Proceedings of the Jangjeon Mathematical Society
23 (2020), No. 4, pp. 541 – 552

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2020.23.4.541

COMMON FIXED POINTS OF MEAN NONEXPANSIVE MAPPINGS IN CAT(0) SPACES

SAFEER HUSSAIN KHAN* AND RITIKA S

ABSTRACT. In this paper, we consider a recently introduced three step iterative process but extended to the case of two mappings in CAT(0) spaces. We give new examples of two mappings which are mean nonexpansive but not nonexpansive and possess a common fixed point. We establish Δ-convergence and weak convergence results for this iterative process for approximating common fixed points of two mean nonexpansive mappings. The results obtained in this paper extend and improve the recent ones announced by many authors in the sense that both our class of mappings and the iterative process are more general. Moreover, our results remain true for the spaces contained in CAT(0) spaces like Banach spaces and R-trees etc.

1. INTRODUCTION

A geodesic path or simply a geodesic joining two pints x and y of a metric space (X, d) is a map c: $[0, l] \subset \mathbb{R} \to X$ such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. A geodesic (or metric) segment joining x and y is defined as the image α of c. A unique geodesic segment is denoted by $[x, y]$. (X, d) is termed as a geodesic space if every two points of X are joined by a geodesic and it is called uniquely geodesic if no more than one geodesic joining x and y can be found for each $x, y \in X$. If $C \subseteq X$ contains every geodesic segment joining any two of its points, it is called convex. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of two things: three points x_1, x_2, x_3 of X called the vertices of Δ and a geodesic segment between each pair of vertices called the edges of Δ. A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) may be defined as a triangle $\bar{\Delta}(x_1, x_2, x_3) = \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ with $d_{\mathbb{E}^2}(\bar{x}_i, \bar{x}_j) = (x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space is called a $CAT(0)$ space if the comparison axiom given below is satisfied by all geodesic triangles.

Date: Submitted: 07 Octobet 2019.

Key words and phrases. CAT(0) spaces, Mean Nonexpansive Mappings, Common Fixed Points, Convergence.

2000 Mathematics Subject Classification. 47H09, 47H10, 47J25
Corresponding author.

Comparison Axiom: A geodesic triangle Δ is said to satisfy the $CAT(0)$ inequality if for all $u, v \in \Delta$ and all comparison points $\overline{u}, \overline{v} \in \bar{\Delta}$, we have $d(u,v) \leq d_{\mathbb{E}^2}(\overline{u}, \overline{v})$.

Let X be a $CAT(0)$ space, $x, u_1, u_2 \in X$ and u_0 be the midpoint of the segment $[u_1, u_2]$, then the following is immediate from the $CAT(0)$ inequality:

$$d(x,u_0)^2 \leq \frac{1}{2}d(x,u_1)^2 + \frac{1}{2}d(x,u_2)^2 - \frac{1}{4}d(u_1,u_2)^2.$$

In fact, it is well-known that a geodesic space is a $CAT(0)$ space if and only if it satisfies the so-called (CN)-inequality.

Definition 1. *For a bounded sequence $\{x_n\}$ in a $CAT(0)$ space X and for $x \in X$, define*

$$r(x, \{x_n\}) = \lim_{n\to\infty} \sup d(x, x_n$$

Then

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$$

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

are respectively called the asymptotic radius and the asymptotic center of $\{x_n\}$. is the set

Remark 1. *$A(\{x_n\})$ consists of exactly one point in $CAT(0)$ spaces (* [4, Proposition 7]*.)*

Definition 2. *A sequence $\{x_n\}$ in a $CAT(0)$ space X is said to Δ-converge to $x \in X$ or x is the Δ-limit of $\{x_n\}$ if x is the unique asymptotic center of $\{u_n\}$ for every sub-sequence $\{u_n\}$ of $\{x_n\}$.*

Mathematically, $\Delta\text{-}\lim x_n = x$. If $\{x_n\}$ Δ-converges to two different points p and p, then $\lim_{n\to\infty} \sup d(x_n, p) = \lim_{n\to\infty} \sup d(x_n, q)$. This means that the Opial property remains true even in $CAT(0)$ spaces.

