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Independent Transversal Dominating Energy of a Graph

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Abstract

Let G be a simple graph and let $V(G)$ be its vertex set. The independent transversal dominating set or $\gamma_{it}-$ set is a dominating set of G which intersects every maximal independent set of G. Using the $\gamma_{it}-$ set, a diagonal matrix is constructed which is similar to the ones available in the literature for covering set, equitable dominating set and so on. Using the idea of signless Laplacian, one can add this diagonal matrix to the usual adjacency matrix of G to obtain new matrix called independent transversal dominating adjacency matrix denoted by $A_{it}(G)$. The sum of all the absolute values of eigenvalues of $A_{it}(G)$ is called independent transversal dominating energy denoted by $E_{it}(G)$. In the present paper, some spectral properties of $A_{it}(G)$ are obtained. Some upper and lower bounds for the largest eigenvalue of $A_{it}(G)$ and $E_{it}(G)$ are derived. $E_{it}(G)$ is obtained for some standard graphs.

Keywords: Spectrum of independent transversal dominating matrix, independent transversal dominating energy of a graph.
AMS subject classification : 05C50.

1 Introduction

All graphs considered in this paper are finite, undirected, loopless, and without multiple edges. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$ with $|V(G)| = n$ and $|E(G)| = m$ called order and size of G respectively. The minimum and maximum degree of a vertex of G are denoted by $\delta(G)$ and $\Delta(G)$ respectively. For more concepts and standard results on graph theory one may refer [3, 4, 6]. A set S of vertices is independent if no two vertices in S are adjacent. An independent set of maximum cardinality is a maximum independent set of G. A subset D of the vertex set V of G is called a dominating set of G if every vertex $v \in V - D$ is adjacent to some vertex in D. The domination number of G, $\gamma(G)$ is the minimum cardinality of a dominating set in G. A minimum dominating set of a graph G is called a γ-set of G [17]. A subset C of $V(G)$ is called covering set of G if every edge of G is incident to at least one vertex of C. The covering number $\alpha_0(G)$ is the minimum cardinality of a covering set [1]. A dominating set D_{it} of V of a graph G is said to be an independent transversal dominating set if D_{it} intersects every maximum independent set of G. The minimum cardinality of an independent transversal dominating set of G is called the independent transversal domination number of G and is denoted by $\gamma_{it}(G)$. Such an independent transversal dominating set D_{it} also called a γ_{it}-set [14]. One can observe that an independent transversal dominating set need not be an independent set and vice versa. Also a γ-set of G need not be γ_{it}-set of G and vice versa. For more information about γ_{it} set one can refer [7].

The concept of graph energy was first introduced by I. Gutman [12], as $E(G) = \sum_{i=1}^{n} |\lambda_i|$, where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of the adjacency matrix $A(G)$. For more details on spectra of various types of graphs and energies with applications please refer to [2, 8, 9, 18, 20].
We quote the following useful results:

Lemma 1.1. *([4])The eigenvalues of the $n \times n$ matrix $aI + bJ$ are a with multiplicity $n-1$, and $a + nb$ with multiplicity 1 (where I is the $n \times n$ unit matrix and J is the $n \times n$ matrix, whose all entries are 1).*

Theorem 1.2. *([16]) Let $a = (a_1, a_2, \cdots, a_n)$ and $b = (b_1, b_2, \cdots, b_n)$ be $n-$ tuples of real numbers satisfying $0 \le m_1 \le a_i \le M_1$ and $0 \le m_2 \le b_i \le M_2$ $(i = 1, 2, \cdots, n)$. Then,*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \frac{n^2}{3}(M_1 M_2 - m_1 m_2)^2.$$

Lemma 1.3. *([20]) Let $n \ge 1$ be an integer and $a_1, a_2, \cdots, a_n$ be some non-negative real numbers such that $a_1 \ge a_2 \ge \cdots \ge a_n$. Then $(a_1 + \cdots + a_n)(a_1 + a_n) \ge a_1^2 + a_2^2 + \cdots + a_n^2 + n a_1 a_n$. Moreover, the equality holds if and only if for some $r \in \{1, 2, \cdots, n\}$, $a_1 = a_2 = \cdots = a_r$ and $a_{r+1} = \cdots = a_n$.*

For a real number x, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Theorem 1.4. *([5]) If $1 \le i \le n$, x_i and y_i are positive real numbers, then*

$$\left| n \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i \right| \le \theta(n)(A - x)(B - y),$$

where x, y, A and B are real constants, that for each i, $1 \le i \le n$, $x \le x_i \le A$ and $y \le y_i \le B$. Further, $\theta(n) = n \lfloor \frac{n}{2} \rfloor (1 - \frac{1}{n} \lfloor \frac{n}{2} \rfloor)$.

Several graph energies were introduced and studied in the literature such as minimum covering energy $E_c(G)$ [1], Laplacian energy [13], signless laplacian energy [10], distance energy [15], minimum equitable dominating energy [21] and status sum energy [23].

In the present paper, some spectral properties of $A_{it}(G)$ are obtained. Some upper and lower bounds for the largest eigenvalue of $A_{it}(G)$ and $E_{it}(G)$ are derived. $E_{it}(G)$ is obtained for some standard graphs.

2 Independent transversal dominating energy of a graph

Let $A_{it}(G) = (a_{ij})_{n\times n}$ be the independent transversal dominating matrix of G, where,

$$a_{ij} = \begin{cases} 1 & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent;} \\ 1 & \text{if } i = j \text{ and } v_i \in D_{it} \text{ ;} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\phi_n(G, \mu)$ denote the characteristic polynomial of $A_{it}(G)$, which is of the form

$$\phi_n(G, \mu) = det(\mu I - A_{it}(G)) = |\mu I - A_{it}(G)| = \alpha_0\mu^n + \alpha_1\mu^{n-1} + \alpha_2\mu^{n-2} + \cdots + \alpha_n,$$

where α_i are integers, $1 \leq i \leq n$. The independent transversal dominating eigenvalues of G are the eigenvalues of $A_{it}(G)$. Since $A_{it}(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. If G has distinct independent transversal dominating eigenvalues $\mu_1, \mu_2, \cdots, \mu_k$ with multiplicities $m_1, m_2, \cdots, m_k$ respectively, then the independent transversal dominating spectrum of G is denoted by $Spec_{it}(G)$ and is given by $Spec_{it}(G) = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_k \\ m_1 & m_2 & \cdots & m_k \end{pmatrix}$.
The independent transversal dominating energy of G is then defined as

$$E_{it}(G) = \sum_{i=1}^{n} | \mu_i | \ .$$

Let us compute the independent transversal dominating energy for the graph $G = G_1$ and exhibit the relationship between $E(G)$, $E_c(G)$ and $E_{it}(G)$ for the graph G_2 and G_3, depicted in Fig. 1.

