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CONGRUENCE RELATIONS FOR THE FUNDAMENTAL UNIT WITH NEGATIVE NORM OF A REAL QUADRATIC FIELD

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Abstract. In this note, we give some explicit congruence relations for the fundamental unit with negative norm of a real quadratic field, by elementary method. Our results extensively solve the remaining case of the main theorem of Zhang and Yue [6], which only considers the case that real quadratic field has odd ideal class number and its fundamental unit has positive norm.

1. Introduction

Let d be a square-free positive integer and $K = \mathbb{Q}(\sqrt{d})$ a real quadratic field. Denote the ideal class number of K by $h(d)$. The famous unsolved Gauss's conjecture asserts that: *there are infinitely many such d with $h(d) = 1$*. It is well known that

$$2 \nmid h(d) \iff d = p,\ 2p \text{ or } p_1p_2, \tag{1}$$

where p, p_1 and p_2 are primes such that $p_1 \equiv p_2 \equiv 3 \pmod 4$.

Let $\varepsilon = x + y\sqrt{d}$ be the fundamental unit of the order $\mathbb{Z}[\sqrt{d}]$, i.e., (x, y) is the least positive integer solution of the Pell equations

$$x^2 - dy^2 = \pm 1.$$

According to [6], we call ε the fundamental integral unit of $K = \mathbb{Q}(\sqrt{d})$.

It is known that

$$2 \nmid h(d) \text{ and } N(\varepsilon) = x^2 - dy^2 = 1 \iff d = p,\ 2p \text{ or } p_1p_2, \tag{2}$$

where p, p_1 and p_2 are primes such that $p \equiv p_1 \equiv p_2 \equiv 3 \pmod 4$ ([2, p. 163]). In this case, Zhang and Yue [6] get some congruence relations for ε:

Theorem 1. ([6, Theorem 1.1]) *Let $2 \nmid h(d)$ and $N(\varepsilon) = 1$.*

(1) *If $d = p$ with $p \equiv 3 \pmod 4$ then $x \equiv 0 \pmod 2$. Moreover, $x \equiv 2 \pmod 4$ if $p \equiv 3 \pmod 8$ and $x \equiv 0 \pmod 4$ if $p \equiv 7 \pmod 8$.*
(2) *If $d = 2p$ with $p \equiv 3 \pmod 4$ then $y \equiv 0 \pmod 2$ and $x + y \equiv 3 \pmod 4$.*
(3) *If $d = p_1p_2$ with $p_1 \equiv p_2 \equiv 3 \pmod 4$ then $x \equiv 3 \pmod 4$ and $y \equiv 0 \pmod 4$.*

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The original proof of Zhang and Yue is based on ramification theory of local fields. Their result is reobtained by Chakraborty and Saikia [1] by ramification theory of global fields. Chakraborty and Saikia [1] also apply their method to study the case of pure cubic field with a power integral basis and get similar congruence results for its fundamental unit. Williams proves a stronger form of Theorem 1 ([5, Theorem 2]) in an elementary way.

In this paper, we consider the remaining case of Theorem 1, i.e., $K = \mathbb{Q}(\sqrt{d})$ has odd class number and its fundamental integral unit ε has norm -1.

By (1) and (2),

$$2 \nmid h(d) \text{ and } N(\varepsilon) = x^2 - dy^2 = -1 \iff d = 2 \text{ or } p, \tag{3}$$

where p is a prime such that $p \equiv 1 \pmod 4$.

In this case, ignoring $d = 2$, we also find similar congruence relations for ε by investigating Table 1 and we state it in the following theorem.