Lemma 1. *Let X be a $CAT(0)$ space. Then*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z)$$

for all $x, y, z \in X$ and $t \in [0,1]$.

Lemma 2. *Every bounded sequence in a complete $CAT(0)$ space has a Δ-convergent sub-sequence.*

Lemma 3. *If C is a closed convex subset of a complete $CAT(0)$ space and if $\{x_n\}$ is a bounded sequence in C then the asymptotic center of $\{x_n\}$ is in C.*

Lemma 4. *If $\{x_n\}$ is a bounded sequence in a complete $CAT(0)$ space X with $A(\{x_n\}) = \{p\}$, $\{v_n\}$ is a sub-sequence of $\{x_n\}$ with $A(\{v_n\}) = \{v\}$ and the sequence $\{d(x_n, v)\}$ converges, then $p = v$.*

Mean nonexpansive mappings in Banach spaces were introduced by Zhang [11]. Existence and uniqueness results for fixed points for such mappings in Banach spaces were also given by him. Since then, some fixed point theorems of such mapping has been studied by many researchers; see, for example, [2, 6, 10, 12]. In 2017, Abkar and Rastgoo [2] studied the mean nonexpansive mappings in the context of $CAT(0)$ spaces. Motivated by their work, in this paper we approximate common fixed points of mean nonexpansive mappings by using a three step iterative process in the framework of $CAT(0)$ spaces.

Definition 3. *Let C be nonempty subset of a $CAT(0)$ space X. A mapping $S : C \to C$ is said to be nonexpansive if $d(Su, Sv) \leq d(u, v)$ for all $u, v \in C$. S is quasi-nonexpansive if $d(Su, p) \leq d(u, p)$ for all $u \in C$ and $p \in F(S)$. Finally, we say S is mean nonexpansive if $d(Su, Sv) \leq ad(u, v) + bd(u, Sv)$ for all*
$u, v \in C$ where a and b are two non-negative real numbers such that $a+b \leq 1$.

If $a = 1$ and $b = 0$, then clearly a mean nonexpansive mapping reduces to a. nonexpansive mapping. It is to be noted that a mean may not be continuous and hence not necessarily nonexpansive as revealed by the following example. As far as common fixed points are concerned, there do exist mean nonexpansive mappings with a common fixed point. Now we give a couple of new examples which reflect this fact.

Example 1. *Suppose that $T : \mathbb{R} \to \mathbb{R}$ is a mapping defined by*

$$Tx = \begin{cases} \frac{\sin x}{5} & if \quad x \in (-\infty, 0] \\ 2 & if \quad x \in (0, \infty) \end{cases}$$

Then T is mean nonexpansive with $a = \frac{1}{3}, b = \frac{2}{3}$ and a fixed point 0. However, it is not a nonexpansive mapping.

Proof. Clearly T has a fixed point 0. Since it is not continuous at $x = 0$, it is not a nonexpansive. To show that T is mean nonexpansive, we proceed as follows.

Case 1: $x \in (-\infty, 0]$

$$\begin{aligned} |Tx - Ty| &= \left| \frac{\sin x}{5} - \frac{\sin y}{5} \right| \\ &= \frac{1}{4} \left| \sin x - \frac{\sin x}{5} - \frac{\sin y}{5} + x + \frac{\sin y}{5} - x - \sin y + \frac{\sin y}{5} \right| \\ &\leq \frac{1}{4} |x - y| + \frac{2}{4} \left| x - \frac{\sin y}{5} \right| + \frac{1}{4} \left| \frac{\sin x}{5} - \frac{\sin y}{5} \right|. \end{aligned}$$

This implies that $\frac{3}{4} \left| \frac{\sin x}{5} - \frac{\sin y}{5} \right| \leq \frac{1}{4} |x - y| + \frac{2}{4} \left| x - \frac{\sin y}{5} \right|$

or $\left| \frac{\sin x}{5} - \frac{\sin y}{5} \right| \leq \frac{1}{3} |x - y| + \frac{2}{3} \left| x - \frac{\sin y}{5} \right|$.

