=2}^{n}\{\Gamma(1-c_j+\zeta_j s)\}^{C_j}}{\prod_{j=m+1}^{q}\{\Gamma(1-d_j+\eta_j s)\}^{D_j}\prod_{j=n+1}^{p}\{\Gamma(c_j-\zeta_j s)\}^{C_j}}.$$

Where the incomplete gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$ defined as follows:

$$\gamma(s,x) = \int_0^x t^{s-1}e^{-t}\,dt \quad (\Re(s)>0; x\geq 0)\,, \tag{5}$$

and

$$\Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}\,dt \quad (x\geq 0; \Re(s)>0 \,\text{when}\, x=0)\,, \tag{6}$$

known as lower and upper incomplete gamma functions respectively.
The incomplete I-functions defined in (1) and (2) are exists for all $x \geq 0$ under the same contour and circumstance defined by Rathie [20]. When $C_1 = 1$, the following decomposition formula also holds true for the incomplete I-functions defined as

$${}^{\gamma}I_{p,q}^{m,n}(z) + {}^{\Gamma}I_{p,q}^{m,n}(z) = I_{p,q}^{m,n}(z),\ (C_1 = 1). \tag{7}$$

The Srivastava polynomials ([25], p.1, equation(1)) is defined as follows:

$$S_v^u(y) = \sum_{l=0}^{[v/u]}\frac{(-v)_{u,l}}{l!}A_{v,l}y^l,\ \ v=0,1,2...,u\in\mathbb{Z}^+, \tag{8}$$

where $A_{v,l}$ is any bounded complex sequence.

Now, we mention the fractional calculus operators associated with the Appell function F_3, which we use in the series to obain the key effects. For $\delta,\delta',\theta,\theta',\xi\in\mathbb{C}$ and $x>0$ with $\Re(\xi)>0$, we have

(9)
$$\begin{aligned}
&\left(I_{0+}^{\delta,\delta',\theta,\theta',\xi}f\right)(y) \\
&= \frac{y^{-\delta}}{\Gamma(\xi)}\int_0^y (y-x)^{\xi-1}x^{-\delta'}F_3\left(\delta,\delta',\theta,\theta';\xi;1-\frac{x}{y},1-\frac{y}{x}\right)f(x)dx
\end{aligned}$$

and

$$(10)\quad \left(I_{-}^{\delta,\delta',\theta,\theta',\xi}f\right)(y)$$
$$=\frac{y^{-\delta'}}{\Gamma(\xi)}\int_y^\infty (x-y)^{\xi-1}x^{-\delta}\,F_3\left(\delta,\delta',\theta,\theta';\xi;1-\frac{y}{x},1-\frac{x}{y}\right)f(x)dx,$$

which are known as left and right hand sided MSM fractional integral operators, respectively [16]. Also the MSM fractional differential operators [23] for left and right hand sided are defined as

$$(11)\quad \left(D_{0+}^{\delta,\delta',\theta,\theta',\xi}f\right)(y)=\left(\frac{d}{dy}\right)^{\alpha}\left(I_{0+}^{-\delta',-\delta,-\theta'+\alpha,-\theta,-\xi+\alpha}f\right)(y)$$

and

$$(12)\quad \left(D_{-}^{\delta,\delta',\theta,\theta',\xi}f\right)(y)=\left(-\frac{d}{dy}\right)^{\alpha}\left(I_{-}^{-\delta',-\delta,-\theta',-\theta+\alpha,-\xi+\alpha}f\right)(y),$$

where $\alpha=[\Re(\xi)]+1$.

Now we shall use the following well-known results [23] in our findings.

Lemma 1.1. *Let* $\delta,\delta',\theta,\theta',\xi,\epsilon\in\mathbb{C};\Re(\xi)>0$ *and* $\Re(\epsilon)>\max.\left\{0,\Re(\delta-\delta'-\theta-\xi),\Re(\delta'-\theta')\right\}$, *then*

$$(13)\quad \left(I_{0+}^{\delta,\delta',\theta,\theta',\xi}t^{\epsilon-1}\right)(y)$$
$$=\frac{\Gamma(\epsilon)\,\Gamma(-\delta'+\theta'+\epsilon)\,\Gamma(-\delta-\delta'-\theta+\xi+\epsilon)y^{-\delta-\delta'+\xi+\epsilon-1}}{\Gamma(\theta'+\epsilon)\,\Gamma(-\delta-\delta'+\xi+\epsilon)\,\Gamma(-\delta'-\theta+\xi+\epsilon)}\,.$$

Lemma 1.2. *Let* $\delta,\delta',\theta,\theta',\xi,\epsilon\in\mathbb{C};\Re(\xi)>0$ *and* $\Re(\epsilon)>\max.\left\{\Re(\theta),\Re(-\delta-\delta'+\xi),\Re(-\delta-\theta'+\xi)\right\}$, *then*

$$(14)\quad \left(I_{-}^{\delta,\delta',\theta,\theta',\xi}t^{-\epsilon}\right)(y)$$
$$=\frac{\Gamma(-\theta+\epsilon)\,\Gamma(\delta+\delta'-\xi+\epsilon)\,\Gamma(\delta+\theta'-\xi+\epsilon)y^{-\delta-\delta'+\xi-\epsilon}}{\Gamma(\epsilon)\,\Gamma(\delta-\theta+\epsilon)\,\Gamma(\delta+\delta'+\theta'-\xi+\epsilon)}\,.$$

Lemma 1.3. *Let* $\delta,\delta',\theta,\theta',\xi,\epsilon\in\mathbb{C}$ *and* $\Re(\epsilon)>\max.\left\{0,\Re(-\delta+\theta),\Re(-\delta-\delta'-\theta'+\xi)\right\}$, *then*

$$(15)\quad \left(D_{0+}^{\delta,\delta',\theta,\theta',\xi}t^{\epsilon-1}\right)(y)$$
$$=\frac{\Gamma(\epsilon)\,\Gamma(\delta-\theta+\epsilon)\,\Gamma(\delta+\delta'+\theta'-\xi+\epsilon)y^{\delta+\delta'-\xi+\epsilon-1}}{\Gamma(-\theta+\epsilon)\,\Gamma(\delta+\delta'-\xi+\epsilon)\,\Gamma(\delta+\theta'-\xi+\epsilon)}\,.$$

