

## TWO VARIABLE DEGENERATE BELL POLYNOMIALS ASSOCIATED WITH POISSON DEGENERATE CENTRAL MOMENTS

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**ABSTRACT.** In this paper, we introduce two variable degenerate Bell polynomials as a generalization of the degenerate Bell polynomials which were studied earlier. Among other things, we show their connections with Poisson degenerate central moments and Charlier polynomials.

### 1. INTRODUCTION

In recent years, the idea of studying degenerate versions of many special polynomials and numbers, which was initiated by Carlitz in [4], received regained interests of some mathematicians and many interesting results were discovered (see [7,8,10,11,13-15]). They have been explored by employing several different tools such as combinatorial methods, generating functions,  $p$ -adic analysis, umbral calculus techniques, differential equations and probability theory.

The aim of this paper is to introduce two variable degenerate Bell polynomials and study their connections with Poisson degenerate central moments and Charlier polynomials. Here we note that those polynomials are generalizations of the single variable degenerate Bell polynomials and in turn these single variable degenerate Bell polynomials are degenerate versions of the single variable Bell polynomials (see [11,13,15]).

The outline of this paper is as follows. In Section 1, as a preparation to the next section we will recall the degenerate exponential functions, the Stirling numbers of the first and second kinds, Bell polynomials and numbers, the degenerate Stirling numbers of the first and second kinds, the Poisson random variable with parameter  $\alpha$ , the degenerate Bell polynomials and numbers, and the Charlier polynomials. In Section 2, we will introduce the two variable degenerate Bell polynomials, express them in terms of the degenerate Bell polynomials and show they reduce to the degenerate Bell polynomials when the two arguments are the same. We will define the Poisson degenerate central moments and prove that they are equal to the two variable degenerate Bell polynomials. Also, we will find an explicit expression for the two variable degenerate Bell polynomials. By letting one of the arguments of the two variable degenerate Bell polynomials equal to zero, we will be led to define Stirling type numbers as the coefficients in the expansion of powers of  $\alpha$ . Then we will express the Stirling type numbers in terms of the degenerate Stirling numbers of the second kind and vice versa. Finally, we will be able to find certain connections between the two variable degenerate Bell polynomials and Charlier polynomials involving degenerate Stirling numbers.

For any  $0 \neq \lambda \in \mathbb{R}$ , the degenerate exponential functions are defined by

$$(1) \quad e_{\lambda}^x(t) = (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \quad (\text{see [7, 13, 14]}),$$

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where  $(x)_{0,\lambda} = 1$ ,  $(x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$ ,  $(n \geq 1)$ .

In (1), we note that  $\lim_{\lambda \rightarrow 0} e_\lambda^x(t) = e^{tx}$ , and let  $e_\lambda(t) = e_\lambda^1(t)$ .

As is well known, the Stirling numbers of the first kind are defined as

$$(2) \quad (x)_n = \sum_{l=0}^n S_1(n,l)x^l, \quad (n \geq 0), \quad (\text{see [8, 10]}),$$

where  $(x)_0 = 1$ ,  $(x)_n = x(x-1)(x-2)\cdots(x-(n-1))$ ,  $(n \geq 1)$ . As the inversion formula of (2), the Stirling numbers of the second kind are defined by

$$(3) \quad x^n = \sum_{l=0}^n S_2(n,l)(x)_l, \quad (n \geq 0), \quad (\text{see [8, 10]}).$$

From (2) and (3), we note that

$$(4) \quad \frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k) \frac{t^n}{n!},$$

and

$$(5) \quad \frac{1}{k!}(e^t - 1)^k = \sum_{n=k}^{\infty} S_2(n,k) \frac{t^n}{n!}, \quad (k \geq 0).$$

The Bell polynomials are given by

$$(6) \quad e^{x(e^t-1)} = \sum_{n=0}^{\infty} \text{Bel}_n(x) \frac{t^n}{n!}, \quad (\text{see [1-3, 5, 6, 17]}).$$

Thus, we have

$$(7) \quad \text{Bel}_n(x) = \sum_{l=0}^n S_2(n,l)x^l = e^{-x} \sum_{k=0}^{\infty} \frac{k^n}{k!} x^k, \quad (\text{see [1-3, 5, 6, 17]}).$$

When  $x = 1$ ,  $\text{Bel}_n = \text{Bel}_n(1)$  are called the Bell numbers.