Case 2: $x \in (0, \infty)$

$$\begin{aligned}|Tx - Ty| &= |2-2| = 0 \\ &\leq \left|\frac{x}{3} - \frac{y}{3} + \frac{2}{3}x - \frac{4}{3}\right| \\ &\leq \frac{1}{3}|x-y| + \frac{2}{3}|x-2|\end{aligned}$$

Hence in both cases, T is mean nonexpansive with $a = \frac{1}{3}, b = \frac{2}{3}$. $\square$

Example 2. *Suppose that* $S : (-2,2) \to \mathbb{R}$ *is a mapping defined by*

$$Sx = \begin{cases} \frac{x}{7} & if \quad x \in (-2,1] \\ 1 & if \quad x \in (1,2) \end{cases}$$

Then S *has a fixed point* 0, *it is mean nonexpansive with* $a = \frac{1}{5}, b = \frac{4}{5}$. *But. it is not a nonexpansive mapping.*

Proof. Clearly S has a fixed point 0. Since it is not continuous at $x = 1$, it is not a nonexpansive. To show that S is mean nonexpansive, we proceed as follows.

Case 1: $x \in (-2,1]$

$$\begin{aligned}|Sx - Sy| &= \left|\frac{x}{7} - \frac{y}{7}\right| \\ &= \frac{1}{6}\left|x - \frac{x}{7} - y + \frac{y}{7}\right| \\ &\leq \frac{1}{6}|x-y| + \frac{4}{6}\left|x - \frac{y}{7}\right| + \frac{1}{6}\left|\frac{x}{7} - \frac{y}{7}\right|.\end{aligned}$$

This implies that $\frac{5}{6}\left|\frac{x}{7} - \frac{y}{7}\right| \leq \frac{1}{6}|x-y| + \frac{4}{6}\left|x - \frac{y}{7}\right|$

or $\left|\frac{x}{7} - \frac{y}{7}\right| \leq \frac{1}{5}|x-y| + \frac{4}{5}\left|x - \frac{y}{7}\right|$.

Case 2: $x \in (0,\infty)$

$$\begin{aligned}|Sx - Sy| &= |1-1| = 0 \\ &\leq \left|\frac{x}{5} - \frac{y}{5} + \frac{4}{5}x - \frac{4}{5}\right| \\ &\leq \frac{1}{5}|x-y| + \frac{4}{5}|x-1|\end{aligned}$$

Hence in both cases, S is mean nonexpansive with $a = \frac{1}{5}, b = \frac{4}{5}$. $\square$

Remark 2. *From the above examples, it is clear that there do exist mean nonexpansive mappings with a common fixed point.*

We now extend the idea of mean nonexpansive mappings to two mappings case as follows.

Let C be nonempty subset of a $CAT(0)$ space X and $T, S : C \to C$ be two mappings. Then T, S are said to be mean nonexpansive mappings if

$$d(Tx,Ty) \leq a_T d(x,y) + b_T d(x,Ty)$$

and

$$d(Sx, Sy) \leq a_S d(x, y) + b_S d(x, Sy)$$

hold respectively for all $x, y \in C$ where a_T, a_S and b_T , b_S are non-negative real numbers such that $a_T + b_T \leq 1$ and $a_S + b_S \leq 1$.