Lemma 1.4. *Let* $\delta, \delta^{'}, \theta, \theta^{'}, \xi, \epsilon \in \mathbb{C}$ *and* $\Re(\epsilon) > \max.\left\{\Re(-\theta^{'}), \Re(\delta^{'} + \theta - \xi), \Re(\delta + \theta^{'} - \xi) + [\Re(\xi)] + 1\right\}$, *then*

$$\left(D_{-}^{\delta,\delta^{'},\theta,\theta^{'},\xi} t^{-\epsilon}\right)(y) = \frac{\Gamma(\theta^{'} + \epsilon)\,\Gamma(-\delta - \delta' + \xi + \epsilon)\,\Gamma(-\delta^{'} - \theta + \xi + \epsilon) y^{\delta+\delta'-\xi+\epsilon}}{\Gamma(\epsilon)\,\Gamma(-\delta^{'} + \theta^{'} + \epsilon)\,\Gamma(-\delta - \delta' - \theta + \xi + \epsilon)}. \tag{16}$$

Next, the left and right hand sided Saigo fractional integral operators [22] are defined for $y > 0$ and $\delta, \theta, \xi, \in \mathbb{C}, \Re(\delta) > 0$, respectively, as follow:

$$\left(I_{0+}^{\delta,\theta,\xi} f\right)(y) = \frac{y^{-\delta-\theta}}{\Gamma(\delta)} \int_0^y (y-t)^{\delta-1}\, {}_2F_1\left(\delta + \theta, -\xi; \delta; 1 - \frac{t}{y}\right) f(t)dt \tag{17}$$

and

$$\left(I_{-}^{\delta,\theta,\xi} f\right)(y) = \frac{1}{\Gamma(\delta)} \int_y^\infty (t-y)^{\delta-1} t^{-\delta-\theta}\, {}_2F_1\left(\delta + \theta, -\xi; \delta; 1 - \frac{y}{t}\right) f(t)dt. \tag{18}$$

Further, the left and right hand sided Saigo fractional differential operators are as follow:

$$\left(D_{0+}^{\delta,\theta,\xi} f\right)(y) = \left(\frac{d}{dy}\right)^{\beta} \left(I_{0+}^{-\delta+\beta,-\theta-\beta,\delta+\xi-\beta} f\right)(y) \tag{19}$$

and

$$\left(D_{-}^{\delta,\theta,\xi} f\right)(y) = \left(-\frac{d}{dy}\right)^{\beta} \left(I_{-}^{-\delta+\beta,-\theta-\beta,\delta+\xi} f\right)(y), \tag{20}$$

where $\beta = [\Re(\delta)] + 1$.

For $\theta = -\delta$ and $\theta = 0$ in (17)-(20), the fractional operators of Riemann-Liouville and Erdelyi-Kober type are obtained respectively (for more details see [13]).

Also, the MSM fractional operators (9)-(12) are connected to Saigo operators (17)-(20)by

$$\left(I_{0+}^{\delta,0,\theta,\theta^{'},\xi} f\right)(y) = \left(I_{0+}^{\xi,\delta-\xi,-\theta} f\right)(y),\ \left(I_{-}^{\delta,0,\theta,\theta^{'},\xi} f\right)(y) = \left(I_{-}^{\xi,\delta-\xi,-\theta} f\right)(y), \tag{21}$$

and

$$\left(D_{0+}^{0,\delta^{'},\theta,\theta^{'},\xi} f\right)(y) = \left(D_{0+}^{\xi,\delta^{'}-\xi,\theta^{'}-\xi} f\right)(y), \tag{22}$$

$$\left(D_{-}^{0,\delta^{'},\theta,\theta^{'},\xi} f\right)(y) = \left(D_{-}^{\xi,\delta^{'}-\xi,\theta^{'}-\xi} f\right)(y).$$

The goal of this analysis is to investigate the differentiation and integration pertaining to the product of Srivastava polynomials and incomplete I-functions, associated with the fractional order Marichev-Saigo-Maeda (MSM) operators. Different cases of the main outcomes are also discussed.

2. Main Results

In this section we investigate the MSM fractional integration and differentiation for the product of Srivastava polynomials and incomplete I-functions. Fisrt we define results for the left-hand sided MSM fractional order integrals of the product of Srivastava polynomials and incomplete I-functions.

Theorem 2.1. *Let $\delta, \delta', \theta, \theta', \xi, \epsilon, k \in \mathbb{C}; \Re(\xi), \tau > 0$ and $\Re(\epsilon) > \max.\left\{0, \Re(\delta' - \theta'), \Re(\delta + \delta' + \theta - \xi)\right\}$. Then, for $y > 0$, we have*

(23)

$$\left(I_{0+}^{\delta,\delta',\theta,\theta',\xi}\, z^{\epsilon-1} S_v^u(z)\, {}^{\Gamma}I_{p,q}^{m,n}\left[kz^{\tau} \,\middle|\, \begin{matrix} (c_1,\zeta_1;C_1:x),(c_j,\zeta_j;C_j)_{2,p} \\ (d_j,\eta_j;D_j)_{1,q} \end{matrix}\right]\right)(y)$$

$$= y^{-\delta-\delta'+\xi+\epsilon-1} \sum_{l=0}^{[v/u]} \frac{(-v)_{u,l}}{l!} A_{v,l}\, y^l \times$$

$$ {}^{\Gamma}I_{p+3,q+3}^{m,n+3}\left[ky^{\tau} \,\middle|\, \begin{matrix} (c_1,\zeta_1;C_1:x),(1-\epsilon-l,\tau;1),(1+\delta'-\theta'-\epsilon-l,\tau;1), \\ (d_j,\eta_j;D_j)_{1,q},(1-\theta'-\epsilon-l,\tau;1),(1+\delta+\delta'-\xi-\epsilon-l,\tau;1), \end{matrix}\right.$$

$$\left.\begin{matrix} ,(1+\delta+\delta'+\theta-\xi-\epsilon-l,\tau;1),(c_j,\zeta_j;C_j)_{2,p} \\ ,(1+\delta'+\theta-\xi-\epsilon-l,\tau;1) \end{matrix}\right].$$

Proof. We start from the L.H.S of (23), and using equation (2) and (8), we have

$$\left(I_{0+}^{\delta,\delta',\theta,\theta',\xi}\, z^{\epsilon-1} \sum_{l=0}^{[v/u]} \frac{(-v)_{u,l}}{l!} A_{v,l} z^l\, \frac{1}{2\pi\iota} \int_L \Phi(s,x)\, k^s z^{\tau s} ds\right)(y).$$

By altering the order of summation and integration, we have

$$\sum_{l=0}^{[v/u]} \frac{(-v)_{u,l}}{l!} A_{v,l}\, \frac{1}{2\pi\iota} \int_L \Phi(s,x)\, k^s \left(I_{0+}^{\delta,\delta',\theta,\theta',\xi}\, z^{\epsilon+l+\tau s-1}\right)(y)\, ds.$$