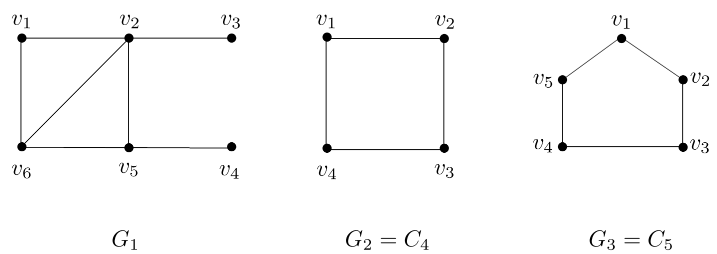

Figure 1 : Graphs considered in the Examples 2.1, 2.2 and 2.3.

Example 2.1. *Let G_1 be a graph with vertices $v_1, v_2, v_3, v_4, v_5, v_6$ (See Fig. 1).*
The following are the independent transversal dominating sets of G_1.
(i) $D^*_{it}(G_1) = \{v_2, v_4, v_5\}$, (ii) $D^{**}_{it}(G_1) = \{v_1, v_3, v_5\}$.
Case(1): *Consider $A^*_{it}(G_1)$ with respect to $D^*_{it}(G_1) = \{v_2, v_4, v_5\}$. Then,*

$$A^*_{it}(G_1) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The characteristic polynomial of $A^{*}_{it}(G_1)$ is $\mu^6 - 3\mu^5 - 4\mu^4 + 8\mu^3 + 5\mu^2 - 3\mu$, the independent transversal dominating eigenvalues are $\mu_1 \cong 3.3761$, $\mu_2 \cong 1.5952$, $\mu_3 \cong 0.4048$, $\mu_4 = 0$, $\mu_5 = -1$ and $\mu_6 \cong -1.376$. Hence, $E^{*}_{it}(G_1) \cong 7.7522$.

Case(2): *Consider $A^{**}_{it}(G_1)$ with respect to $D^{**}_{it}(G_1) = \{v_1, v_3, v_5\}$. Then,*

$$A^{**}_{it}(G_1) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The characteristic polynomial of $A^{**}_{it}(G_1)$ is $\mu^6 - 3\mu^5 - 4\mu^4 + 10\mu^3 + 5\mu^2 - 8\mu - 2$, the independent transversal dominating eigenvalues are $\mu_1 \cong 3.2042$, $\mu_2 \cong 1.4142$, $\mu_3 \cong 1.1717$, $\mu_4 \cong -0.2331$, $\mu_5 \cong -1.1428$ and $\mu_6 \cong -1.4142$. Hence, $E^{**}_{it}(G_1) \cong 8.5802$.

The Example 2.1 shows the fact that the independent transversal dominating energy of a graph depends on the choice of independent transversal dominating set.
Therefore, the independent transversal dominating energy is not graph invariant.

Example 2.2. *Let $G_2 = C_4$ be a graph with vertices v_1, v_2, v_3, v_4 (See Fig. 1).*
Independent transversal dominating sets of G_2 are $\{v_1, v_2\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_3, v_4\}$.
Minimum covering sets of G_2 are $\{v_1, v_3\}, \{v_2, v_4\}$.
This example shows that independent transversal dominating sets and minimum covering sets need not be same. On computing the corresponding energies we get the following observations:
(i) $\alpha_0(C_4) = \gamma_{it}(C_4) = 2$, (ii) $E(C_4) = 4$, $E_c(C_4) = \sqrt{17} + 1$ and $E_{it}(C_4) = \sqrt{5} + 3$ (iii) $E_{it}(G_2)$ is different from $E(G_2)$ and $E_c(G_2)$.

Example 2.3. *Let $G_3 = C_5$ be a graph with vertices v_1, v_2, v_3, v_4, v_5 (See Fig. 1).*
Independent transversal dominating sets of G_3 are $\{v_1, v_2, v_5\}, \{v_1, v_4, v_5\}, \{v_1, v_2, v_3\}$.
Minimum covering sets of G_3 are $\{v_1, v_3, v_4\}, \{v_1, v_2, v_4\}, \{v_2, v_4, v_5\}, \{v_2, v_3, v_5\}$.
Clearly, independent transversal dominating sets and minimum covering sets need not be same. On computing the corresponding energies we get the following observations:
(i) $\alpha_0(C_5) = \gamma_{it}(C_5) = 3$, (ii) Largest eigenvalue of $A_{it}(G_3)$ and $A_c(G_3)$ are very near and 1 is the common eigenvalue, (iii) $E_{it}(G_3)$ is different from $E(G_3)$ and $E_c(G_3)$.

These observations motivate us to workout the spectral properties of independent transversal dominating matrix of G and also $E_{it}(G)$ which will be carried out in the next sections.

3 Some spectral properties of $A_{it}(G)$

Theorem 3.1. *For any (n, m) graph G with independent transversal domination number γ_{it} and with the characteristic polynomial of $A_{it}(G)$, $\phi_n(G,\mu) = \alpha_0\mu^n + \alpha_1\mu^{n-1} + \alpha_2\mu^{n-2} + \cdots + \alpha_n$, the following statements are true:*
(i) $\alpha_0 = 1$, (ii) $\alpha_1 = -\gamma_{it}$, (iii) $\alpha_2 = \binom{\gamma_{it}}{2} - m$, (iv) $\alpha_3 = m\gamma_{it} - \sum_{v_i \in D_{it}} d(v_i) - 2t - \binom{\gamma_{it}}{3}$,
where t is the number of triangles in G.