Theorem 2. *Let $K = \mathbb{Q}(\sqrt{p})$ be a real quadratic field, where $p \equiv 1 \pmod 4$ is a prime. Denote the fundamental integral unit of K by $\varepsilon = x + y\sqrt{p}$. Then, we have*

$$y \equiv 1 \pmod 4 \text{ and } \begin{cases} x \equiv 0 \pmod 4 \text{ if } p \equiv 1 \pmod 8, \\ x \equiv 2 \pmod 4 \text{ if } p \equiv 5 \pmod 8. \end{cases}$$

Let $\varepsilon_0 = (x_0 + y_0\sqrt{d})/2$ be the fundamental unit of $K = \mathbb{Q}(\sqrt{d})$, i.e., (x_0, y_0) is the least positive integer solution of the Pell equations

$$x_0^2 - dy_0^2 = \pm 4.$$

It is well-known that $\varepsilon = \varepsilon_0$ if $d \equiv 2, 3 \pmod 4$ and $\varepsilon = \varepsilon_0$ or ε_0^3 if $d \equiv 1 \pmod 4$.

From Table 1, we also find some congruence relations for the fundamental unit ε_0 in case that $\varepsilon_0 \neq \varepsilon$. In fact, we have the following theorem.

Theorem 3. *Let $K = \mathbb{Q}(\sqrt{p})$ be a real quadratic field, where $p \equiv 1 \pmod 4$ is a prime. Denote the fundamental unit of K by $\varepsilon_0 = (x_0 + y_0\sqrt{p})/2$. Assume both x_0 and y_0 are odd. Then we have*

$$p \equiv 5 \pmod 8 \text{ and } y_0 \equiv 1 \pmod 4.$$

From Theorem 3, we note that if $p \equiv 1 \pmod 8$, then the fundamental unit ε_0 of $\mathbb{Q}(\sqrt{p})$ is integral, i.e., $\varepsilon_0 \in \mathbb{Z}[\sqrt{p}]$. On the other hand, for $p \equiv 5 \pmod 8$, ε_0 may belong or not belong to $\mathbb{Z}[\sqrt{p}]$. So, we ask the following intriguing question.

Intriguing Question: For which primes $p \equiv 5 \pmod 8$, the fundamental unit ε_0 of $\mathbb{Q}(\sqrt{p})$ belongs to $\mathbb{Z}[\sqrt{p}]$? Equivalently, for which primes $p \equiv 5 \pmod 8$, the negative Pell equation $x^2 - py^2 = -4$ has no odd integer solution? How does such prime distributes in the primes $p \equiv 5 \pmod 8$?

To find some clue to the above question, we compute the fundamental unit ε_0 of $\mathbb{Q}(\sqrt{p})$ with prime $p < 10^6$ and decide whether $\varepsilon_0 \in \mathbb{Z}[\sqrt{p}]$ or not? We list some computation results in section 3.

Finally, we remove the condition $2 \nmid h(d)$ and consider the general case: $K = \mathbb{Q}(\sqrt{d})$ with $N(\varepsilon) = -1$. We also get similar results:

Theorem 4. *Let $K = \mathbb{Q}(\sqrt{d})$ be a real quadratic field whose fundamental unit ε_0 has norm -1. Let $\varepsilon = x + y\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$ be its fundamental integral unit. Then $d \equiv 1, 2 \pmod 4$ and we have*

(1) *If* $d \equiv 2 \pmod 4$, *then* $\varepsilon = \varepsilon_0$. *In this case*

$$d \equiv 2 \pmod 8,\ x \equiv 1 \pmod 2 \text{ and } y \equiv 1 \pmod 4.$$

(2) *If* $d \equiv 1 \pmod 4$, *then* $\varepsilon = \varepsilon_0$ *or* ε_0^3. *In this case*

$$y \equiv 1 \pmod 4 \text{ and } \begin{cases} x \equiv 0 \pmod 4 \text{ if } d \equiv 1 \pmod 8, \\ x \equiv 2 \pmod 4 \text{ if } d \equiv 5 \pmod 8. \end{cases}$$

Additionally, if $\varepsilon_0 = (x_0 + y_0\sqrt{d})/2 \neq \varepsilon$, then

$$d \equiv 5 \pmod 8,\ x_0 \equiv 1 \pmod 2 \text{ and } y_0 \equiv 1 \pmod 4.$$

The rest of paper is organized as follows. In section 2, we give the proofs of Theorem 2, Theorem 3 and Theorem 4, which are elementary. The computation tables are listed in section 3.