In 2016, Thakur et al. [9] established a new three step iterative process in Banach spaces and they showed by an example that this iteration is much faster than many other iterations. Recently, Ritika [8] modified this iterative process to the case of two mappings for generalized quasi-contractive type operators in $CAT(0)$ spaces as follows: Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and $S, T : C \to C$ be two mappings. Then the sequence $\{x_n\}$ in C is defined as: For $x_1 \in C$,

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)Sz_n \oplus \alpha_n Ty_n \\ y_n &= (1-\beta_n)z_n \oplus \beta_n Tz_n \\ z_n &= (1-\gamma_n)x_n \oplus \gamma_n Sx_n \end{aligned} \tag{1.1}$$

where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are sequences of positive numbers in $(0, 1)$. If we take $S = T$ in the above iterative process (1.1), then this process reduces to Thakur et al. [9] iterative process in $CAT(0)$ space settings.

In this paper, we approximate common fixed points of two mean nonexpansive mappings by using above iterative process (1.1). Our result extend and improve many results in the existing literature due to Abbas and Nazir [1], Abkar and Rastgoo [2], Agarwal et al. [3], Thakur et al. [9], Wu and Zhang [10], Zhang [11], Zhou and Cui [12] and many others in the sense that both our class of mappings and the iterative process are more general. Moreover, our results remain true for the spaces contained in $CAT(0)$ spaces like Banach spaces and R-trees etc.

2. Main results

Let X be a complete $CAT(0)$ space and $T : X \to X$ be a self mapping of X. Suppose $F(T) = \{p \in C : Tp = p\}$ is the set of fixed points of T. Therefore, $F(S) \cap F(T) = \{p \in C : Tp = p = Sp\}$.

We need the following useful and well known lemma to prove our main results in this paper.

Lemma 5. *Let p be a given point in a $CAT(0)$ space X and $\{\alpha_n\}$ be a sequence in a closed interval $[a, b]$ with $0 < a \leq b < 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences in X such that* $\limsup_{n\to\infty} d(x_n, p) \leq c$, $\limsup_{n\to\infty} d(y_n, p) \leq c$ *and* $\lim\limits_{n\to\infty} d((1-\alpha_n)x_n \oplus \alpha_n y_n, p) = c$ *for some* $c \geq 0$. *Then* $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$.

For the rest of the paper, unless specified otherwise, X is a complete $CAT(0)$ space and C a nonempty closed convex bounded subset of a X

Lemma 6. *A a mean nonexpansive mapping with $b < 1$ has a fixed point.*

Lemma 7. *If $\{x_n\}$ is a sequence in C such that $\lim_{n\to\infty} d(Tx_n, x_n) = 0$ and $\Delta - \lim x_n = p$, then p is a fixed point of the mean nonexpansive mapping $T : C \to C$ with $b < 1$ ..*

Lemma 8. *Suppose that $S, T : C \to C$ are mean nonexpansive mappings with $b_S \neq 1, \frac{1}{2}$ and $b_T \neq 1, \frac{1}{2}$. Then $F(S) \cap F(T) \neq \phi$ if and only if the sequence $\{x_n\}_{n=0}^{\infty}$ defined by (1.1) is bounded and $\lim_{n\to\infty} d(Sx_n, x_n) = 0 = \lim_{n\to\infty} d(Tx_n, x_n) = 0$.*

Proof. Assume that $F(S) \cap F(T) \neq \phi$. Let $p \in F(S) \cap F(T)$. Since S and T are mean nonexpansive mappings.

$$\begin{aligned} d(x_{n+1}, p) &= d((1-\alpha_n)Sz_n \oplus \alpha_n Ty_n, p) \\ &\leq (1-\alpha_n)d(Sz_n, p) + \alpha_n d(Ty_n, p) \\ &\leq (1-\alpha_n)[a_S d(z_n, p) + b_S d(z_n, p)] \\ &+ \alpha_n[a_T d(y_n, p) + b_T d(y_n, p)] \\ &\leq (1-\alpha_n)d(z_n, p) + \alpha_n d(y_n, p) \end{aligned} \tag{2.1}$$