Now by applying the Lemma 1.1, we have

(24)

$$\sum_{l=0}^{[v/u]} \frac{(-v)_{u,l}}{l!} A_{v,l}\, \frac{1}{2\pi\iota} \int_L \Phi(s,x)\, k^s\, y^{-\delta-\delta'+\xi+\epsilon+l+\tau s-1} \times$$

$$\frac{\Gamma(\epsilon+l+\tau s)\,\Gamma(-\delta'+\theta'+\epsilon+l+\tau s)\,\Gamma(-\delta-\delta'-\theta+\xi+\epsilon+l+\tau s)}{\Gamma(\theta'+\epsilon+l+\tau s)\,\Gamma(-\delta-\delta'+\xi+\epsilon+l+\tau s)\,\Gamma(-\delta'-\theta+\xi+\epsilon+l+\tau s)}\, ds.$$

Using (2) in (24), we get (23). □

Similarly the following result can also be obtained for ${}^{\gamma}I_{p,q}^{m,n}$, we stated here without proof.

Theorem 2.2. *Let* $\delta, \delta', \theta, \theta', \xi, \epsilon, k \in \mathbb{C}; \Re(\xi), \tau > 0$ *and* $\Re(\epsilon) > \max.\left\{0, \Re(\delta' - \theta'), \Re(\delta + \delta' + \theta - \xi)\right\}$. *Then, for* $y > 0$, *we have*

(25)

$$\left(I_{0+}^{\delta,\delta',\theta,\theta',\xi}\, z^{\epsilon-1} S_v^u(z)\ {}^{\gamma}I_{p,q}^{m,n}\left[kz^{\tau}\,\middle|\,\begin{matrix}(c_1,\zeta_1;C_1:x),(c_j,\zeta_j;C_j)_{2,p}\\(d_j,\eta_j;D_j)_{1,q}\end{matrix}\right]\right)(y)$$
$$= y^{-\delta-\delta'+\xi+\epsilon-1}\sum_{l=0}^{[v/u]}\frac{(-v)_{u,l}}{l!}A_{v,l}\, y^l \times$$
$${}^{\gamma}I_{p+3,q+3}^{m,n+3}\left[ky^{\tau}\,\middle|\,\begin{matrix}(c_1,\zeta_1;C_1:x),(1-\epsilon-l,\tau;1),(1+\delta'-\theta'-\epsilon-l,\tau;1),\\(d_j,\eta_j;D_j)_{1,q},(1-\theta'-\epsilon-l,\tau;1),(1+\delta+\delta'-\xi-\epsilon-l,\tau;1),\end{matrix}\right.$$
$$\left.\begin{matrix},(1+\delta+\delta'+\theta-\xi-\epsilon-l,\tau;1),(c_j,\zeta_j;C_j)_{2,p}\\,(1+\delta'+\theta-\xi-\epsilon-l,\tau;1)\end{matrix}\right].$$

Now we obtain the results for right-hand sided MSM fractional order integrals of the product of Srivastava polynomials and incomplete I-functions.

Theorem 2.3. *Let* $\delta, \delta', \theta, \theta', \xi, \epsilon, k \in \mathbb{C}; \Re(\xi), \tau > 0$ *and* $\Re(\epsilon) > \max.\left\{\Re(\theta), \Re(-\delta - \delta' + \xi), \Re(-\delta - \theta' + \xi)\right\}$. *Then, for* $y > 0$, *we have*

(26)

$$\left(I_{-}^{\delta,\delta',\theta,\theta',\xi}\, z^{-\epsilon} S_v^u(z)\ {}^{\Gamma}I_{p,q}^{m,n}\left[kz^{-\tau}\,\middle|\,\begin{matrix}(c_1,\zeta_1;C_1:x),(c_j,\zeta_j;C_j)_{2,p}\\(d_j,\eta_j;D_j)_{1,q}\end{matrix}\right]\right)(y)$$
$$= y^{-\delta-\delta'+\xi-\epsilon}\sum_{l=0}^{[v/u]}\frac{(-v)_{u,l}}{l!}A_{v,l}\, y^l \times$$
$${}^{\Gamma}I_{p+3,q+3}^{m,n+3}\left[ky^{-\tau}\,\middle|\,\begin{matrix}(c_1,\zeta_1;C_1:x),(1+\theta+l-\epsilon,\tau;1),(1-\delta-\delta'+\xi+l-\epsilon,\tau;1),\\(d_j,\eta_j;D_j)_{1,q},(1+l-\epsilon,\tau;1),\end{matrix}\right.$$
$$\left.\begin{matrix}(1-\delta-\theta'+\xi+l-\epsilon,\tau;1),(c_j,\zeta_j;C_j)_{2,p}\\(1-\delta+\theta+l-\epsilon,\tau;1),(1-\delta-\delta'-\theta'+\xi+l-\epsilon,\tau;1)\end{matrix}\right].$$

Proof. We start from the L.H.S of (26), and using equation (2) and (8), we have

$$\left(I_{-}^{\delta,\delta',\theta,\theta',\xi}\, z^{-\epsilon}\sum_{l=0}^{[v/u]}\frac{(-v)_{u,l}}{l!}A_{v,l}z^l\ \frac{1}{2\pi\iota}\int_L \Phi(s,x)\,k^s z^{-\tau s}ds\right)(y).$$

By altering the order of summation and integration, we have

$$\sum_{l=0}^{[v/u]}\frac{(-v)_{u,l}}{l!}A_{v,l}\ \frac{1}{2\pi\iota}\int_L \Phi(s,x)\,k^s\left(I_{-}^{\delta,\delta',\theta,\theta',\xi}\, z^{-(\epsilon-l+\tau s)}\right)(y)\,ds.$$

Now by applying the Lemma 1.2, we have

(27)

$$\sum_{l=0}^{[v/u]} \frac{(-v)_{u,l}}{l!} A_{v,l} \, \frac{1}{2\pi\iota} \int_L \Phi(s,x)\, k^s \, y^{-\delta-\delta'+\xi-\epsilon+l-\tau s} \times$$
$$\frac{\Gamma(-\theta+\epsilon-l+\tau s)\,\Gamma(\delta+\delta'-\xi+\epsilon-l+\tau s)\,\Gamma(\delta+\theta'-\xi+\epsilon-l+\tau s)}{\Gamma(\epsilon-l+\tau s)\,\Gamma(\delta-\theta+\epsilon-l+\tau s)\,\Gamma(\delta+\delta'+\theta'-\xi+\epsilon-l+\tau s)}\, ds.$$