2. Proof of Main results

Proof of Theorem 2: The proof is based on the unique factorization property of the ring of Gaussian integers $\mathbb{Z}[i]$. It is well known that the primes of $\mathbb{Z}[i]$ are of three types,

(1) $1 + i$
(2) integer primes $q \equiv 3 \pmod 4$
(3) factors π and $\overline{\pi}$ of the integer primes $q \equiv 1 \pmod 4$ (i.e., $\pi\overline{\pi} = q$)

together with all multiples of these types by units (for example see [3, p. 120-121]).

Now consider the negative Pell equation $x^2 - py^2 = -1$. Modulo it by 4, we deduce that $2 \mid x$ and $2 \nmid y$ as $p \equiv 1 \pmod 4$.

Change $x^2 - py^2 = -1$ to

$$(x+i)(x-i) = py^2 \tag{4}$$

and consider the prime factorization of $x + i$ in $\mathbb{Z}[i]$. Clearly, for integer primes $q \equiv 3 \pmod 4$, $q \nmid (x + i)$. Since y is odd, we get $(1 + i) \nmid (x + i)$. Therefore, the prime factors of $x + i$ are prime factors π of integer primes $q \equiv 1 \pmod 4$. Also, π and $\overline{\pi}$ can not both divide $x + i$. This implies that

$$x + i = i^a \pi_1^{b_1} \pi_2^{b_2} \cdots \pi_s^{b_s}, \tag{5}$$

where $q_j = \pi_j\overline{\pi}_j$ $(1 \leq j \leq s)$ are distinct integer primes congruent to 1 modulo 4, and $a, b_1, b_2, \ldots, b_s$ are positive integers.

Combining (4) with (5), we get

$$py^2 = (x+i)\overline{(x+i)} = q_1^{b_1} q_2^{b_2} \cdots q_n^{b_n}. \tag{6}$$

This implies that the prime factors of y are all congruent to 1 modulo 4. Hence $y \equiv 1 \pmod 4$.

Since $y^2 \equiv 1 \pmod 8$

$$x^2 = py^2 - 1 \equiv \begin{cases} 0 \pmod 8 \text{ if } p \equiv 1 \pmod 8, \\ 4 \pmod 8 \text{ if } p \equiv 5 \pmod 8. \end{cases}$$

Therefore, we get the desired result.

Proof of Theorem 3: (x_0, y_0) satisfies the negative Pell equation:

$$x_0^2 - py_0^2 = -4. \tag{7}$$

By assumption, both x_0^2 and $y_0^2 \equiv 1 \pmod 8$. This implies that $p \equiv 5 \pmod 8$.

Reformulate (7) as

$$(x_0 + 2i)(x_0 - 2i) = py_0^2. \tag{8}$$

As $2 \nmid y_0$ and $q \nmid (x_0 + 2i)$ for any odd integer prime q, the prime factorization of $x_0 + 2i$ in $\mathbb{Z}[i]$ is

$$x_0 + 2i = i^a \pi_1^{b_1} \pi_2^{b_2} \cdots \pi_s^{b_s}, \tag{9}$$

where $q_j = \pi_j \overline{\pi}_j$ $(1 \leqslant j \leqslant s)$ are distinct integer primes congruent to 1 modulo 4, and $a, b_1, b_2, \ldots, b_s$ are positive integers. Combining (8) and (9), we get

$$py_0^2 = (x_0 + 2i)\overline{(x_0 + 2i)} = q_1^{b_1} q_2^{b_2} \cdots q_s^{b_s}. \tag{10}$$

Hence $y_0 \equiv 1 \pmod 4$.

Proof of Theorem 4: Now assume the fundamental unit of $K = \mathbb{Q}(\sqrt{d})$ has norm -1. Under this condition, the negative Pell equation $x^2 - dy^2 = -1$ is solvable, which implies that dy^2 is a sum of squares. Therefore, $d \equiv 1 \pmod 4$ or $d \equiv 2 \pmod 4$.

