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ONE-DIMENSIONAL PSEUDOREPRESENTATIONS WITH SMALL DEFECT THAT ARE TRIVIAL ON A NORMAL SUBGROUP

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Abstract. We prove that every one-dimensional pseudorepresentation of a group with a sufficiently small defect that is trivial on a normal subgroup is defined by a one-dimensional pseudorepresentation of the corresponding quotient group.

§ 1. Introduction

Let G be a group, let N be a normal subgroup of G, and let π be a one-dimensional pseudorepresentation of G, i.e., $\pi\colon G \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and

(1) $|\pi(gh)-\pi(g)\pi(h)| \le \varepsilon, \qquad g,h \in G, \quad \text{and} \quad \pi(g^k) = \pi(g)^k, \qquad k \in \mathbb{Z}.$

The minimum number ε satisfying (1) is called the *defect* of the pseudorepresentation π. A pseudorepresentation is said to be *pure* if its restriction to every amenable subgroup of G is an ordinary complex character of the subgroup. For the generalities concerning pseudorepresentations, see [1–5]; for the specific features concerning one-dimensional pseudorepresentations, see [6].

Suppose that the restriction of the pseudorepresentation π to the normal subgroup N is a sufficiently small perturbation of the mapping taking N to one. The main result of the present note is that $\pi(N) = \{1\}$ and there is a one-dimensional pseudorepresentation ρ of G/N such that

(2) $$\pi(g) = \rho(gN), \qquad g \in G.$$

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§ 2. Preliminaries

Lemma. *Let G be a group and let π be a one-dimensional pure pseudorepresentation of G. If*

$$|\pi(g) - 1| < \sqrt{3} \quad \textit{for all} \quad g \in G,$$

then $\pi(g)$ is identically equal to one for all $g \in G$.

Proof. Let $g \in G$, $g \neq e_G$, where e_G stands for the identity element of G. Let $G(g)$ be the cyclic subgroup of G generated by g. By assumption, the restriction of π to $G(g)$ is an ordinary complex character χ of $G(g)$, which satisfies the inequality

$$|\chi(h) - 1| < \sqrt{3} \quad \text{for all} \quad h \in G(g).$$

Therefore, χ is bounded, and therefore unitary; however, the image $\chi(G(g))$ of $G(g)$ is a subgroup of the unit circle; if it contains no elements at a distance from 1 which is greater than or equal to $\sqrt{3}$, then the subgroup is $\{1\}$, which completes the proof.

§ 3. Main theorem

Theorem. *Let G be a group, let N be a normal subgroup of G, and let π be a pure one-dimensional representation of G whose defect is less than 1. Let*

$$|\pi(n) - 1| < \sqrt{3} \quad \textit{for all} \quad n \in N.$$

Then there is a pure one-dimensional representation ρ of G/N such that

$$\pi(g) = \rho(gN) \quad \textit{for all} \quad g \in G.$$

Proof. It follows from the assumption of the theorem and from the above lemma that

$$\pi(n) = 1 \quad \text{for every} \quad n \in N.$$

Let us consider the relationship between the cyclic groups generated by the numbers $\pi(gn)$ and $\pi(g)$ for any $g \in G$ and $n \in N$. For every integer k we have

$$\pi(gn)^k = \pi((gn)^k) = \pi(g^k n_{g,k}) = \pi(g)^k + \delta(g, n, k),$$

where $|\delta(g,n,k)| < 1$. As is well known, two complex characters whose difference is pointwise less than one are equal. Thus,

$$\pi(gn)^k = \pi(g)^k$$

for all integers k and all $g \in G$ and $n \in N$. Therefore, π is constant on the cosets by N. Introducing a one-dimensional mapping

$$\rho : G/N \to \mathbb{C}^*$$

by the rule

$$\rho(gN) = \pi(g) \quad \text{for all} \quad g \in G,$$

we can immediately see that ρ is well defined and satisfies the conditions imposed on pseudorepresentations, this pseudorepresentation is pure and satisfies (2) by the very definition. This completes the proof of the theorem.

§ 4. Discussion

Corollary. *Let G be a group, let N be a normal subgroup of G, and let π be a one-dimensional representation of G whose defect is less than* 0.24*. Let*

$$|\pi(n) - 1| < \sqrt{3} \quad \textit{for all} \quad n \in N.$$

Then there is a pure one-dimensional representation ρ of G/N such that

$$\pi(g) = \rho(gN) \quad \textit{for all} \quad g \in G.$$

Proof. The proof follows immediately from the theorem and from the fact that a one-dimensional pseudorepresentation of a group whose defect is less than 0.24 is pure [6].

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