Using (2) in (27), we get (26). □

Theorem 2.4. *Let* $\delta, \delta', \theta, \theta', \xi, \epsilon, k \in \mathbb{C}; \Re(\xi), \tau > 0$ *and* $\Re(\epsilon) > \max.\left\{\Re(\theta), \Re(-\delta-\delta'+\xi), \Re(-\delta-\theta'+\xi)\right\}$. *Then, for* $y > 0$, *we have*

(28)

$$\left(I_{-}^{\delta,\delta',\theta,\theta',\xi}\, z^{-\epsilon} S_v^u(z)\; {}^{\gamma}I_{p,q}^{m,n}\left[kz^{-\tau} \,\middle|\, \begin{matrix} (c_1,\zeta_1;C_1:x),(c_j,\zeta_j;C_j)_{2,p} \\ (d_j,\eta_j;D_j)_{1,q} \end{matrix}\right]\right)(y)$$
$$= y^{-\delta-\delta'+\xi-\epsilon} \sum_{l=0}^{[v/u]} \frac{(-v)_{u,l}}{l!} A_{v,l}\, y^l \times$$
$${}^{\gamma}I_{p+3,q+3}^{m,n+3}\left[ky^{-\tau} \,\middle|\, \begin{matrix} (c_1,\zeta_1;C_1:x),(1+\theta+l-\epsilon,\tau;1),(1-\delta-\delta'+\xi+l-\epsilon,\tau;1), \\ (d_j,\eta_j;D_j)_{1,q},(1+l-\epsilon,\tau;1), \end{matrix}\right.$$
$$\left.\begin{matrix} (1-\delta-\theta'+\xi+l-\epsilon,\tau;1),(c_j,\zeta_j;C_j)_{2,p} \\ (1-\delta+\theta+l-\epsilon,\tau;1),(1-\delta-\delta'-\theta'+\xi+l-\epsilon,\tau;1) \end{matrix}\right].$$

Now the upcoming results for the left-hand sided MSM fractional order derivative of the product of Srivastava polynomials and incomplete I-functions.

Theorem 2.5. *Let* $\delta, \delta', \theta, \theta', \xi, \epsilon, k \in \mathbb{C}; \tau > 0$ *and* $\Re(\epsilon) > \max.\left\{0, \Re(-\delta+\theta), \Re(-\delta-\delta'-\theta'+\xi)\right\}$. *Then, for* $y > 0$, *we have*

(29)

$$\left(D_{0+}^{\delta,\delta',\theta,\theta',\xi}\, z^{\epsilon-1} S_v^u(z)\; {}^{\Gamma}I_{p,q}^{m,n}\left[kz^{\tau} \,\middle|\, \begin{matrix} (c_1,\zeta_1;C_1:x),(c_j,\zeta_j;C_j)_{2,p} \\ (d_j,\eta_j;D_j)_{1,q} \end{matrix}\right]\right)(y)$$
$$= y^{\delta+\delta'-\xi+\epsilon-1} \sum_{l=0}^{[v/u]} \frac{(-v)_{u,l}}{l!} A_{v,l}\, y^l \times$$
$${}^{\Gamma}I_{p+3,q+3}^{m,n+3}\left[ky^{\tau} \,\middle|\, \begin{matrix} (c_1,\zeta_1;C_1:x),(1-\epsilon-l,\tau;1),(1-\delta+\theta-\epsilon-l,\tau;1), \\ (d_j,\eta_j;D_j)_{1,q},(1+\theta-\epsilon-l,\tau;1), \end{matrix}\right.$$
$$\left.\begin{matrix} ,(1-\delta-\delta'-\theta'+\xi-\epsilon-l,\tau;1),(c_j,\zeta_j;C_j)_{2,p} \\ ,(1-\delta-\delta'+\xi-\epsilon-l,\tau;1),(1-\delta-\theta'+\xi-\epsilon-l,\tau;1) \end{matrix}\right].$$

Proof. We start from the L.H.S of (29), and using equation (2) and (8), we have

$$\left(D_{0+}^{\delta,\delta',\theta,\theta',\xi}\, z^{\epsilon-1}\sum_{l=0}^{[v/u]}\frac{(-v)_{u,l}}{l!}A_{v,l}z^{l}\,\frac{1}{2\pi\iota}\int_{L}\Phi(s,x)\,k^{s}z^{\tau s}ds\right)(y).$$

By altering the order of summation and integration, we have

$$\sum_{l=0}^{[v/u]}\frac{(-v)_{u,l}}{l!}A_{v,l}\,\frac{1}{2\pi\iota}\int_{L}\Phi(s,x)\,k^{s}\left(D_{0+}^{\delta,\delta',\theta,\theta',\xi}\, z^{\epsilon+l+\tau s-1}\right)(y)\,ds.$$

Now by applying the Lemma 1.3, we have

(30)

$$\sum_{l=0}^{[v/u]}\frac{(-v)_{u,l}}{l!}A_{v,l}\,\frac{1}{2\pi\iota}\int_{L}\Phi(s,x)\,k^{s}\,y^{\delta+\delta'-\xi+\epsilon+l+\tau s-1}\times$$
$$\frac{\Gamma(\epsilon+l+\tau s)\,\Gamma(\delta-\theta+\epsilon+l+\tau s)\,\Gamma(\delta+\delta'+\theta'-\xi+\epsilon+l+\tau s)}{\Gamma(-\theta+\epsilon+l+\tau s)\,\Gamma(\delta+\delta'-\xi+\epsilon+l+\tau s)\,\Gamma(\delta+\theta'-\xi+\epsilon+l+\tau s)}\,ds.$$

Using (2) in (30), we get (29). □

Theorem 2.6. *Let* $\delta,\delta',\theta,\theta',\xi,\epsilon,k\in\mathbb{C};\tau>0$ *and* $\Re(\epsilon)>\max.\left\{0,\Re(-\delta+\theta),\Re(-\delta-\delta'-\theta'+\xi)\right\}$. *Then, for* $y>0$, *we have*

(31)

$$\left(D_{0+}^{\delta,\delta',\theta,\theta',\xi}\, z^{\epsilon-1}S_{v}^{u}(z)\;{}^{\gamma}I_{p,q}^{m,n}\left[kz^{\tau}\,\middle|\,\begin{matrix}(c_1,\zeta_1;C_1:x),(c_j,\zeta_j;C_j)_{2,p}\\(d_j,\eta_j;D_j)_{1,q}\end{matrix}\right]\right)(y)$$
$$=y^{\delta+\delta'-\xi+\epsilon-1}\sum_{l=0}^{[v/u]}\frac{(-v)_{u,l}}{l!}A_{v,l}\,y^{l}\times$$
$${}^{\gamma}I_{p+3,q+3}^{m,n+3}\left[ky^{\tau}\,\middle|\,\begin{matrix}(c_1,\zeta_1;C_1:x),(1-\epsilon-l,\tau;1),(1-\delta+\theta-\epsilon-l,\tau;1),\\(d_j,\eta_j;D_j)_{1,q},(1+\theta-\epsilon-l,\tau;1),\end{matrix}\right.$$
$$\left.\begin{matrix},(1-\delta-\delta'-\theta'+\xi-\epsilon-l,\tau;1),(c_j,\zeta_j;C_j)_{2,p}\\,(1-\delta-\delta'+\xi-\epsilon-l,\tau;1),(1-\delta-\theta'+\xi-\epsilon-l,\tau;1)\end{matrix}\right].$$

Now following results for the right-hand sided MSM fractional order derivative of the product of Srivastava polynomials and incomplete I-functions.