Proof. (i)Since $\phi_n(G,\mu) = det(\mu I - A_{it}(G))$, it immediately follows that $\alpha_0 = 1$.
(ii) Let $\mu_1, \mu_2, \cdots, \mu_n$ be the n eigenvalues of $A_{it}(G)$. So we may write

$$\begin{aligned}\phi_n(G,\mu) &= (\mu - \mu_1)(\mu - \mu_2)\cdots(\mu - \mu_n)\\ &= \mu^n - (\mu_1 + \mu_2 + \cdots + \mu_n)\mu^{n-1} + \cdots + (-1)^n \mu_1\mu_2\cdots\mu_n.\end{aligned}$$

It is clear that, $\alpha_1 = -\gamma_{it}$.
(iii) $(-1)^2\alpha_2$ is equal to the sum of the determinants of all 2×2 principal sub matrices of $A_{it}(G)$, that is

$$\begin{aligned}(-1)^2\alpha_2 &= \sum_{1\le i<j\le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}\\ &= \sum_{1\le i<j\le n} (a_{ii}a_{jj} - a_{ij}a_{ji})\\ &= \sum_{1\le i<j\le n} a_{ii}a_{jj} - \sum_{1\le i<j\le n} a_{ij}a_{ji}\\ &= \binom{\gamma_{it}}{2} - m.\end{aligned}$$

(iv) We have

$$\begin{aligned}(-1)^3\alpha_3 &= \sum_{1\le i<j<k\le n} \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix}\\ &= \sum_{1\le i<j<k\le n} [a_{ii}(a_{jj}a_{kk} - a_{kj}a_{jk}) - a_{ij}(a_{ji}a_{kk} - a_{ki}a_{jk})\\ &+ a_{ik}(a_{ji}a_{kj} - a_{ki}a_{jj})]\\ &= \sum_{1\le i<j<k\le n} a_{ii}a_{jj}a_{kk} + 2\sum_{1\le i<j<k\le n} a_{ij}a_{jk}a_{ki}\\ &- \sum_{1\le i<j<k\le n} [a_{ii}a_{jk}a_{kj} + a_{jj}a_{ik}a_{ki} + a_{kk}a_{ij}a_{ji}]\\ &= \binom{\gamma_{it}}{3} + 2t - \sum_{1\le i<j<k\le n} [a_{ii}a_{jk}a_{kj} + a_{jj}a_{ik}a_{ki} + a_{kk}a_{ij}a_{ji}]\end{aligned}$$

$$= \binom{\gamma_{it}}{3} + 2t - \left[\sum_{i=1}^{n} a_{ii}\right]\left[\sum_{1\le j<k\le n} a_{jk}\right] + \left[\sum_{i=1}^{n} a_{ii}\right]\left[\sum_{k=1,k\neq i}^{n} a_{ik}\right]$$
$$= \binom{\gamma_{it}}{3} + 2t - m\gamma_{it} + \sum_{v_i\in D_{it}} d(v_i).$$

□

Theorem 3.2. *Let G be a (n,m) graph and $\mu_1, \mu_2, \mu_3, \cdots, \mu_n$ are the eigenvalues of $A_{it}(G)$. Then, (i) $\sum_{i=1}^{n}\mu_i = \gamma_{it}$, (ii) $\sum_{1\le i<j\le n}\mu_i\mu_j = \binom{\gamma_{it}}{2} - m$, (iii) $\sum_{i=1}^{n}\mu_i^2 = 2m + \gamma_{it}$.*

Proof. (i) Clearly, $\sum_{i=1}^{n}\mu_i = trace[A_{it}(G)] = \gamma_{it}$.
(ii) From the Theorem 3.1, it is clear that

$$\sum_{1\le i<j\le n}\mu_i\mu_j = \alpha_2 = \binom{\gamma_{it}}{2} - m. \tag{3.1}$$

(iii) Consider

$$(\gamma_{it})^2 = (\mu_1 + \mu_2 + \cdots + \mu_n)^2$$
$$= \sum_{i=1}^{n}\mu_i^2 + 2\sum_{1\le i<j\le n}\mu_i\mu_j$$
$$= \sum_{i=1}^{n}\mu_i^2 + \gamma_{it}^2 - \gamma_{it} - 2m \quad [from\ (3.1)].$$

Therefore

$$\sum_{i=1}^{n}\mu_i^2 = 2m + \gamma_{it}.$$

□

Theorem 3.3. *If $A_{it}(G)$ is non singular, then the minimum independent transversal dominating energy is always greater than or equal to n.*

Proof. Let $\mu_1, \mu_2, \mu_3, \cdots, \mu_n$ be the eigenvalues of $A_{it}(G)$. Since $A_{it}(G)$ is non singular we must have $\mu_i \neq 0$ for all i, $1 \le i \le n$.
By using the Arithmetic and Geometric mean inequality, we obtain

$$\frac{|\mu_1| + |\mu_2| + \cdots + |\mu_n|}{n} \ge (|\mu_1|\mu_1||\mu_2|\cdots|\mu_n|)^{\frac{1}{n}}$$
$$|\mu_1| + |\mu_2| + \cdots + |\mu_n| \ge n\,|detA_{it}(G)|^{\frac{1}{n}}$$
$$E_{it}(G) \ge n\,|detA_{it}(G)|^{\frac{1}{n}}\ .$$

Since $det(A_{it}(G)) \neq 0$, we have $\mu_i \neq 0$ for all $1 \le i \le n$. Hence $|detA_{it}(G)|^{\frac{1}{n}} \ge 1$.
Therefore,

$$E_{it}(G) \geq n.$$

□

Theorem 3.4. *Let G_1 and G_2 be the graphs of same order n, with size m and m' respectively. Let γ_{it} and γ'_{it} be the independent transversal domination number of G_1 and G_2 respectively. If $\mu_1, \mu_2, \cdots, \mu_n$ and $\mu'_1, \mu'_2, \mu'_3, \cdots, \mu'_n$ are the eigenvalues of $A_{it}(G_1)$ and $A_{it}(G_2)$ respectively, then*

$$\sum_{i=1}^{n} \mu_i \mu'_i \leq \sqrt{(2m + \gamma_{it})(2m' + \gamma'_{it})}.$$

Proof. Consider the Cauchy-Schwartz inequality

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right).$$

For $1 \leq i \leq n$, replacing $x_i = \mu_i$ and $y_i = \mu'_i$ in the above inequality, we obtain

$$\left(\sum_{i=1}^{n} \mu_i \mu'_i\right)^2 \leq \left(\sum_{i=1}^{n} \mu_i^2\right)\left(\sum_{i=1}^{n} (\mu'_i)^2\right)$$
$$= (2m + \gamma_{it})(2m' + \gamma'_{it}) \qquad (by \;\; using \;\; Theorem \;\; 3.2).$$

Therefore

$$\sum_{i=1}^{n} \mu_i \mu'_i \leq \sqrt{(2m + \gamma_{it})(2m' + \gamma'_{it})}.$$

□

Theorem 3.5. *Let G be a (n, m) graph. If $\mu_1, \mu_2, \cdots, \mu_n$ are the eigenvalues of $A_{it}(G)$ with $\mu_1 \geq \mu_2 \geq \mu_3 \geq, \cdots, \geq \mu_n$, then*

$$\bar{d} + \tfrac{\gamma_{it}}{n} \leq \mu_1 \leq \Delta(G) + 1,$$

where $\bar{d}$ is the average degree of all the vertices in G.