The degenerate Stirling numbers of the second kind are defined by

$$(8) \quad (x)_{n,\lambda} = \sum_{l=0}^n S_{2,\lambda}(n,l)(x)_l, \quad (n \geq 0), \quad (\text{see [7, 10, 13-15]}).$$

As the inversion formula of (8), the degenerate Stirling numbers of the first kind are given by

$$(9) \quad (x)_n = \sum_{l=0}^n S_{1,\lambda}(n,l)(x)_{l,\lambda}, \quad (n \geq 0), \quad (\text{see [7, 14]}).$$

Let  $\log_\lambda(t)$  be the compositional inverse of  $e_\lambda(t)$  such that  $e_\lambda(\log_\lambda t) = \log_\lambda(e_\lambda(t)) = t$ . Then we note that

$$(10) \quad \log_\lambda(1+t) = \frac{1}{\lambda}((1+t)^\lambda - 1) = \sum_{n=1}^{\infty} \lambda^{n-1}(1)_{n,\frac{1}{\lambda}} \frac{t^n}{n!}, \quad (\text{see [7, 14]}).$$

From (10), we note that  $\lim_{\lambda \rightarrow 0} \log_\lambda(1+t) = \log(1+t)$ . By (8) and (9), we easily get

$$(11) \quad \frac{1}{k!}(e_\lambda(t) - 1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k) \frac{t^n}{n!}, \quad (\text{see [7, 10, 13-15]}),$$

and

$$(12) \quad \frac{1}{k!}(\log_\lambda(1+t))^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k) \frac{t^n}{n!}, \quad (\text{see [7, 14]}),$$

where  $k$  is a nonnegative integer.

A random variable  $X$  is a real valued function defined on a sample space. If  $X$  takes any values in a countable set, then  $X$  is called a discrete random variable. For a discrete random variable  $X$ , the probability mass function  $p(a)$  of  $X$  is defined by

$$p(a) = P\{X = a\}, \quad (\text{see [13, 19]}).$$

A random variable  $X$  taking on one of the values  $0, 1, 2, \dots$  is said to be the Poisson random variable with parameter  $\alpha (> 0)$  if the probability mass function of  $X$  is given by

$$(13) \quad p(i) = P\{X = i\} = e^{-\alpha} \frac{\alpha^i}{i!}, \quad i = 0, 1, 2, \dots$$

For  $n \geq 1$ , the quantity  $E[X^n]$  of the Poisson random variable  $X$  with parameter  $\alpha (> 0)$ , which is called the  $n$ -th moment of  $X$ , is given by

$$(14) \quad E[X^n] = \sum_{i=0}^{\infty} i^n p(i) = e^{-\alpha} \sum_{i=0}^{\infty} \frac{i^n}{i!} \alpha^i = \text{Bel}_n(\alpha).$$

Recently, Kim-Kim-Dolgy introduced the degenerate Bell polynomials given by

$$(15) \quad e^{x(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} \text{Bel}_{n,\lambda}(x) \frac{t^n}{n!}, \quad (\text{see [11, 13, 15]}).$$

When  $x = 1$ ,  $\text{Bel}_{n,\lambda} = \text{Bel}_{n,\lambda}(1)$  are called the degenerate Bell numbers. It is not difficult to show that  $\lim_{\lambda \rightarrow 0} \text{Bel}_{n,\lambda}(x) = \text{Bel}_n(x)$ .