Theorem 2.7. *Let* $\delta,\delta',\theta,\theta',\xi,\epsilon,k\in\mathbb{C};\tau>0$ *and* $\Re(\epsilon)>\max.\left\{\Re(-\theta'),\Re(\delta'+\theta-\xi),\Re(\delta+\theta'-\xi)+[\Re(\xi)]+1\right\}$. *Then, for*

$y > 0$, *we have*

(32)

$$\left(D_{-}^{\delta,\delta',\theta,\theta',\xi} z^{-\epsilon} S_v^u(z)\, {}^{\Gamma}I_{p,q}^{m,n}\left[kz^{-\tau}\left|\begin{array}{c}(c_1,\zeta_1;C_1:x),(c_j,\zeta_j;C_j)_{2,p}\\(d_j,\eta_j;D_j)_{1,q}\end{array}\right.\right]\right)(y)$$
$$= y^{\delta+\delta'-\xi-\epsilon}\sum_{l=0}^{[v/u]}\frac{(-v)_{u,l}}{l!}A_{v,l}\,y^l\times$$
$${}^{\Gamma}I_{p+3,q+3}^{m,n+3}\left[ky^{-\tau}\left|\begin{array}{c}(c_1,\zeta_1;C_1:x),(1-\theta'+l-\epsilon,\tau;1),(1+\delta+\delta'-\xi+l-\epsilon,\tau;1),\\(d_j,\eta_j;D_j)_{1,q},(1+l-\epsilon,\tau;1),\end{array}\right.\right.$$
$$\left.\left.\begin{array}{c},(1+\delta'+\theta-\xi+l-\epsilon,\tau;1),(c_j,\zeta_j;C_j)_{2,p}\\,(1+\delta'-\theta'+l-\epsilon,\tau;1),(1+\delta+\delta'+\theta-\xi+l-\epsilon,\tau;1)\end{array}\right.\right].$$

Proof. We start from the L.H.S of (32), and using equation (2) and (8), we have

$$\left(D_{-}^{\delta,\delta',\theta,\theta',\xi} z^{-\epsilon}\sum_{l=0}^{[v/u]}\frac{(-v)_{u,l}}{l!}A_{v,l}z^l\,\frac{1}{2\pi\iota}\int_L\Phi(s,x)\,k^s z^{-\tau s}ds\right)(y).$$

By altering the order of summation and integration, we have

$$\sum_{l=0}^{[v/u]}\frac{(-v)_{u,l}}{l!}A_{v,l}\,\frac{1}{2\pi\iota}\int_L\Phi(s,x)\,k^s\left(D_{-}^{\delta,\delta',\theta,\theta',\xi} z^{-(\epsilon-l+\tau s)}\right)(y)\,ds.$$

Now by applying the Lemma 1.4, we have

(33)

$$\sum_{l=0}^{[v/u]}\frac{(-v)_{u,l}}{l!}A_{v,l}\,\frac{1}{2\pi\iota}\int_L\Phi(s,x)\,k^s\,y^{\delta+\delta'-\xi-\epsilon+l-\tau s}\times$$
$$\frac{\Gamma(\theta'+\epsilon-l+\tau s)\,\Gamma(-\delta-\delta'+\xi+\epsilon-l+\tau s)\,\Gamma(-\delta'-\theta+\xi+\epsilon-l+\tau s)}{\Gamma(\epsilon-l+\tau s)\,\Gamma(-\delta'+\theta'+\epsilon-l+\tau s)\,\Gamma(-\delta-\delta'-\theta+\xi+\epsilon-l+\tau s)}\,ds.$$

Using (2) in (33), we get (32). □

Theorem 2.8. *Let* $\delta,\delta',\theta,\theta',\xi,\epsilon,k\in\mathbb{C};\tau>0$ *and* $\Re(\epsilon)>\max.\left\{\Re(-\theta'),\Re(\delta'+\theta-\xi),\Re(\delta+\theta'-\xi)+[\Re(\xi)]+1\right\}$. *Then, for*

$y > 0$, we have

(34)

$$\left(D_{-}^{\delta,\delta',\theta,\theta',\xi}\, z^{-\epsilon} S_v^u(z)\ {}^{\gamma}I_{p,q}^{m,n}\left[kz^{-\tau}\left|\begin{array}{c}(c_1,\zeta_1;C_1:x),(c_j,\zeta_j;C_j)_{2,p}\\(d_j,\eta_j;D_j)_{1,q}\end{array}\right.\right]\right)(y)$$

$$= y^{\delta+\delta'-\xi-\epsilon}\sum_{l=0}^{[v/u]}\frac{(-v)_{u,l}}{l!}A_{v,l}\,y^l\times$$

$$ {}^{\gamma}I_{p+3,q+3}^{m,n+3}\left[ky^{-\tau}\left|\begin{array}{c}(c_1,\zeta_1;C_1:x),(1-\theta'+l-\epsilon,\tau;1),(1+\delta+\delta'-\xi+l-\epsilon,\tau;1),\\(d_j,\eta_j;D_j)_{1,q},(1+l-\epsilon,\tau;1),\end{array}\right.\right.$$

$$\left.\begin{array}{r},(1+\delta'+\theta-\xi+l-\epsilon,\tau;1),(c_j,\zeta_j;C_j)_{2,p}\\,(1+\delta'-\theta'+l-\epsilon,\tau;1),(1+\delta+\delta'+\theta-\xi+l-\epsilon,\tau;1)\end{array}\right].$$

3. Special cases

In this section we define some special cases by rearranging the parameters in view of (21) and (22). As if we rearrange the parameters involved in Theorem (2.1), we have the following results in the form of Corollary (3.1) to (3.3) for left- hand side Saigo operators for fractional Integrals.