Assuming $d \equiv 2 \pmod 4$, from the negative Pell equation $x^2 - dy^2 = -1$, we deduce that $2 \nmid x$. As $x^2 \equiv 1 \pmod 8$, we get $2 \nmid y$. From

$$(x + i)(x - i) = dy^2,$$

we deduce that the prime factorization of $x + i$ in $\mathbb{Z}[i]$ is

$$(x + i) = i^a (1 + i) \pi_1^{b_1} \pi_2^{b_2} \cdots \pi_s^{b_s},$$

where $q_j = \pi_j \overline{\pi_j}$ $(1 \leqslant j \leqslant s)$ are distinct integer primes congruent to 1 modulo 4, and $a, b_1, \ldots, b_s$ are positive integers. Therefore,

$$2q_1^{b_1} \cdots q_s^{b_s} = dy^2.$$

This implies that

$$d \equiv 2 \pmod 8 \text{ and } y \equiv 1 \pmod 4.$$

If $d \equiv 1 \pmod 4$, the proofs in this case are similar to those of Theorem 2 and Theorem 3. So, we omit it.

3. Tables and Conjectures

Computing by Mathematica 11.2, we get the following table which lists the fundamental unit ε_0 and the fundamental integral unit ε of $\mathbb{Q}(\sqrt{p})$ with primes $p = 2$, or $p \equiv 1 \pmod 4$ and $p < 100$.

We will write the ratio of

$$R(x) = \frac{\#\{\ p < x,\ p \equiv 5 \pmod 8), \epsilon_0 \notin \mathbb{Z}[\sqrt{p}]\ \}}{\#\{\ p < x,\ p \equiv 5 \pmod 8)\ \}}$$

in the table below.

True primes p with $p < 10^6$

5, 13, 29, 53, 61, 109, 149, 157, 173, 181, 229, 277, 293, 317, 397, 421, 461, 509, 541, 557, 613, 653, 661, 733, 773, 797, 821, 853, 941, 1013, 1021, 1061, 1069, 1093, 1109, 1117, 1181, 1229, 1237, 1277, 1373, 1381, 1429, 1453, 1493,

Table 1.

p	2	5	13	17	29	37
ε_0	$1+\sqrt{2}$	$(1+\sqrt{5})/2$	$(3+\sqrt{13})/2$	$4+\sqrt{17}$	$(5+\sqrt{29})/2$	$6+\sqrt{37}$
ε	$1+\sqrt{2}$	$2+\sqrt{5}$	$18+5\sqrt{13}$	$4+\sqrt{17}$	$70+13\sqrt{29}$	$6+\sqrt{37}$
p	41	53	61	73	89	97
ε_0	$32+5\sqrt{41}$	$(7+\sqrt{53})/2$	$(39+5\sqrt{61})/2$	$1068+125\sqrt{73}$	$500+53\sqrt{89}$	$5604+569\sqrt{97}$
ε	$32+5\sqrt{41}$	$182+25\sqrt{53}$	$29718+3805\sqrt{61}$	$1068+125\sqrt{73}$	$500+53\sqrt{89}$	$5604+569\sqrt{97}$

Table 2.

x	$R(x)$	x	$R(x)$	x	$R(x)$
1×10^5	0.691121	1×10^6	0.684044	1×10^7	0.678816
2×10^5	0.682554	2×10^6	0.681823	2×10^7	0.678041
3×10^5	0.686034	3×10^6	0.682211	3×10^7	0.677387
4×10^5	0.684793	4×10^6	0.681359	4×10^7	0.676518
5×10^5	0.685486	5×10^6	0.681516	5×10^7	0.676087
6×10^5	0.687205	6×10^6	0.681409	6×10^7	0.675645
7×10^5	0.684976	7×10^6	0.680767	7×10^7	0.675491
8×10^5	0.685657	8×10^6	0.679168	8×10^7	0.675297
9×10^5	0.685753	9×10^6	0.679262	9×10^7	0.675200