From (15), we note that

$$(16) \quad \text{Bel}_{n,\lambda}(x) = \sum_{l=0}^n x^l S_{2,\lambda}(n, l) = e^{-x} \sum_{k=0}^{\infty} \frac{(k)_{n,\lambda}}{k!} x^k, \quad (\text{see [11, 13, 15]}).$$

It is well known that Charlier polynomials  $C_n(x; \alpha)$  are defined by

$$(17) \quad e^{-\alpha}(1+t)^x = \sum_{n=0}^{\infty} C_n(x; \alpha) \frac{t^n}{n!}, \quad (\text{see [16 - 18]}),$$

where  $x, t, \alpha \in \mathbb{R}$ . We observe that

$$(18) \quad \begin{aligned} e^{-\alpha}(1+t)^x &= \sum_{m=0}^{\infty} (-\alpha)^m \frac{t^m}{m!} \sum_{l=0}^{\infty} x^l \frac{(\log(1+t))^l}{l!} \\ &= \sum_{m=0}^{\infty} (-\alpha)^m \frac{t^m}{m!} \sum_{k=0}^{\infty} \sum_{l=0}^k x^l S_1(k, l) \frac{t^k}{k!} \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n (-\alpha)^{n-k} \binom{n}{k} \sum_{l=0}^k x^l S_1(k, l) \right) \frac{t^n}{n!}. \end{aligned}$$

Thus, by (17) and (18), we get

$$(19) \quad C_n(x; \alpha) = \sum_{l=0}^n \left( \sum_{k=l}^n \binom{n}{k} (-1)^{n-k} \alpha^{n-k} S_1(n, k) \right) x^l, \quad (n \geq 0).$$

## 2. TWO VARIABLE DEGENERATE BELL POLYNOMIALS

For  $\alpha, \beta \in \mathbb{R}$ , let us consider *two variable degenerate Bell polynomials* which are given by

$$(20) \quad e^{\alpha(e_\lambda(t)-1)} e_\lambda^{\beta-\alpha}(t) = \sum_{n=0}^{\infty} \text{Bel}_{n,\lambda}(\alpha, \beta) \frac{t^n}{n!}.$$

From the left hand side of (20), we have

$$(21) \quad \begin{aligned} e^{\alpha(e_\lambda(t)-1)} e_\lambda^{\beta-\alpha}(t) &= \sum_{l=0}^{\infty} \text{Bel}_{l,\lambda}(\alpha) \frac{t^l}{l!} \sum_{m=0}^{\infty} (\beta - \alpha)_{m,\lambda} \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} \left( \sum_{l=0}^n \binom{n}{l} \text{Bel}_{l,\lambda}(\alpha) (\beta - \alpha)_{n-l,\lambda} \right) \frac{t^n}{n!}. \end{aligned}$$

Therefore, by (20) and (21), we obtain the following theorem.

**Theorem 1.** For  $n \geq 0$ , we have

$$\text{Bel}_{n,\lambda}(\alpha, \beta) = \sum_{l=0}^n \binom{n}{l} \text{Bel}_{l,\lambda}(\alpha) (\beta - \alpha)_{n-l,\lambda}$$

In particular,

$$\text{Bel}_{n,\lambda}(\alpha, \alpha) = \text{Bel}_{n,\lambda}(\alpha), \quad (n \geq 0).$$

Let  $X$  be a Poisson random variable with parameter  $\alpha (> 0)$ . Then we consider the *Poisson degenerate central moments* given by  $E[(X - \alpha)_{n,\lambda}]$ , ( $n \geq 0$ ),

where  $(X - \alpha)_{0,\lambda} = 1$ ,  $(X - \alpha)_{n,\lambda} = (X - \alpha)(X - \alpha - \lambda) \cdots (X - \alpha - (n-1)\lambda)$ , ( $n \geq 1$ ).