Corollary 3.1. *Let $\delta,\theta,\xi,\epsilon,k\in\mathbb{C};\Re(\delta),\tau>0$ and $\Re(\epsilon)>\max.\{0,\Re(\theta-\xi)\}$. Therefore, for $y>0$, we have*

$$\left(I_{0+}^{\delta,\theta,\xi}\, z^{\epsilon-1} S_v^u(z)\ {}^{\Gamma}I_{p,q}^{m,n}\left[kz^{\tau}\left|\begin{array}{c}(c_1,\zeta_1;C_1:x),(c_j,\zeta_j;C_j)_{2,p}\\(d_j,\eta_j;D_j)_{1,q}\end{array}\right.\right]\right)(y) \tag{35}$$

$$= y^{-\theta+\epsilon-1}\sum_{l=0}^{[v/u]}\frac{(-v)_{u,l}}{l!}A_{v,l}\,y^l\times$$

$$ {}^{\Gamma}I_{p+2,q+2}^{m,n+2}\left[ky^{\tau}\left|\begin{array}{c}(c_1,\zeta_1;C_1:x),(1-\epsilon-l,\tau;1),\\(d_j,\eta_j;D_j)_{1,q},(1+\theta-\epsilon-l,\tau;1),\end{array}\right.\right.$$

$$\left.\begin{array}{r},(1+\theta-\xi-\epsilon-l,\tau;1),(c_j,\zeta_j;C_j)_{2,p}\\,(1-\delta-\xi-\epsilon-l,\tau;1)\end{array}\right].$$

Also the Riemann-Liouville fractional integral type image formulas can be obtained as follows:

Corollary 3.2. *Let $\delta,\xi,\epsilon,k\in\mathbb{C};\Re(\delta),\tau>0$ and $\Re(\epsilon)>\max.\{0,\Re(-\delta-\xi)\}$. Therefore, for $y>0$, we have*

$$\left(I_{0+}^{\delta,-\delta,\xi}\, z^{\epsilon-1} S_v^u(z)\ {}^{\Gamma}I_{p,q}^{m,n}\left[kz^{\tau}\left|\begin{array}{c}(c_1,\zeta_1;C_1:x),(c_j,\zeta_j;C_j)_{2,p}\\(d_j,\eta_j;D_j)_{1,q}\end{array}\right.\right]\right)(y) \tag{36}$$

$$= y^{\delta+\epsilon-1}\sum_{l=0}^{[v/u]}\frac{(-v)_{u,l}}{l!}A_{v,l}\,y^l\times$$

$$ {}^{\Gamma}I_{p+1,q+1}^{m,n+1}\left[ky^{\tau}\left|\begin{array}{c}(c_1,\zeta_1;C_1:x),(1-\epsilon-l,\tau;1),(c_j,\zeta_j;C_j)_{2,p}\\(d_j,\eta_j;D_j)_{1,q},(1-\delta-\epsilon-l,\tau;1)\end{array}\right.\right].$$

The Erdelyi-Kober fractional integral can also be introduce by Corollary (3.1) as follows:

Corollary 3.3. *Let $\delta, \xi, \epsilon, k \in \mathbb{C}; \Re(\delta), \tau > 0$ and $\Re(\epsilon) > \max.\{0, \Re(-\xi)\}$. Therefore, for $y > 0$, we have*

(37)

$$\left(I^{+}_{\xi,\delta}\, z^{\epsilon-1} S^{u}_{v}(z)\; {}^{\Gamma}I^{m,n}_{p,q}\left[kz^{\tau}\,\middle|\, \begin{matrix} (c_1,\zeta_1;C_1:x),(c_j,\zeta_j;C_j)_{2,p} \\ (d_j,\eta_j;D_j)_{1,q} \end{matrix}\right]\right)(y)$$

$$= y^{\epsilon-1}\sum_{l=0}^{[v/u]}\frac{(-v)_{u,l}}{l!}A_{v,l}\,y^{l}\times$$

$${}^{\Gamma}I^{m,n+1}_{p+1,q+1}\left[ky^{\tau}\,\middle|\, \begin{matrix} (c_1,\zeta_1;C_1:x),(1-\xi-\epsilon-l,\tau;1),(c_j,\zeta_j;C_j)_{2,p} \\ (d_j,\eta_j;D_j)_{1,q},(1-\delta-\xi-\epsilon-l,\tau;1) \end{matrix}\right].$$

Now by rearranging the parameters involved in Theorem (2.3), we have the following results in the form of Corollary (3.4) to (3.6) for right- hand side Saigo operators for fractional Integrals.

Corollary 3.4. *Let $\delta, \theta, \xi, \epsilon, k \in \mathbb{C}; \Re(\delta), \tau > 0$ and $\Re(\epsilon) > \max.\{\Re(-\theta), \Re(-\xi)\}$. Therefore, for $y > 0$, we have*

$$\left(I^{\delta,\theta,\xi}_{-}\, z^{-\epsilon} S^{u}_{v}(z)\; {}^{\Gamma}I^{m,n}_{p,q}\left[kz^{-\tau}\,\middle|\, \begin{matrix} (c_1,\zeta_1;C_1:x),(c_j,\zeta_j;C_j)_{2,p} \\ (d_j,\eta_j;D_j)_{1,q} \end{matrix}\right]\right)(y) \tag{38}$$

$$= y^{-\theta-\epsilon}\sum_{l=0}^{[v/u]}\frac{(-v)_{u,l}}{l!}A_{v,l}\,y^{l}\times$$

$${}^{\Gamma}I^{m,n+2}_{p+2,q+2}\left[ky^{-\tau}\,\middle|\, \begin{matrix} (c_1,\zeta_1;C_1:x),(1-\xi-\epsilon-l,\tau;1), \\ (d_j,\eta_j;D_j)_{1,q},(1-\epsilon-l,\tau;1), \end{matrix}\right.$$

$$\left.\begin{matrix} (1-\theta-\epsilon-l,\tau;1),(c_j,\zeta_j;C_j)_{2,p} \\ ,(1-\delta-\theta-\xi-\epsilon-l,\tau;1), \end{matrix}\right].$$

Also the Riemann-Liouville fractional integral type image formulas can be obtained as follows:

Corollary 3.5. *Let $\delta, \xi, \epsilon, k \in \mathbb{C}; \Re(\delta), \tau > 0$ and $\Re(\epsilon) > \max.\{\Re(\delta), \Re(-\xi)\}$. Therefore, for $y > 0$, we have*