Proof. Let $A_{it}(G)$ be the independent transversal dominating matrix of G and let x be an eigenvector of $A_{it}(G)$ corresponding to μ_1. Then, $A_{it}x = \mu_1 x$. From the i^{th} equation of this vector equation we get

$$\mu_1 x_i = \sum_{v_i \sim v_j} x_j, \qquad i = 1, 2, \cdots, n,$$

where $v_i \sim v_j$ denotes that v_i is adjacent to v_j. If $x_k > 0$ is the maximum coordinate of x, then

$$\mu_1 x_i = \sum_{v_i \sim v_j} x_j \leq (\Delta(G) + 1)x_k.$$

Therefore

$$\mu_1 \leq \Delta(G) + 1. \tag{3.2}$$

To prove the lower bound, consider the extremal representation

$$\mu_1(A_{it}) = \max_{\|V\|=1}\{x^{'}A_{it}x\} = \max_{x\neq 0}\{\frac{x^{'}A_{it}x}{x^{'}x}\}$$

This implies that

$$\begin{aligned}\mu_1 &\geq \frac{J_1^{'}A_{it}J_1}{J_1^{'}J_1}\\ &= \frac{2m+\gamma_{it}}{n} = \bar{d} + \frac{\gamma_{it}}{n},\end{aligned} \tag{3.3}$$

where J_1 is the n by 1 matrix, each of whose entries is 1.
The Theorem follows from (3.2) and (3.3). □

4 Bounds on the independent transversal dominating energy of a graph

In this section, we derive certain upper and lowerbounds for $E_{it}(G)$. The following theorems are motivated by standard theorems in [[12, 18, 19]].

Theorem 4.1. *Let G be a (n, m) graph. Then,* $E_{it}(G) \geq \bar{d} + \frac{\gamma_{it}}{n}$, *where* $\bar{d}$ *denotes the average degree of the vertices in G.*

Proof. Consider

$$\begin{aligned}E_{it}(G) = \sum_{i=1}^{n} \mid \mu_i \mid = \mu_1 + \sum_{i=2}^{n} \mid \mu_i \mid \geq \mu_1 + \left|\sum_{i=2}^{n}\mu_i\right| \\ = \mu_1 + \mid \gamma_{it} - \mu_1 \mid \\ \geq \bar{d} + \frac{\gamma_{it}}{n} + \mid \gamma_{it} - \mu_1 \mid \qquad (\textit{By using theorem } 3.5) \\ \geq \bar{d} + \frac{\gamma_{it}}{n}.\end{aligned}$$

□

Remark 4.2. *1. Equality in Theorem (4.1) occurs if the graph G is complete.*
2. For any graph G we have, $E_{it}(G) \geq \bar{d} + \frac{\gamma_{it}}{n}$ *and* $\bar{d} \geq \delta(G)$. *This implies that* $E_{it}(G) \geq \delta(G) + \frac{\gamma_{it}}{n}$.

Theorem 4.3. *Let G be a (n, m) graph with* $P = \mid detA_{it}(G) \mid$ *is the absolute value of the determinant of* $A_{it}(G)$. *Then,*

$$\sqrt{2m + \gamma_{it} + n(n-1)P^{\frac{2}{n}}} \leq E_{it}(G) \leq \gamma_{it} + \sqrt{2mn + \gamma_{it}(n - \gamma_{it})}.$$

Proof. Consider

$$\begin{aligned} E_{it}(G) - \gamma_{it} &= \left(|\mu_1| - \frac{\gamma_{it}}{n}\right) + \left(|\mu_2| - \frac{\gamma_{it}}{n}\right) + \cdots + \left(|\mu_n| - \frac{\gamma_{it}}{n}\right) \\ &\leq \left|\mu_1 - \frac{\gamma_{it}}{n}\right| + \left|\mu_2 - \frac{\gamma_{it}}{n}\right| + \cdots + \left|\mu_n - \frac{\gamma_{it}}{n}\right| \\ &\leq \sqrt{\sum_{i=1}^{n}\left(\mu_i - \frac{\gamma_{it}}{n}\right)^2 \sqrt{n}} \qquad (By\ \ using\ \ Cauchy - Schwartz\ \ inequality) \\ &= \sqrt{2mn + n\gamma_{it} - \gamma_{it}^2}. \end{aligned}$$

Therefore

$$E_{it}(G) \leq \gamma_{it} + \sqrt{2mn + \gamma_{it}(n - \gamma_{it})}. \tag{4.1}$$

From the Arithmetic-Geometric inequality, we obtain

$$\begin{aligned} \frac{1}{n(n-1)} \sum_{i \neq j} \mid \mu_i \mid\mid \mu_j \mid &\geq \left(\prod_{i \neq j} \mid \mu_i \mid\mid \mu_j \mid \right)^{\frac{1}{n(n-1)}} \\ &\geq \left(\prod_{i=1}^{n} \mid \mu_i \mid^{2(n-1)} \right)^{\frac{1}{n(n-1)}} \\ &= \left| \prod_{i=1}^{n} \mu_i \right|^{\frac{2}{n}}. \end{aligned}$$

Thus

$$\sum_{i \neq j} \mid \mu_i \mid\mid \mu_j \mid \geq n(n-1)P^{\frac{2}{n}}.$$

Now consider

$$\begin{aligned} (E_{it}(G))^2 &= \left(\sum_{i=1}^{n} \mid \mu_i \mid \right)^2 \\ &= \sum_{i=1}^{n} (\mid \mu_i \mid)^2 + \sum_{i \neq j} \mid \mu_i \mid\mid \mu_j \mid \\ &\geq 2m + \gamma_{it} + n(n-1)P^{\frac{2}{n}} \quad (from\ \ (3.2)\ and\ (4.2)). \end{aligned}$$

Therefore

$$E_{it}(G) \geq \sqrt{2m + \gamma_{it} + n(n-1)P^{\frac{2}{n}}}. \tag{4.2}$$

From (4.1) and (4.3), the theorem follows. □

The following result exhibits the relationship between $E_{it}(G)$ and $E(G)$ in terms of γ_{it}.