Note that

$$\lim_{\lambda \rightarrow 0} E[(X - \alpha)_{n,\lambda}] = E[(X - \alpha)^n].$$

We observe that

$$(22) \quad \begin{aligned} E[e_\lambda^{X-\alpha+\beta}(t)] &= \sum_{k=0}^{\infty} e_\lambda^{k-\alpha+\beta}(t) p(k) \\ &= e_\lambda^{\beta-\alpha}(t) e^{-\alpha} \sum_{k=0}^{\infty} e_\lambda^k(t) \frac{\alpha^k}{k!} \\ &= e_\lambda^{\beta-\alpha}(t) e^{\alpha(e_\lambda(t)-1)} \\ &= \sum_{n=0}^{\infty} \text{Bel}_{n,\lambda}(\alpha, \beta) \frac{t^n}{n!}. \end{aligned}$$

On the other hand,

$$(23) \quad \begin{aligned} E[e_\lambda^{X-\alpha+\beta}(t)] &= E \left[ \sum_{n=0}^{\infty} (X - \alpha + \beta)_{n,\lambda} \frac{t^n}{n!} \right] \\ &= \sum_{n=0}^{\infty} E[(X - \alpha + \beta)_{n,\lambda}] \frac{t^n}{n!}. \end{aligned}$$

Therefore, by (22) and (23), we obtain the following theorem.

**Theorem 2.** For  $n \geq 0$ , we have

$$E[(X - \alpha + \beta)_{n,\lambda}] = \text{Bel}_{n,\lambda}(\alpha, \beta).$$

In particular,

$$E[(X - \alpha + \beta)^n] = \sum_{l=0}^n \binom{n}{l} \text{Bel}_l(\alpha) (\beta - \alpha)^{n-l}.$$

It is easy to show that

$$(24) \quad (x+y)_{n,\lambda} = \sum_{i=0}^n \binom{n}{i} (x)_{i,\lambda} (y)_{n-i,\lambda}, \quad (n \geq 0).$$

By (24), we get

$$(25) \quad E[(X - \alpha + \beta)_{n,\lambda}] = \sum_{k=0}^n \binom{n}{k} (\beta - \alpha)_{n-k,\lambda} E[(X)_{k,\lambda}].$$

From (13) and (15), we have

$$(26) \quad \begin{aligned} E[e_\lambda^X(t)] &= \sum_{n=0}^\infty e_\lambda^n(t) p(n) = e^{-\alpha} \sum_{n=0}^\infty e_\lambda^n(t) \frac{\alpha^n}{n!} \\ &= e^{\alpha(e_\lambda(t)-1)} = \sum_{n=0}^\infty \text{Bel}_{n,\lambda}(\alpha) \frac{t^n}{n!}. \end{aligned}$$

On the other hand,

$$(27) \quad E[e_\lambda^X(t)] = E\left[\sum_{n=0}^\infty (X)_{n,\lambda} \frac{t^n}{n!}\right] = \sum_{n=0}^\infty E[(X)_{n,\lambda}] \frac{t^n}{n!}.$$

By (26) and (27), we get

$$(28) \quad \text{Bel}_{n,\lambda}(\alpha) = E[(X)_{n,\lambda}], \quad (n \geq 0).$$

From Theorem 2, (25) and (28), we have

$$(29) \quad \begin{aligned} \text{Bel}_{n,\lambda}(\alpha, \beta) &= E[(X - \alpha + \beta)_{n,\lambda}] = \sum_{k=0}^n \binom{n}{k} (\beta - \alpha)_{n-k,\lambda} E[(X)_{k,\lambda}] \\ &= \sum_{k=0}^n \binom{n}{k} (\beta - \alpha)_{n-k,\lambda} \text{Bel}_{k,\lambda}(\alpha) \\ &= \sum_{k=0}^n \binom{n}{k} (\beta - \alpha)_{n-k,\lambda} \sum_{i=0}^k \alpha^i S_{2,\lambda}(k, i) \\ &= \sum_{i=0}^n \alpha^i \sum_{k=i}^n \binom{n}{k} (\beta - \alpha)_{n-k,\lambda} S_{2,\lambda}(k, i), \quad (n \geq 0). \end{aligned}$$

Therefore, by (29), we obtain the following theorem.