(39)

$$\left(I^{\delta,-\delta,\xi}_{-}\, z^{-\epsilon} S^{u}_{v}(z)\; {}^{\Gamma}I^{m,n}_{p,q}\left[kz^{-\tau}\,\middle|\, \begin{matrix} (c_1,\zeta_1;C_1:x),(c_j,\zeta_j;C_j)_{2,p} \\ (d_j,\eta_j;D_j)_{1,q} \end{matrix}\right]\right)(y)$$

$$= y^{\delta-\epsilon}\sum_{l=0}^{[v/u]}\frac{(-v)_{u,l}}{l!}A_{v,l}\,y^{l}\times$$

$${}^{\Gamma}I^{m,n+1}_{p+1,q+1}\left[ky^{-\tau}\,\middle|\, \begin{matrix} (c_1,\zeta_1;C_1:x),(1+\delta-\epsilon-l,\tau;1),(c_j,\zeta_j;C_j)_{2,p} \\ (d_j,\eta_j;D_j)_{1,q},(1-\epsilon-l,\tau;1) \end{matrix}\right].$$

The right-hand Erdelyi-Kober fractional integration $\kappa^{-}_{\xi,\delta}\left(=I^{\delta,0,\xi}_{-}\right)$ can also be introduce by Corollary (3.4) as follows:

Corollary 3.6. *Let $\delta, \xi, \epsilon, k \in \mathbb{C}; \Re(\delta), \tau > 0$ and $\Re(\epsilon) > \max.\{0, \Re(-\xi)\}$. Therefore, for $y > 0$, we have*

(40)
$$\left(\kappa^{-}_{\xi,\delta}\, z^{-\epsilon} S^u_v(z)\ {}^{\Gamma}I^{m,n}_{p,q}\left[kz^{-\tau}\left|\begin{array}{c}(c_1,\zeta_1;C_1:x),(c_j,\zeta_j;C_j)_{2,p}\\(d_j,\eta_j;D_j)_{1,q}\end{array}\right.\right]\right)(y)$$
$$= y^{-\epsilon}\sum_{l=0}^{[v/u]}\frac{(-v)_{u,l}}{l!}A_{v,l}\,y^l\times$$
$${}^{\Gamma}I^{m,n+1}_{p+1,q+1}\left[ky^{-\tau}\left|\begin{array}{c}(c_1,\zeta_1;C_1:x),(1-\xi-\epsilon-l,\tau;1),(c_j,\zeta_j;C_j)_{2,p}\\(d_j,\eta_j;D_j)_{1,q},(1-\delta-\xi-\epsilon-l,\tau;1)\end{array}\right.\right].$$

Now by rearranging the parameters involved in Theorem (2.5), we have the following results in the form of Corollary (3.7) to (3.9) for left- hand side Saigo operators for fractional derivatives.

Corollary 3.7. *Let $\delta, \theta, \xi, \epsilon, k \in \mathbb{C}; \tau > 0$ and $\Re(\epsilon) > \max.\{0, \Re(-\delta-\theta-\xi)\}$. Therefore, for $y > 0$, we have*

$$(41)\quad \left(D^{\delta,\theta,\xi}_{0+}\, z^{\epsilon-1} S^u_v(z)\ {}^{\Gamma}I^{m,n}_{p,q}\left[kz^{\tau}\left|\begin{array}{c}(c_1,\zeta_1;C_1:x),(c_j,\zeta_j;C_j)_{2,p}\\(d_j,\eta_j;D_j)_{1,q}\end{array}\right.\right]\right)(y)$$
$$= y^{\theta+\epsilon-1}\sum_{l=0}^{[v/u]}\frac{(-v)_{u,l}}{l!}A_{v,l}\,y^l\times$$
$${}^{\Gamma}I^{m,n+2}_{p+2,q+2}\left[ky^{\tau}\left|\begin{array}{c}(c_1,\zeta_1;C_1:x),(1-\epsilon-l,\tau;1),\\(d_j,\eta_j;D_j)_{1,q},(1-\theta-\epsilon-l,\tau;1),\end{array}\right.\begin{array}{c}(1-\delta-\theta-\xi-\epsilon-l,\tau;1),(c_j,\zeta_j;C_j)_{2,p}\\,(1-\xi-\epsilon-l,\tau;1)\end{array}\right].$$

Also the Riemann-Liouville fractional derivative type image formulas can be obtained as follows:

Corollary 3.8. *Let $\delta, \xi, \epsilon, k \in \mathbb{C}; \tau > 0$ and $\Re(\epsilon) > \max.\{0, \Re(-\xi)\}$. Therefore, for $y > 0$, we have*

$$(42)\quad \left(D^{\delta,-\delta,\xi}_{0+}\, z^{\epsilon-1} S^u_v(z)\ {}^{\Gamma}I^{m,n}_{p,q}\left[kz^{\tau}\left|\begin{array}{c}(c_1,\zeta_1;C_1:x),(c_j,\zeta_j;C_j)_{2,p}\\(d_j,\eta_j;D_j)_{1,q}\end{array}\right.\right]\right)(y)$$
$$= y^{-\delta+\epsilon-1}\sum_{l=0}^{[v/u]}\frac{(-v)_{u,l}}{l!}A_{v,l}\,y^l\times$$
$${}^{\Gamma}I^{m,n+1}_{p+1,q+1}\left[ky^{\tau}\left|\begin{array}{c}(c_1,\zeta_1;C_1:x),(1-\epsilon-l,\tau;1),(c_j,\zeta_j;C_j)_{2,p}\\(d_j,\eta_j;D_j)_{1,q},(1-\theta-\epsilon-l,\tau;1)\end{array}\right.\right].$$

The right-hand side Erdelyi-Kober fractional derivative $D^{+}_{\xi,\delta}\left(= D^{\delta,0,\xi}_{0+}\right)$ can also be introduce by Corollary (3.7) as follows:

Corollary 3.9. *Let $\delta, \xi, \epsilon, k \in \mathbb{C}; \tau > 0$ and $\Re(\epsilon) > \max.\{0, \Re(-\delta-\xi)\}$. Therefore, for $y > 0$, we have*

(43)

$$\left(D^{+}_{\xi,\delta}\, z^{\epsilon-1} S^u_v(z)\ {}^{\Gamma}I^{m,n}_{p,q}\left[kz^{\tau}\,\middle|\, \begin{matrix}(c_1,\zeta_1;C_1:x),(c_j,\zeta_j;C_j)_{2,p}\\(d_j,\eta_j;D_j)_{1,q}\end{matrix}\right]\right)(y)$$
$$= y^{\epsilon-1}\sum_{l=0}^{[v/u]}\frac{(-v)_{u,l}}{l!}A_{v,l}\,y^l\times$$
$${}^{\Gamma}I^{m,n+1}_{p+1,q+1}\left[ky^{\tau}\,\middle|\, \begin{matrix}(c_1,\zeta_1;C_1:x),(1-\delta-\xi-\epsilon-l,\tau;1),(c_j,\zeta_j;C_j)_{2,p}\\(d_j,\eta_j;D_j)_{1,q},(1-\xi-\epsilon-l,\tau;1)\end{matrix}\right].$$

Now by rearranging the parameters involved in Theorem (2.7), we have the following results from Corollary (3.10) to (3.12) for right- hand side Saigo operators for fractional derivatives.