Corollary 4.4. *Let G be a (n, m) graph. Then,*

$$\sqrt{E(G) + \gamma_{it}} \leq E_{it}(G) \leq \gamma_{it} + \sqrt{\tfrac{n}{2}E(G)^2 + \gamma_{it}(n - \gamma_{it})}.$$

Proof. For any graph G, using the fact that $2\sqrt{m} \leq E(G) \leq 2m$ ([18], Theorem 5.2) and the Theorem 4.3, we have

$$\begin{aligned}(E_{it}(G))^2 \geq 2m + \gamma_{it} + n(n-1)P^{\frac{2}{n}} &\geq 2m + \gamma_{it} \\ &\geq E(G) + \gamma_{it}.\end{aligned}$$

So we get

$$E_{it}(G) \geq \sqrt{E(G) + \gamma_{it}}. \tag{4.3}$$

Again from the Theorem 4.3, we have

$$\begin{aligned}E_{it}(G) &\leq \gamma_{it} + \sqrt{2mn + \gamma_{it}(n - \gamma_{it})} \\ &\leq \gamma_{it} + \sqrt{\frac{n}{2}E(G)^2 + \gamma_{it}(n - \gamma_{it})}. \end{aligned} \tag{4.4}$$

From (4.4) and (4.5), the Theorem follows. □

Theorem 4.5. *Let G be a (n, m) graph and $\mu_1, \mu_2, \mu_3, \cdots, \mu_n$ are the eigenvalues of $A_{it}(G)$ with $\mu_1 \geq \mu_2 \geq \mu_3, ... \geq \mu_n$. Then*

$$E_{it}(G) \geq \left(\frac{|\mu_n|}{\mu_1}\right) E(G).$$

Proof. For any graph G we have $E(G) \leq \sqrt{2mn}$ ([19], p.642) and in the Lemma 1.3, for $1 \leq i \leq n$, replacing $a_1 = |\mu_1|$, $a_2 = |\mu_2|$, $a_3 = |\mu_3| \cdots a_n = |\mu_n|$, we obtain

$$\begin{aligned}E_{it}(G)(|\mu_1| + |\mu_n|) &\geq 2m + \gamma_{it} + n|\mu_1||\mu_n| \\ E_{it}(G) &\geq \frac{2m + \gamma_{it} + n|\mu_1||\mu_n|}{|\mu_1| + |\mu_n|} \\ &\geq \frac{2\sqrt{(2m + \gamma_{it})n|\mu_1||\mu_n|}}{|\mu_1| + |\mu_n|} \qquad (From\ Arithmetic - Geometric\ mean\ inequality) \\ &\geq \frac{2\sqrt{2mn}\sqrt{|\mu_1||\mu_n|}}{|\mu_1| + |\mu_n|} \\ &\geq \frac{2E(G)\sqrt{|\mu_1||\mu_n|}}{|\mu_1| + |\mu_n|} \\ &\geq \left(\frac{|\mu_n|}{\mu_1}\right) E(G).\end{aligned}$$

Hence the proof. □

Theorem 4.6. *Let G be a (n, m) graph and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be the non increasing sequence of eigenvalues of $A_{it}(G)$. Then,*

$$E_{it}(G) \geq \sqrt{n(2m + \gamma_{it}) - \tfrac{n^2}{3}(\mid \mu_1 \mid - \mid \mu_n \mid)^2}.$$

Proof. Let us apply the Theorem 1.2 where we replace $a_i = 1$, $b_i =\mid \mu_i \mid$, for $1 \leq i \leq n$, $M_1 = 1$, $m_1 = 1$, $M_2 =\mid \mu_1 \mid$ and $m_2 =\mid \mu_n \mid$. Then we obtain

$$\sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} \mid \mu_i \mid^2 - \left(\sum_{i=1}^{n} \mid \mu_i \mid\right)^2 \leq \frac{n^2}{3}(\mid \mu_1 \mid - \mid \mu_n \mid)^2$$
$$n(2m + \gamma_{it}) - (E_{it}(G))^2 \leq \frac{n^2}{3}(\mid \mu_1 \mid - \mid \mu_n \mid)^2.$$

Therefore

$$E_{it}(G) \geq \sqrt{n(2m + \gamma_{it}) - \tfrac{n^2}{3}(\mid \mu_1 \mid - \mid \mu_n \mid)^2}.$$

□

Theorem 4.7. *Let G be a (n, m) graph and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be the non increasing sequence of eigenvalues of $A_{it}(G)$. Then,*

$$E_{it}(G) \geq \sqrt{n(2m + \gamma_{it}) - \theta(n)(\mid \mu_1 \mid - \mid \mu_n \mid)^2},$$

where $\theta(n) = n\lfloor \frac{n}{2} \rfloor(1 - \frac{1}{n}\lfloor \frac{n}{2} \rfloor)$.

Proof. For all $1 \leq i \leq n$, assume that, $x_i = |\mu_i| = y_i$, $x = |\mu_n| = y$, $A = |\mu_1| = B$. By using the Theorem 1.4, we obtain

$$\left| n\sum_{i=1}^{n} \mu_i^2 - \sum_{i=1}^{n} |\mu_i| \sum_{i=1}^{n} |\mu_i| \right| \leq \theta(n)(|\mu_1| - |\mu_n|)(|\mu_1| - |\mu_n|)$$
$$\left| n\sum_{i=1}^{n} \mu_i^2 - \left(\sum_{i=1}^{n} |\mu_i|\right)^2 \right| \leq \theta(n)(|\mu_1| - |\mu_n|)^2.$$

Since, $E_{it}(G) = \sum_{i=1}^{n} |\mu_i|$ and $\sum_{i=1}^{n} \mu_i^2 = 2m + \gamma_{it}$, the above inequality becomes

$$n(2m + \gamma_{it}) - (E_{it}(G))^2 \leq \theta(n)(|\mu_1| - |\mu_n|)^2.$$

Therefore

$$E_{it}(G) \geq \sqrt{n(2m + \gamma_{it}) - \theta(n)(\mid \mu_1 \mid - \mid \mu_n \mid)^2}.$$

□

5 Independent transversal dominating energy of some standard graphs

In this section, $E_{it}(G)$ is computed for some standard graphs.