**Theorem 3.** For  $n \geq 0$ , we have

$$\text{Bel}_{n,\lambda}(\alpha, \beta) = \sum_{i=0}^n \alpha^i \sum_{k=i}^n \binom{n}{k} (\beta - \alpha)_{n-k,\lambda} S_{2,\lambda}(k, i).$$

Let us take  $\beta = 0$  in Theorem 2. Then we have

$$(30) \quad E[(X - \alpha)_{n,\lambda}] = \text{Bel}_{n,\lambda}(\alpha, 0), \quad (n \geq 0).$$

In light of (16), we define *Stirling type numbers*  $S_\lambda(n, k)$  associated with  $\text{Bel}_{n,\lambda}(\alpha, 0)$  as

$$(31) \quad E[(X - \alpha)_{n,\lambda}] = \text{Bel}_{n,\lambda}(\alpha, 0) = \sum_{k=0}^n S_\lambda(n, k) \alpha^k, \quad (n \geq 0).$$

When  $\alpha = 1$ , the Poisson degenerate central moments are given by

$$(32) \quad \text{Bel}_{n,\lambda}(1, 0) = E[(X - 1)_{n,\lambda}] = \sum_{k=0}^n S_\lambda(n, k), \quad (n \geq 0).$$

By Theorem 2 and (24), we easily get

$$\begin{aligned}
 (33) \quad \text{Bel}_{n,\lambda}(\alpha, \beta) &= E[(X - \alpha + \beta)_{n,\lambda}] = \sum_{k=0}^n \binom{n}{k} (\beta)_{n-k,\lambda} E[(X - \alpha)_{k,\lambda}] \\
 &= \sum_{k=0}^n \binom{n}{k} (\beta)_{n-k,\lambda} \text{Bel}_{k,\lambda}(\alpha, 0) = \sum_{k=0}^n \binom{n}{k} (\beta)_{n-k,\lambda} \sum_{l=0}^k S_\lambda(k, l) \alpha^l \\
 &= \sum_{l=0}^n \alpha^l \sum_{k=l}^n \binom{n}{k} (\beta)_{n-k} S_k(k, l).
 \end{aligned}$$

From (29), we note that

$$\begin{aligned}
 (34) \quad \text{Bel}_{n,\lambda}(\alpha, \beta) &= \sum_{k=0}^n \binom{n}{k} (\beta - \alpha)_{k,\lambda} \text{Bel}_{n-k,\lambda}(\alpha) \\
 &= \sum_{k=0}^n \binom{n}{k} \sum_{l=0}^k \binom{k}{l} (\beta)_{l,\lambda} (-\alpha)_{k-l,\lambda} \sum_{i=0}^{n-k} \alpha^i S_{2,\lambda}(n-k, i) \\
 &= \sum_{k=0}^n \sum_{l=0}^k \binom{n}{l} \binom{n-l}{n-k} (\beta)_{l,\lambda} (-\alpha)_{k-l,\lambda} \sum_{i=0}^{n-k} \alpha^i S_{2,\lambda}(n-k, i) \\
 &= \sum_{l=0}^n \sum_{b=0}^{n-l} \binom{n}{l} \binom{n-l}{b} (\beta)_{l,\lambda} (-\alpha)_{n-b-l,\lambda} \sum_{i=0}^b \alpha^i S_{2,\lambda}(b, i) \\
 &= \sum_{l=0}^n \sum_{b=0}^l \binom{n}{n-l} \binom{l}{b} (\beta)_{n-l,\lambda} (-\alpha)_{l-b,\lambda} \sum_{i=0}^b \alpha^i S_{2,\lambda}(b, i) \\
 &= \sum_{l=0}^n \sum_{b=0}^l \binom{n}{l} \binom{l}{l-b} (\beta)_{n-l,\lambda} (-\alpha)_{b,\lambda} \sum_{i=0}^{l-b} \alpha^i S_{2,\lambda}(l-b, i) \\
 &= \sum_{l=0}^n \sum_{b=0}^l \binom{n}{l} \binom{l}{b} (\beta)_{n-l,\lambda} (-\alpha)_{b,\lambda} \sum_{i=b}^l \alpha^{i-b} S_{2,\lambda}(l-b, i-b) \\
 &= \sum_{l=0}^n \binom{n}{l} (\beta)_{n-l,\lambda} \sum_{i=0}^l \alpha^i \sum_{b=0}^i \binom{l}{b} (-1)^b \langle 1 \rangle_{b, \frac{\lambda}{\alpha}} S_{2,\lambda}(l-b, i-b),
 \end{aligned}$$