1549, 1597, 1621, 1637, 1669, 1693, 1709, 1733, 1741, 1789, 1877, 1933, 1997, 2029, 2053, 2141, 2213, 2237, 2293, 2309, 2333, 2381, 2389, 2437, 2477, 2549, 2557, 2677, 2693, 2741, 2749, 2789, 2861, 2909, 2957, 3037, 3061, 3221, 3229, 3253, 3373, 3389, 3461, 3469, 3517, 3533, 3541, 3557, 3581, 3613, 3637, 3677, 3701, 3733, 3917, 3989, 4157, 4229, 4253, 4261, 4349, 4373, 4397, 4493, 4517, 4549, 4597, 4621, 4637, 4789, 4861, 4877, 4909, 4957, 4973, 5101, 5189, 5333, 5381, 5413, 5437, 5501, 5573, 5581, 5653, 5693, 5701, 5717, 5741, 5821, 5861, 5869, 5981, 6029, 6037, 6053, 6101, 6133, 6173, 6197, 6269, 6277, 6301, 6317, 6389, 6397, 6421, 6581, 6637, 6661, 6701, 6709, 6733, 6781, 6829, 6917, 6949, 6997, 7069, 7229, 7237, 7253, 7333, 7349, 7477, 7549, 7573, 7621, 7741, 7789, 7853, 7901, 7933, 8053, 8069, 8093, 8117, 8221, 8237, 8269, 8293, 8317, 8389, 8429, 8501, 8573, 8677, 8693, 8741, 8821, 8893, 8933, 8941, 9013, 9029, 9133, 9157, 9173, 9181, 9349, 9397, 9413, 9421, 9437, 9461, 9533, 9629, 9677, 9733, 9749, 9781, 9949, 10037, 10093, 10133, 10141, 10181, 10253, 10301, 10357, 10429, 10453, 10477, 10501, 10589, 10613, 10709, 10781, 10837, 10853, 10909, 10949, 10957, 10973, 11069, 11093, 11117, 11149, 11173, 11197, 11261, 11317, 11597, 11621, 11677, 11701, 11717, 11789, 11813, 11821, 11933, 11941, 11981, 12037, 12109, 12149, 12157, 12197, 12253, 12269, 12277, 12301, 12413, 12421, 12437, 12541, 12589, 12613, 12637, 12757, 12781, 12821, 12853, 12893, 12917, 13037, 13093, 13109, 13229, 13381, 13397, 13421, 13469, 13597, 13613, 13669, 13693, 13709, 13757, 13789, 13829, 13877, 13901, 13933, 13997, 14029, 14197, 14221, 14293, 14341, 14389, 14437, 14461, 14533, 14549, 14557, 14621, 14629, 14669, 14717, 14797, 14813, 14869, 15061, 15077, 15149, 15269, 15349, 15373, 15413, 15461, 15493, 15541, 15581, 15629, 15733, 15749, 15773, 15797, 15901, 16493, 16573, 16661, 16069, 16141, 16189, 16229, 16333, 16349, 16381, 16421, 16453, 16829, 16981, 17021, 17029, 17077, 17189, 17293, 17317, 17341, 17477, 17509,

17581, 17597, 17669, 17749, 17981, 18013, 18061, 18077, 18133, 18181, 18229,
18253, 18301, 18341, 18413, 18461, 18541, 18637, 18661, 18749, 18757, 18773,
18797, 18869, 18917, 18973, 19013, 19037, 19069, 19141, 19157, 19181, 19237,
19301, 19309, 19333, 19373, 19381, 19421, 19429, 19469, 19477, 19501, 19541,
19597, 19661, 19949, 20021, 20101, 20269, 20341, 20389, 20509, 20533, 20549,
20693, 20717, 20773, 20981, 21061, 21101, 21149, 21157, 21221, 21341, 21397,
21557, 21589, 21613, 21661, 21701, 21757, 21821, 21893, 21997, 22013, 22037,
22093, 22109, 22277, 22381, 22397, 22453, 22541, 22549, 22573, 22621, 22637,
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23293, 23333, 23357, 23509, 23549, 23557, 23629, 23669, 23677, 23789, 23813,
23869, 23893, 23909, 23981, 24029, 24077, 24109, 24133, 24197, 24229, 24317,
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25013, 25037, 25117, 25237, 25301, 25309, 25357, 25453, 25621, 25717, 25733,
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33637, 33749, 33757, 33773, 33797, 33893, 33941, 33997, 34061, 34213, 34301,
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False primes p with $p < 10^6$

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