Corollary 3.10. *Let $\delta, \theta, \xi, \epsilon, k \in \mathbb{C}; \tau > 0$ and*
$\Re(\epsilon) > \max.\{\Re(-\delta-\xi), \Re(\theta)+[\Re(\delta)]+1\}$. Therefore, for $y > 0$, we have

$$(44)\quad \left(D^{\delta,\theta,\xi}_{-}\, z^{-\epsilon} S^u_v(z)\ {}^{\Gamma}I^{m,n}_{p,q}\left[kz^{-\tau}\,\middle|\, \begin{matrix}(c_1,\zeta_1;C_1:x),(c_j,\zeta_j;C_j)_{2,p}\\(d_j,\eta_j;D_j)_{1,q}\end{matrix}\right]\right)(y)$$
$$= y^{\theta-\epsilon}\sum_{l=0}^{[v/u]}\frac{(-v)_{u,l}}{l!}A_{v,l}\,y^l\times$$
$${}^{\Gamma}I^{m,n+2}_{p+2,q+2}\left[ky^{-\tau}\,\middle|\, \begin{matrix}(c_1,\zeta_1;C_1:x),(1-\delta-\xi-\epsilon-l,\tau;1),\\(d_j,\eta_j;D_j)_{1,q},(1-\epsilon-l,\tau;1),\end{matrix}\right.$$
$$\left.\begin{matrix}(1+\theta-\epsilon-l,\tau;1),(c_j,\zeta_j;C_j)_{2,p}\\,(1+\theta-\xi-\epsilon-l,\tau;1),\end{matrix}\right].$$

Also the Riemann-Liouville fractional derivative type image formulas can be obtained as follows:

Corollary 3.11. *Let $\delta, \xi, \epsilon, k \in \mathbb{C}; \tau > 0$ and*
$\Re(\epsilon) > \max.\{\Re(-\delta-\xi), \Re(-\delta)+[\Re(\delta)]+1\}$. Therefore, for $y > 0$, we have

(45)

$$\left(D^{\delta,-\delta,\xi}_{-}\, z^{-\epsilon} S^u_v(z)\ {}^{\Gamma}I^{m,n}_{p,q}\left[kz^{-\tau}\,\middle|\, \begin{matrix}(c_1,\zeta_1;C_1:x),(c_j,\zeta_j;C_j)_{2,p}\\(d_j,\eta_j;D_j)_{1,q}\end{matrix}\right]\right)(y)$$
$$= y^{-\delta-\epsilon}\sum_{l=0}^{[v/u]}\frac{(-v)_{u,l}}{l!}A_{v,l}\,y^l\times$$
$${}^{\Gamma}I^{m,n+1}_{p+1,q+1}\left[ky^{-\tau}\,\middle|\, \begin{matrix}(c_1,\zeta_1;C_1:x),(1+\theta-\epsilon-l,\tau;1),(c_j,\zeta_j;C_j)_{2,p}\\(d_j,\eta_j;D_j)_{1,q},(1-\epsilon-l,\tau;1)\end{matrix}\right].$$

The right-hand Erdelyi-Kober fractional differentiation $D^{-}_{\xi,\delta}\left(= D^{\delta,0,\xi}_{-}\right)$ can also be introduce by corresponding Corollary (3.10) as follows:

Corollary 3.12. *Let* $\delta, \xi, \epsilon, k \in \mathbb{C}; \tau > 0$ *and* $\Re(\epsilon) > \max.\{\Re(-\delta-\xi), [\Re(\delta)]+1\}$. *Therefore, for* $y > 0$, *we have*

(46)

$$\left(D^{-}_{\xi,\delta}\, z^{-\epsilon} S^{u}_{v}(z)\ {}^{\Gamma}I^{m,n}_{p,q}\left[kz^{-\tau}\,\middle|\,\begin{matrix}(c_1,\zeta_1;C_1:x),(c_j,\zeta_j;C_j)_{2,p}\\(d_j,\eta_j;D_j)_{1,q}\end{matrix}\right]\right)(y)$$

$$= y^{-\epsilon}\sum_{l=0}^{[v/u]}\frac{(-v)_{u,l}}{l!}A_{v,l}\,y^{l}\times$$

$$^{\Gamma}I^{m,n+1}_{p+1,q+1}\left[ky^{-\tau}\,\middle|\,\begin{matrix}(c_1,\zeta_1;C_1:x),(1-\delta-\xi-\epsilon-l,\tau;1),(c_j,\zeta_j;C_j)_{2,p}\\(d_j,\eta_j;D_j)_{1,q},(1-\xi-\epsilon-l,\tau;1)\end{matrix}\right].$$

Similar results can also be obtained for ${}^{\gamma}I^{m,n}_{p,q}$.

4. Concluding remarks and discussion

In this article, we have investigated the Marichev-Saigo-Maeda (MSM) fractional order differentiation and integral operators for the function involving the product of Srivastava polynomials and incomplete I-functions. Moreover, their special cases are also depicted in the corollaries by taking suitable values of the parameters which are associated with Saigo, Riemann-Liouville, and Erdelyi-Kober fractional operators. Also by subtituting the particular values to the coefficient $A_{v,l}$, one can obtain various image formulas which comprise the polynomials viz. Laguerre, Hermite, Jacobi and many others. As if we substitute $v = 0$, then $S^u_v(z)$ reduces to unity, i.e., $S^u_0(z) \to 1$ and we obtain the results due to Jangid et al. [10]. Also, if we put $u = 2$ and $A_{v,l} = (-1)^l$, then equation (8) reduces to

$$S^2_v(z) \to z^{\frac{v}{2}} H_v\left(\frac{1}{2\sqrt{z}}\right),$$

where $H_v(z)$ is the well known Hermite polynomials defined as

$$H_v(z) = \sum_{l=0}^{[v/u]}(-1)^l\frac{v!}{l!(v-2l)!}(2z)^{v-2l}.$$

Eventually, it can easily be define the extensive representation of various special functions that are used in applied sciences.

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