where  $\langle x \rangle_{0,\lambda} = 1$ ,  $\langle x \rangle_{n,\lambda} = x(x+\lambda)(x+2\lambda)\cdots(x+(n-1)\lambda)$ , ( $n \geq 1$ ).

Thus, by (34), we get

$$\begin{aligned}
 (35) \quad \text{Bel}_{n,\lambda}(\alpha, 0) &= \sum_{i=0}^n \alpha^i \sum_{b=0}^i \binom{n}{b} (-1)^b \langle 1 \rangle_{b, \frac{\lambda}{\alpha}} S_{2,\lambda}(n-b, i-b) \\
 &= E[(X - \alpha)_{n,\lambda}], \quad (n \geq 0).
 \end{aligned}$$

Therefore, by (31) and (35), we obtain the following theorem.

**Theorem 4.** For  $n, i \geq 0$ , we have

$$S_\lambda(n, i) = \sum_{b=0}^i \binom{n}{b} (-1)^b \langle 1 \rangle_{b, \frac{\lambda}{\alpha}} S_{2,\lambda}(n-b, i-b),$$

where  $\langle x \rangle_{0,\lambda} = 1$ ,  $\langle x \rangle_{x,\lambda} = x(x+\lambda)\cdots(x+(n-1)\lambda)$ , ( $n \geq 1$ ).

Now, we consider the inversion formula of Theorem 4. We observe that

$$(36) \quad \sum_{k=0}^n \alpha^k S_{2,\lambda}(n, k) = \text{Bel}_{n,\lambda}(\alpha) = \text{Bel}_{n,\lambda}(\alpha, \alpha) = E[(X - \alpha + \alpha)_{n,\lambda}]$$

$$\begin{aligned}
 &= \sum_{l=0}^n \binom{n}{l} (\alpha)_{n-l,\lambda} E[(X - \alpha)_{l,\lambda}] = \sum_{l=0}^n \binom{n}{l} (\alpha)_{n-l,\lambda} \sum_{k=0}^l \alpha^k S_{\lambda}(l, k) \\
 &= \sum_{l=0}^n \binom{n}{n-l} (\alpha)_{l,\lambda} \sum_{k=0}^{n-l} \alpha^k S_{\lambda}(n-l, k) = \sum_{l=0}^n \binom{n}{l} (\alpha)_{l,\lambda} \sum_{k=l}^n \alpha^{k-l} S_{\lambda}(n-l, k-l) \\
 &= \sum_{k=0}^n \alpha^k \sum_{l=0}^k \binom{n}{l} \alpha^{-l} (\alpha)_{l,\lambda} S_{\lambda}(n-k, k-l) = \sum_{k=0}^n \alpha^k \sum_{l=0}^k \binom{n}{l} (1)_{l, \frac{\lambda}{\alpha}} S_{\lambda}(n-l, k-l).
 \end{aligned}$$

Therefore, by comparing the coefficients on both sides of (36), we obtain the following theorem.