$$\begin{aligned} d(y_n, p) &= d((1-\beta_n)z_n \oplus \beta_n Tz_n, p) \\ &\leq (1-\beta_n)d(z_n, p) + \beta_n d(Tz_n, p) \\ &\leq (1-\beta_n)d(z_n, p) + \beta_n[a_T d(z_n, p) + b_T d(z_n, p)] \\ &\leq (1-\beta_n)d(z_n, p) + \beta_n d(z_n, p) \\ &= d(z_n, p) \\ &= d((1-\gamma_n)x_n \oplus \gamma_n Sx_n, p) \\ &\leq (1-\gamma_n)d(x_n, p) + \gamma_n d(Sx_n, p) \\ &\leq (1-\gamma_n)d(x_n, p) + \gamma_n[a_S d(x_n, p) + b_S d(x_n, p)] \\ &\leq (1-\gamma_n)d(x_n, p) + \gamma_n d(x_n, p) \\ &= d(x_n, p) \end{aligned}$$

By using above inequalities in (2.1), we get

$$\begin{aligned} d(x_{n+1}, p) &\leq (1-\alpha_n)d(z_n, p) + \alpha_n d(y_n, p) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(x_n, p) \\ &= d(x_n, p) \end{aligned}$$

That is, $d(x_{n+1}, p) \leq d(x_n, p)$ for all $n \in \mathbb{N}$, which shows that the sequence $\{d(x_n, p)\}$ is decreasing and bounded below so that $\lim_{n\to\infty} d(x_n, p)$ exists. Without loss of generality, let us assume that

$$\lim_{n\to\infty} d(x_n, p) = c.$$

Now, we will prove that $\lim_{n\to\infty} d(Sx_n, x_n) = 0$.

From the above calculations, it follows that

$$\begin{aligned} d(x_{n+1},p) &\leq (1-\alpha_n)d(z_n,p)+\alpha_n d(y_n,p) \\ &\leq (1-\alpha_n)d(z_n,p)+\alpha_n d(z_n,p) \\ &= d(z_n,p) \end{aligned}$$

Thus we have

$$d(x_{n+1},p) \leq d(z_n,p) \leq d(x_n,p)$$

which gives

$$c = \lim_{n\to\infty} d(z_n,p) = \lim_{n\to\infty} d((1-\gamma_n)x_n \oplus \gamma_n Sx_n, p). \tag{2.2}$$

Next,

$$\begin{aligned} \limsup_{n\to\infty} d(Sx_n,p) &\leq \limsup_{n\to\infty}[a_S d(x_n,p)+b_S d(x_n,p)] \\ &\leq \limsup_{n\to\infty} d(x_n,p) \\ &\leq c \end{aligned} \tag{2.3}$$

and

$$\limsup_{n\to\infty} d(x_n,p) \leq c. \tag{2.4}$$

Thus, by using Lemma 5, together with (2.2), (2.3) and (2.4), we have $\lim_{n\to\infty} d(Sx_n,x_n)=0$.

Now, we prove that $\lim_{n\to\infty} d(Tx_n,x_n)=0$.
For this note that

$$d(x_n,Tx_n) \leq d(x_n,Sx_n)+d(Sx_n,Sz_n)+d(Sz_n,Ty_n)+d(Ty_n,Tx_n). \tag{2.5}$$

The limit of the first term on the right side has already been proven zero. We now prove that the limit of all the three remaining terms is also zero. Firstly, we take

$$\begin{aligned} d(Sx_n,Sz_n) &\leq a_S d(x_n,z_n)+b_S d(x_n,Sz_n) \\ &\leq a_S d(x_n,z_n)+b_S[d(x_n,Sx_n)+d(Sx_n,Sz_n)]. \end{aligned}$$