Theorem 5.1. *Let K_n be the complete graph with $\gamma_{it}(K_n) = n$. Then, $E_{it}(K_n) = n$.*

Proof. Let the vertex set of K_n be $V = \{v_1, v_2, \cdots, v_n\}$ and the independent transversal dominating set be $D_{it} = \{v_1, v_2, \cdots, v_n\}$. Then, $A_{it}(K_n) = J_n$, where J_n is an $n \times n$ matrix with all the entries are $1's$. Fix $a = 0$ and $b = 1$ in the Lemma 1.1, we obtain $Spec_{it}(K_n) = \begin{pmatrix} n & 0 \\ 1 & n-1 \end{pmatrix}$.

Therefore $E_{it}(K_n) = n$. □

Definition 5.2. *The cocktail party graph is denoted by $K_{n\times 2}$ is a graph having the vertex set $V = \bigcup_{i=1}^{n} \{u_i, v_i\}$ and an edge set $E = \{u_iu_j, v_iv_j, i \neq j\} \bigcup \{u_iv_j, v_iu_j, i \leq i < j \leq n\}$.*

Theorem 5.3. *Let $K_{n\times 2}$ be the cocktail party graph with $\gamma_{it}(K_{n\times 2}) = n$. Then,*

$$E_{it}(K_{n\times 2}) = (2 + \sqrt{5})n - (1 + \sqrt{5}).$$

Proof. Let the vertex set of $K_{n\times 2}$ be $V = \{u_1, v_1, u_2, v_2, \cdots, u_n, v_n\}$ and the independent transversal dominating set $D_{it} = \{u_1, u_2, u_3, \cdots, u_n\}$. Then,

$$A_{it}(K_{n\times 2}) = \begin{bmatrix} 1 & 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & \cdots & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & \cdots & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 \end{bmatrix}_{2n \times 2n}.$$

Consider

$$\phi_n(K_{n\times 2}, \mu) = \mid \mu I - A_{it}(K_{n\times 2}) \mid = \begin{vmatrix} \mu-1 & 0 & -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ 0 & \mu & -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ -1 & -1 & \mu-1 & 0 & \cdots & -1 & -1 & -1 & -1 \\ -1 & -1 & 0 & \mu & \cdots & -1 & -1 & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & -1 & -1 & \cdots & \mu-1 & 0 & -1 & -1 \\ -1 & -1 & -1 & -1 & \cdots & 0 & \mu & -1 & -1 \\ -1 & -1 & -1 & -1 & \cdots & -1 & -1 & \mu-1 & 0 \\ -1 & -1 & -1 & -1 & \cdots & -1 & -1 & 0 & \mu \end{vmatrix}_{2n\times 2n}.$$

Now replace C_1 by $C_1' = C_1 + C_2 + \cdots + C_n$ and then replace R_{2k+1}' by $R_{2k+1}'' = R_{2k+1}' - R_1'$ and R_{2k+2}' by $R_{2k+2}'' = R_{2k+2}' - R_2'$ for $k = 1, 2, \cdots, n-1$. We get

$$= \begin{vmatrix} \mu-(2n-1) & 0 & -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ \mu-(2n-2) & \mu & -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ 0 & -1 & \mu & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1-\mu & 1 & \mu+1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & -1 & 0 & 0 & \cdots & \mu & 1 & 0 & 0 \\ 0 & -1-\mu & 0 & 0 & \cdots & 1 & \mu+1 & 0 & 0 \\ 0 & -1 & 0 & 0 & \cdots & 0 & 0 & \mu & 1 \\ 0 & -1-\mu & 0 & 0 & \cdots & 0 & 0 & 1 & \mu+1 \end{vmatrix}_{2n\times 2n}.$$

The characteristic polynomial is

$$\phi_n(K_{n\times 2},\mu) = (\mu^2+\mu+1)^{n-1}[\mu^2+(1-2n)\mu+(n-1)].$$

The Characteristic equation is

$$\phi_n(K_{n\times 2},\mu) = (\mu^2+\mu+1)^{n-1}[\mu^2+(1-2n)\mu+(n-1)] = 0$$

$$Spec_{it}(K_{n\times 2}) = \begin{pmatrix} \frac{(2n-1)+\sqrt{4n^2-8n+5}}{2} & \frac{\sqrt{5}-1}{2} & \frac{(2n-1)-\sqrt{4n^2-8n+5}}{2} & \frac{-\sqrt{5}-1}{2} \\ 1 & n-1 & 1 & n-1 \end{pmatrix}.$$

Therefore

$$E_{it}(K_{n\times 2}) = (2+\sqrt{5})n - (1+\sqrt{5}).$$

□

Definition 5.4. *For an integer $n \geq 3$, S_n^0 denotes the crown graph graph with the vertex set $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ and an edge set $\{u_i v_i : 1 \leq i, j \leq n, i \neq j\}$. Therefore, S_n^0 coincides with the complete bipartite graph $K_{n,n}$ with the horizontal edges removed.*

Theorem 5.5. *Let S_n^0 be the crown graph with $\gamma_{it}(S_n^0) = 2$. Then,*

$$E_{it}(S_n^0) = 2(n-2) + \sqrt{n^2-2n+5} + \sqrt{n^2+2n-3}.$$

Further

$$\lfloor E_{it}(S_n^0) \rfloor = 4(n-1).$$

Proof. Let the vertex set of S_n^0 be $V = \{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ and the independent transversal dominating set $D_{it} = \{u_1, v_1\}$. Then,

$$A_{it}(S_n^0) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{2n\times 2n}.$$

Consider

$$\phi_n(S_n^0,\mu) = \mid \mu I - A_{it}(S_n^0) \mid = \begin{vmatrix} \mu-1 & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \mu & 0 & \cdots & 0 & -1 & 0 & -1 & \cdots & -1 \\ 0 & 0 & \mu & \cdots & 0 & -1 & -1 & 0 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu & -1 & -1 & -1 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \mu-1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & -1 & \cdots & -1 & 0 & \mu & 0 & \cdots & 0 \\ -1 & -1 & 0 & \cdots & -1 & 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & \mu \end{vmatrix}_{2n\times 2n}.$$