**Theorem 5.** For  $n, k \geq 0$ , we have

$$S_{2,\lambda}(n, k) = \sum_{l=0}^k \binom{n}{l} (1)_{l, \frac{\lambda}{\alpha}} S_{\lambda}(n-l, k-l).$$

We note that the left hand side of (17) is given by

$$\begin{aligned}
 (37) \quad e^{-\alpha t} (1+t)^x &= e^{-\alpha t} e_{\lambda}^x(\log_{\lambda}(1+t)) = e^{-\alpha t} \sum_{l=0}^{\infty} (x)_{l,\lambda} \frac{1}{l!} (\log_{\lambda}(1+t))^l \\
 &= e^{-\alpha t} \sum_{l=0}^{\infty} (x)_{l,\lambda} \sum_{k=l}^{\infty} S_{1,\lambda}(k, l) \frac{t^k}{k!} = \sum_{m=0}^{\infty} (-\alpha)^m \frac{t^m}{m!} \sum_{k=0}^{\infty} \sum_{l=0}^k (x)_{l,\lambda} S_{1,\lambda}(k, l) \frac{t^k}{k!} \\
 &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n \binom{n}{k} (-\alpha)^{n-k} \sum_{l=0}^k (x)_{l,\lambda} S_{1,\lambda}(k, l) \right) \frac{t^n}{n!}.
 \end{aligned}$$

Therefore, by (17) and (37), we obtain the following lemma.

**Lemma 6.** For  $n \geq 0$ , we have

$$C_n(x; \alpha) = \sum_{l=0}^n (x)_{l,\lambda} \sum_{k=l}^n \binom{n}{k} (-\alpha)^{n-k} S_{1,\lambda}(k, l).$$

Replacing  $t$  by  $e_{\lambda}(t) - 1$  and  $x$  by  $-\alpha$  in (17), we get

$$\begin{aligned}
 (38) \quad e^{\alpha(e_{\lambda}(t)-1)} e_{\lambda}^{-\alpha}(t) &= \sum_{k=0}^{\infty} C_k(-\alpha; -\alpha) \frac{1}{k!} (e_{\lambda}(t) - 1)^k \\
 &= \sum_{k=0}^{\infty} C_k(-\alpha; -\alpha) \sum_{n=k}^{\infty} S_{2,\lambda}(n, k) \frac{t^n}{n!} \\
 &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n C_k(-\alpha, -\alpha) S_{2,\lambda}(n, k) \right) \frac{t^n}{n!}.
 \end{aligned}$$

On the other hand,

$$(39) \quad e^{\alpha(e_{\lambda}(t)-1)} e_{\lambda}^{-\alpha}(t) = \sum_{n=0}^{\infty} \text{Bel}_{n,\lambda}(\alpha, 0) \frac{t^n}{n!}.$$

In particular, for  $x = 0$  in (17), we get

$$e^{-\alpha t} = \sum_{k=0}^{\infty} C_k(0; \alpha) \frac{t^k}{k!}.$$

By replacing  $t$  by  $1 - e_\lambda(t)$ , we get

$$\begin{aligned}
 (40) \quad e^{\alpha(e_\lambda(t)-1)} &= \sum_{k=0}^{\infty} C_k(0; \alpha) \frac{1}{k!} (1 - e_\lambda(t))^k \\
 &= \sum_{k=0}^{\infty} (-1)^k C_k(0; \alpha) \sum_{n=k}^{\infty} S_{2,\lambda}(n, k) \frac{t^n}{n!} \\
 &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n (-1)^k C_k(0; \alpha) S_{2,\lambda}(n, k) \right) \frac{t^n}{n!}
 \end{aligned}$$

Therefore, by (38), (39) and (40), we obtain the following lemma.

**Lemma 7.** For  $n \geq 0$ , we have

$$\text{Bel}_{n,\lambda}(\alpha, 0) = \sum_{k=0}^n C_k(-\alpha; -\alpha) S_{2,\lambda}(n, k),$$

and

$$\text{Bel}_{n,\lambda}(\alpha) = \sum_{k=0}^n (-1)^k C_k(0; \alpha) S_{2,\lambda}(n, k).$$

From (17), we get

$$(41) \quad e^{\alpha t} (1+t)^{\beta-\alpha} = \sum_{n=0}^{\infty} C_n(\beta - \alpha; -\alpha) \frac{t^n}{n!}.$$