This implies that $(1-b_S)d(Sx_n,Sz_n) \leq a_S d(x_n,z_n)+b_S d(x_n,Sx_n)$ so that

$$d(Sx_n,Sz_n) \leq \frac{a_S}{1-b_S}d(x_n,z_n)+\frac{b_S}{1-b_S}d(x_n,Sx_n).$$

But then $\lim_{n\to\infty} d(x_n,Sx_n)=0$ and, in turn,

$$\lim_{n\to\infty} d(x_n,z_n) = \lim_{n\to\infty} d(x_n,(1-\gamma_n)x_n\oplus\gamma_n Sx_n)=0$$

give

$$\lim_{n\to\infty} d(Sx_n,Sz_n)=0. \tag{2.6}$$

Now

$$\limsup_{n\to\infty} d(Sz_n,p) \leq \limsup_{n\to\infty} d(z_n,p) \leq c$$

and similarly

$$\limsup_{n\to\infty} d(Ty_n, p) \le c$$

combined with

$$c = \lim_{n\to\infty} d(x_{n+1}, p) = \lim_{n\to\infty} d((1-\alpha_n)Sz_n \oplus \alpha_n Ty_n, p).$$

give by Lemma 5 that

$$\lim_{n\to\infty} d(Sz_n, Ty_n) = 0. \tag{2.7}$$

This also gives by virtue of

$$\begin{aligned} d(x_{n+1}, p) &= d((1-\alpha_n)Sz_n \oplus \alpha_n Ty_n, p) \\ &\le (1-\alpha_n)d(Sz_n, p) + \alpha_n d(Ty_n, p) \\ &\le (1-\alpha_n)[d(Sz_n, Ty_n) + d(Ty_n, p)] + \alpha_n d(Ty_n, p) \\ &= (1-\alpha_n)d(Sz_n, Ty_n) + d(Ty_n, p) \\ &\le (1-\alpha_n)d(Sz_n, Ty_n) + [a_T d(y_n, p) + b_T d(y_n, p)] \\ &\le (1-\alpha_n)d(Sz_n, Ty_n) + d(y_n, p) \end{aligned}$$

that

$$c \le \liminf_{n\to\infty} d(y_n, p).$$

Moreover,

$$\limsup_{n\to\infty} d(y_n, p) \le \limsup_{n\to\infty} d(x_n, p) \le c,$$

and hence

$$c = \lim_{n\to\infty} d(y_n, p) = \lim_{n\to\infty} d((1-\beta_n)z_n \oplus \beta_n Tz_n, p)$$

gives, by appealing to Lemma 5, that

$$\lim_{n\to\infty} d(z_n, Tz_n) = 0.$$

Further,

$$\begin{aligned} d(x_n, y_n) &= d(x_n, (1-\beta_n)z_n \oplus \beta_n Tz_n) \\ &\le (1-\beta_n)d(x_n, z_n) + \beta_n d(x_n, Tz_n) \\ &\le (1-\beta_n)d(x_n, z_n) + \beta_n[d(x_n, z_n) + d(z_n, Tz_n)] \\ &\le d(x_n, z_n) + \beta_n d(z_n, Tz_n). \end{aligned}$$

But $\lim_{n\to\infty} d(x_n, z_n) = 0 = \lim_{n\to\infty} d(z_n, Tz_n)$,therefore

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Now

$$d(Ty_n, Tx_n) \leq a_T d(x_n, y_n) + b_T d(x_n, Ty_n)$$
$$\leq a_T d(x_n, y_n) + b_T [d(x_n, Tx_n) + d(Tx_n, Ty_n)]$$
$$\Rightarrow (1 - b_T) d(Ty_n, Tx_n) \leq a_T d(x_n, y_n) + b_T d(x_n, Tx_n)$$
$$\Rightarrow d(Ty_n, Tx_n) \leq \frac{a_T}{1 - b_T} d(x_n, y_n) + \frac{b_T}{1 - b_T} d(x_n, Tx_n).$$

and so

$$\limsup_{n\to\infty} d(Ty_n, Tx_n) \leq \limsup_{n\to\infty} \frac{b_T}{1 - b_T} d(x_n, Tx_n). \tag{2.8}$$