Replace C_1 by $C_1^{'} = C_1 + C_2 + \cdots + C_{2n}$, we get

$$= \begin{vmatrix} \mu-n & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ \mu-n+1 & \mu & 0 & \cdots & 0 & -1 & 0 & -1 & \cdots & -1 \\ \mu-n+1 & 0 & \mu & \cdots & 0 & -1 & -1 & 0 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu-n+1 & 0 & 0 & \cdots & \mu & -1 & -1 & -1 & \cdots & 0 \\ \mu-n & -1 & -1 & \cdots & -1 & \mu-1 & 0 & 0 & \cdots & 0 \\ \mu-n+1 & 0 & -1 & \cdots & -1 & 0 & \mu & 0 & \cdots & 0 \\ \mu-n+1 & -1 & 0 & \cdots & -1 & 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu-n+1 & -1 & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & \mu \end{vmatrix}_{2n\times 2n}.$$

Now replace $R_k^{'}$ by $R_k^{''} = R_k^{'} - R_2^{'}$ $(k = 3, 4, \cdots, n)$, $R_t^{'}$ by $R_t^{''} = R_t^{'} - R_{n+2}^{'}$ $(t = n+3, n+4, \cdots, 2n)$ and $R_{n+1}^{''} = R_{n+1}^{'} - R_1^{'}$. We obtain

$$= \begin{vmatrix} \mu-n & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ \mu-n+1 & \mu & 0 & \cdots & 0 & -1 & 0 & -1 & \cdots & -1 \\ 0 & -\mu & \mu & \cdots & 0 & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -\mu & 0 & \cdots & \mu & 0 & -1 & 0 & \cdots & 1 \\ 0 & -1 & -1 & \cdots & -1 & \mu-1 & 1 & 1 & \cdots & 1 \\ \mu-n+1 & 0 & -1 & \cdots & -1 & 0 & \mu & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & -\mu & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & 0 & \cdots & 1 & 0 & -\mu & 0 & \cdots & \mu \end{vmatrix}_{2n\times 2n}.$$

Again, replace R''_{n+2} by $R'''_{n+2} = R''_{n+2} - R''_2$. We get

$$= \begin{vmatrix} \mu-n & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ \mu-n+1 & \mu & 0 & \cdots & 0 & -1 & 0 & -1 & \cdots & -1 \\ 0 & -\mu & \mu & \cdots & 0 & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -\mu & 0 & \cdots & \mu & 0 & -1 & 0 & \cdots & 1 \\ 0 & -1 & -1 & \cdots & -1 & \mu-1 & 1 & 1 & \cdots & 1 \\ 0 & -\mu & -1 & \cdots & -1 & 1 & \mu & 1 & \cdots & 1 \\ 0 & -1 & 1 & \cdots & 0 & 0 & -\mu & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & 0 & \cdots & 1 & 0 & -\mu & 0 & \cdots & \mu \end{vmatrix}_{2n\times 2n}.$$

The characteristic polynomial is

$$\phi_n(S_n^0, \mu) = (\mu-1)^{n-2}(\mu+1)^{n-2}(\mu^2 + (1-n)\mu - 1)(\mu^2 + (n-3)\mu + (3-2n)).$$

The characteristic equation is

$$(\mu-1)^{n-2}(\mu+1)^{n-2}(\mu^2 + (1-n)\mu - 1)(\mu^2 + (n-3)\mu + (3-2n)) = 0.$$

$$Spec_{it}(S_n^0) = \begin{pmatrix} 1 & -1 & \frac{(n-1)+\sqrt{n^2-2n+5}}{2} & \frac{(3-n)+\sqrt{n^2+2n-3}}{2} & \frac{(n-1)-\sqrt{n^2-2n+5}}{2} & \frac{(3-n)-\sqrt{n^2+2n-3}}{2} \\ n-2 & n-2 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

So we obtain

$$E_{it}(S_n^0) = 2(n-2) + \sqrt{n^2-2n+5} + \sqrt{n^2+2n-3}.$$

For $n \geq 3$, consider a function $f(n) = \frac{2(n-2)+\sqrt{n^2-2n+5}+\sqrt{n^2+2n-3}}{4(n-1)}$ is a decreasing function on $[3, \infty)$. We can also see that $1 = \lim_{n\to\infty} f(n) < f(n) \leq f(3) \cong 1.0365$ it leads to get $\lfloor f(n) \rfloor = 1$. Hence

$$\lfloor E_{it}(S_n^0) \rfloor = 4(n-1).$$

□

Theorem 5.6. *For the integers m, $n \geq 2$, let $K_{m,n}$ denote the complete bipartite graph with $\gamma_{it}(K_{m,n}) = 2$. Then, the characteristic polynomial of $A_{it}(K_{m,n})$ is $\phi_n(K_{m,n}, \mu) = \mu^{n+n-4}P_4(\mu)$, where $P_4(\mu) = \mu^4 - 2\mu^3 - (mn-1)\mu^2 + (2mn - n - m)\mu - (m-1)(n-1)$ and the spectrum is $Spec_{it}(K_{m,n}) = \begin{pmatrix} \mu_1 & \mu_2 & \mu_3 & 0 & \mu_{m+n} \\ 1 & 1 & 1 & m+n-4 & 1 \end{pmatrix}$, where (i) $2\left(\frac{mn+1}{m+n}\right) \leq \mu_1 \leq \left(\frac{m+n+|m-n|}{2}\right) + 1$. (ii) $0 < \mu_2 + \mu_3 < 2$, $\mu_{m+n} < -1$ and $\mu_1 - 2 < |\mu_{m+n}| < \mu_1$. (iii) $E_{it}(K_{m,n}) = 2(1 + |\mu_{m+n}|)$. (iv) $2(\mu_1 - 1) < E_{it}(K_{m,n}) < 2(\mu_1 + 1)$.*

Proof. Let the vertex set of $K_{m,n}$ be $V = \{u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_n\}$ and the independent transversal dominating set $D_{it} = \{u_1, v_1\}$.
Consider $\phi_n(K_{m,n}, \mu) =| \mu I - A_{it}(K_{m,n}) |= \begin{vmatrix} P_m & -J^T_{m\times n} \\ -J_{n\times m} & P_n \end{vmatrix} = |P_m||P_n - JP_m^{-1}J^T|$,

where $P_m = \begin{bmatrix} \mu-1 & 0 & 0 & \cdots & 0 \\ 0 & \mu & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu \end{bmatrix}_{m\times m}$ and $J_{n\times m}$ is the matrix with all entries equal to unity.