Replacing  $t$  by  $e_\lambda(t) - 1$  in (41), we get

$$\begin{aligned}
 (42) \quad e^{\alpha(e_\lambda(t)-1)} e_\lambda^{\beta-\alpha}(t) &= \sum_{k=0}^{\infty} C_k(\beta - \alpha; -\alpha) \frac{1}{k!} (e_\lambda(t) - 1)^k \\
 &= \sum_{k=0}^{\infty} C_k(\beta - \alpha; -\alpha) \sum_{n=k}^{\infty} S_{2,\lambda}(n, k) \frac{t^n}{n!} \\
 &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n C_k(\beta - \alpha; -\alpha) S_{2,\lambda}(n, k) \right) \frac{t^n}{n!}.
 \end{aligned}$$

Therefore, by (20) and (42), we obtain the following theorem.

**Theorem 8.** For  $n \geq 0$ , we have

$$\text{Bel}_{n,\lambda}(\alpha, \beta) = \sum_{k=0}^n C_k(\beta - \alpha; -\alpha) S_{2,\lambda}(n, k).$$

By replacing  $t$  by  $\log_\lambda(1+t)$  in (20), we get

$$\begin{aligned}
 (43) \quad e^{\alpha t} (1+t)^{\beta-\alpha} &= \sum_{k=0}^{\infty} \text{Bel}_{k,\lambda}(\alpha, \beta) \frac{1}{k!} (\log_\lambda(1+t))^k \\
 &= \sum_{k=0}^{\infty} \text{Bel}_{k,\lambda}(\alpha, \beta) \sum_{n=k}^{\infty} S_{1,\lambda}(n, k) \frac{t^n}{n!} \\
 &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n \text{Bel}_{k,\lambda}(\alpha, \beta) S_{1,\lambda}(n, k) \right) \frac{t^n}{n!}.
 \end{aligned}$$

On the other hand,

$$(44) \quad e^{\alpha t} (1+t)^{\beta-\alpha} = \sum_{n=0}^{\infty} C_n(\beta - \alpha; -\alpha) \frac{t^n}{n!}.$$



Therefore, by (43) and (44), we obtain the following theorem.

**Theorem 9.** For  $n \geq 0$ , we have

$$C_n(\beta - \alpha; -\alpha) = \sum_{k=0}^n \text{Bel}_{k,\lambda}(\alpha, \beta) S_{1,\lambda}(n, k).$$

### 3. CONCLUSION

In this paper, we introduced the two variable degenerate Bell polynomials as a generalization of the single variable degenerate Bell polynomials which were previously studied under a different name of partially degenerate Bell polynomials. Then, among other things, we showed their connections with Poisson degenerate central moments and Charlier polynomials. Their details and results are as in the following. In Theorem 1, we expressed the two variable degenerate Bell polynomials in terms of the degenerate Bell polynomials and showed they reduce to the degenerate Bell polynomials when the two arguments are the same. Then we defined the Poisson degenerate central moments and proved that they are equal to the two variable degenerate Bell polynomials. We found an explicit expression for the two variable degenerate Bell polynomials in Theorem 3. By letting one of the arguments of the two variable degenerate Bell polynomials equal to zero, we were led to define Stirling type numbers as the coefficients in the expansion of powers of  $\alpha$ . Then we expressed the Stirling type numbers in terms of the degenerate Stirling numbers of the second kind and vice versa in Theorems 4 and 5. Finally, we found certain connections between the two variable degenerate Bell polynomials and Charlier polynomials involving degenerate Stirling numbers in Theorems 8 and 9.

There are three immediate possible applications of our results to probability, differential equations and symmetry. As to applications to probability, we saw in this paper a connection between the two variable degenerate Bell polynomials and the Poisson degenerate central moments. For the other two possible applications of our results, we let the reader refer to [12 and the references therein].

As one of our future projects, we would like to continue to study degenerate versions of certain special polynomials and numbers and their applications to physics, science and engineering as well as mathematics.

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