Making use of (2.6), (2.7) and (2.8) in (2.5), we have

$$\limsup_{n\to\infty} d(x_n, Tx_n) \leq \limsup_{n\to\infty} \frac{b_T}{1 - b_T} d(x_n, Tx_n)$$

This yields $\frac{1-2b_T}{1-b_T} \limsup_{n\to\infty} d(x_n, Tx_n) \leq 0$. Thus $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$.

Summing up, $\lim\limits_{n\to\infty} d(Sx_n, x_n) = 0 = \lim\limits_{n\to\infty} d(Tx_n, x_n) = 0$ as required.

Conversely, suppose that $\{x_n\}$ is bounded and $\lim\limits_{n\to\infty} d(Sx_n, x_n) = 0 = \lim\limits_{n\to\infty} d(Tx_n, x_n)$.

In view of [5], the function $r(u, \{x_n\}) = \limsup_{n\to\infty} d(x_n, u), \forall u \in C$ is a non-negative,continuous and convex function of u.

Moreover, for $\|u_k\| \to \infty$, we have $r(u_k, \{x_n\}) \to \infty$. Then one can find a $u_0 \in C$ such that $r(u_0, \{x_n\}) = r_0 = \min_{u\in C} r(u, \{x_n\})$.

If we set $A_C(\{x_n\}) = \{u \in C : \limsup_{n\to\infty} d(x_n, u) = r_0\}$, then $A_C(\{x_n\})$ is T-invariant, indeed, for $p \in A_C(\{x_n\})$, we have

$$\limsup_{n\to\infty} d(Tp, x_n) \leq \limsup_{n\to\infty} [d(Tp, Tx_n) + d(Tx_n, x_n)]$$
$$\leq \limsup_{n\to\infty} d(x_n, p) + \limsup_{n\to\infty} d(Tx_n, x_n)$$
$$\leq \limsup_{n\to\infty} d(x_n, p).$$

Thus, $Tp \in A_C(\{x_n\})$, by the same arguments we can prove that $A_C(\{x_n\})$ is S-invariant. Since X is complete $CAT(0)$ space, $A_C(\{x_n\})$ consists of a single point p which is a common fixed point for both T and S. □

Now, we prove weak convergence result for two mean nonexpansive mappings using the iterative process (1.1) in $CAT(0)$ spaces.

Theorem 1. *Let $S, T : C \to C$ be mean nonexpansive mappings with $b_S \neq 1, \frac{1}{2}$ and $b_T \neq 1, \frac{1}{2}$ such that $F(S) \cap F(T) \neq \phi$. Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ defined by (1.1) converge weakly to a common fixed point of S and T. On top, $\{x_n\}, \{y_n\}$ and $\{z_n\}$ have the same weak limit.*

Proof. For a given $p \in F(S) \cap F(T)$, we have $\limsup_{n\to\infty} d(x_n, p)$ exists. Let p_1 and p_2 be two weak limits of $\{x_n\}$, next we show that $p_1 = p_2$. For this we need to show that $p_1, p_2 \in F(S) \cap F(T)$. Indeed, if we assume that $p_1 \notin F(T)$, that is, $Tp_1 \neq p_1$, and $\{x_{n_p}\}$ (respctively $\{x_{n_q}\}$) an arbitrary sub-sequence of $\{x_n\}$ which converges weakly to p_1 (respectively p_2), then by the Opial's property we must have

$$\limsup_{p\to\infty} d(x_{n_p}, p_1) \leq \limsup_{p\to\infty} d(x_{n_p}, Tp_1).$$