On simplifying, we get the characteristic polynomial is

$$\phi_n(K_{m,n}, \mu) = \mu^{m+n-4}\left\{\mu^4 - 2\mu^3 - (mn-1)\mu^2 + (2mn-n-m)\mu - (m-1)(n-1)\right\}. \tag{5.1}$$

The characteristic equation is

$$\mu^{m+n-4}\left\{\mu^4 - 2\mu^3 - (mn-1)\mu^2 + (2mn-n-m)\mu - (m-1)(n-1)\right\} = 0.$$

Consider the biquadratic equation $P_4(x) = x^4 - 2x^3 - (mn-1)x^2 + (2mn-n-m)x - (m-1)(n-1) = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$.
(i) By using the Theorem 3.5, immediately we get $2\left(\frac{mn+1}{m+n}\right) \leq \mu_1 \leq \left(\frac{m+n+|m-n|}{2}\right) + 1$.
(ii) Since $\mu_1 \geq 2\left(\frac{mn+1}{m+n}\right) > 2$, so one of the roots is positive and strictly greater than 2 and let $\alpha = \mu_1 > 2$. We know that $\alpha\beta\gamma\delta = -(m-1)(n-1)$ then there is atleast one negative root say $\delta < 0$. Now $|P_4(1)| = |1-\alpha||1-\beta||1-\gamma||1-\delta| = 1$ but $|1-\beta||1-\gamma||1-\delta| = \frac{1}{|1-\alpha|} < 1$ $(\because \alpha > 2)$ so there is atleast one quantity less than 1 and let it be $|1-\beta| < 1$ which implies that $0 < \beta < 2$. Now we have $\alpha > 0$, $\beta > 0$, $\delta < 0$ and $\alpha\beta\gamma\delta < 0$ it gives $\gamma > 0$. Since $\mu_1 \geq \mu_2 \cdots \geq \mu_n \geq \cdots \geq \mu_{m+n}$ and $\alpha + \beta + \gamma + \delta = 2$ without loss of generality let $\alpha = \mu_1$, $\beta = \mu_2$, $\gamma = \mu_3$ and $\delta = \mu_{m+n}$. Since $K_{m,n}$ is a connected graph so $A(K_{m,n})$ is irreducible, but $A_{it}(K_{m,n})$ is same as $A(K_{m,n})$ except two diagonal entries which are 1's, $A_{it}(K_{m,n})$ is also irreducible(Godsil and Royle [11]). As a consequence μ_1 is a spectral radius and hence $|\mu_{m+n}| < \mu_1$. But $|\mu_{m+n}| = (\mu_1 - 2) + \mu_2 + \mu_3$ and $|\mu_{m+n}| < \mu_1$ we obtain $\mu_{m+n} < -1 (\because P_4(-1) < 0)$, $0 < \mu_2 + \mu_3 < 2$ and $\mu_1 - 2 < |\mu_{m+n}| < \mu_1$.
(iii) It is clear that $E_{it}(K_{m,n}) = \mu_1 + \mu_2 + \mu_3 + |\mu_{m+n}| = 2(1 + |\mu_{m+n}|)$.
(iv) Since $\mu_1 - 2 < |\mu_{m+n}| < \mu_1$, its easy to obtain $2(\mu_1 - 1) < E_{it}(K_{m,n}) < 2(\mu_1 + 1)$.
Hence the theorem. □

Corollary 5.7. *For an integer $n \geq 2$, let $K_{1,n-1}$ denote the star graph with $\gamma_{it}(K_{1,n-1}) = 2$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be the non increasing sequence of eigenvalues of $A_{it}(K_{1,n-1})$. Then, $E_{it}(K_{1,n-1}) = 2(1+ | \mu_n |)$.*

Proof. Replace $m = 1$ and $n = n-1$ in (5.1), we get the characteristic polynomial of $K_{1,n-1}$ is $\phi_n(K_{1,n-1},\mu) = \mu^{n-3}\left[\mu^3 - 2\mu^2 + (2-n)\mu + (n-2)\right]$. Since the coefficients of μ^2 is negative and coefficients of constant is positive in the cubic equation $\mu^3 - 2\mu^2 + (2-n)\mu + (n-2) = 0$, so it admits exactly one negative and two positive real roots. Without loss of generality, let the roots be μ_1, μ_2 and μ_n. Since $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, so we must have $\mu_1 > 0$, $\mu_2 > 0$ and $\mu_n < 0$. Therefore $\mid \mu_1 \mid + \mid \mu_2 \mid + \mid \mu_n \mid = 2(\mid \mu_n \mid +1)$.
Hence the proof. □

Corollary 5.8. *For $n \geq 2$, let $K_{n,n}$ denotes the complete bipartite graph on $2n$ vertices. Then,* $E_{it}(K_{n,n}) = n + 1 + \sqrt{n^2+2n-3}$.

Proof. Replace $m = n$ in (5.1), we get the characteristic polynomial of $K_{n,n}$ is

$$\phi_n(K_{n,n},\mu) = \mu^{2n-4}\left[\mu^4 - 2\mu^3 - (n^2-1)\mu^2 + 2n(n-1)\mu - (n-1)^2\right].$$

The characteristic equation is

$$\mu^{2n-4}\left[\mu^4 - 2\mu^3 - (n^2-1)\mu^2 + 2n(n-1)\mu - (n-1)^2\right] = 0.$$

$$Spec_{it}(K_{n,n}) = \begin{pmatrix} \frac{(n+1)+\sqrt{n^2-2n+5}}{2} & \frac{(1-n)+\sqrt{n^2+2n-3}}{2} & \frac{(n+1)-\sqrt{n^2-2n+5}}{2} & 0 & \frac{(1-n)-\sqrt{n^2+2n-3}}{2} \\ 1 & 1 & 1 & 2n-4 & 1 \end{pmatrix}.$$

Therefore

$$E_{it}(K_{n,n}) = n + 1 + \sqrt{n^2+2n-3}.$$

□

In the literature, minimum covering color energy of a graph [22] and minimum equitable dominating color energy of a graph [21] and so on are studied. This paper also gives a scope to define similar color energy and work on it.

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