But T is mean nonexpansive mapping, so

$$\begin{aligned}\limsup_{p\to\infty} d(Tx_{n_p}, Tp_1) &\leq a_T \limsup_{p\to\infty} d(x_{n_p}, p_1) + b_T \limsup_{p\to\infty} d(x_{n_p}, Tp_1) \\ &\leq a_T \limsup_{p\to\infty} d(x_{n_p}, p_1) + b_T \limsup_{p\to\infty} d(x_{n_p}, Tx_{n_p}) \\ &+ b_T \limsup_{p\to\infty} d(Tx_{n_p}, Tp_1).\end{aligned}$$

Thus, we obtain

$$(1-b_T) \limsup_{p\to\infty} d(Tx_{n_p}, Tp_1) \leq a_T \limsup_{p\to\infty} d(x_{n_p}, p_1) + b_T \limsup_{p\to\infty} d(x_{n_p}, Tx_{n_p})$$

This gives

$$\limsup_{p\to\infty} d(Tx_{n_p}, Tp_1) \leq \frac{a_T}{1-b_T} \limsup_{p\to\infty} d(x_{n_p}, p_1) + \frac{b_T}{1-b_T} \limsup_{p\to\infty} d(x_{n_p}, Tx_{n_p}).$$

Using above Lemma 8, we obtain

$$\limsup_{p\to\infty} d(Tx_{n_p}, Tp_1) \leq \limsup_{p\to\infty} d(x_{n_p}, p_1).$$

This leads to the contradiction, so we must have $Tp_1 = p_1$.
By symmetry of p_1 and p_2, we deduce that $Tp_2 = p_2$. Now it remains to show that $p_1 = p_2$.
Assume that $p_1 \neq p_2$, then by Opial's condition, we obtain

$$\begin{aligned}\lim_{n\to\infty} d(x_n, p_1) &= \lim_{p\to\infty} d(x_{n_p}, p_1) \\ &< \lim_{p\to\infty} d(x_{n_p}, p_2) \\ &= \lim_{n\to\infty} d(x_n, p_2) \\ &= \lim_{q\to\infty} d(x_{n_q}, p_2) \\ &< \lim_{n\to\infty} d(x_{n_q}, p_1) \\ &= \lim_{n\to\infty} d(x_n, p_1)\end{aligned}$$

This is a contradiction and thus $p_1 = p_2$.
Similarly, we can show that $p_1 \in F(S)$. Since $\{x_{n_p}\}$ is chosen in an arbitrary fashion, we get the weak convergence of $\{x_n\}$ to a common fixed point of S and T. Let us denote by $W(\{u_n\})$, the set of weak limits of the sequence $\{u_n\}$.
To give a final touch towards the completeness of the proof, we show that $W(\{x_n\}) = W(\{y_n\}) = W(\{z_n\})$.

From previous Lemma 8, we know that $\lim_{n\to\infty} d(y_n, p)$ and $\lim_{n\to\infty} d(z_n, p)$ both exist. So, from above same procedure, we show that $\{y_n\}$, $\{z_n\}$ converge weakly and $W(\{y_n\}) \subset F(S) \cap F(T)$, $W(\{z_n\}) \subset F(S) \cap F(T)$.
Since $\lim_{n\to\infty} d(x_n, y_n) = 0$, $\lim_{n\to\infty} d(x_n, z_n) = 0$, then $\lim_{n\to\infty} d(Tx_n, x_n) = 0$ and the result is completely established. □

Now, we give Δ-convergence result for two mean nonexpansive mappings using the iterative process (1.1) in $CAT(0)$ spaces.

Theorem 2. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and $S, T : C \to C$ be mean nonexpansive mappings with $b_S \neq 1, \frac{1}{2}$ and $b_T \neq 1, \frac{1}{2}$. Then the sequence $\{x_n\}$ defined by (1.1) is a Δ-convergent to a common fixed point $p \in F(S) \cap F(T)$.*

Proof. Define $\omega_w(x_n) = \cup_{\{v_n\}\subseteq\{x_n\}} A(\{v_n\}) \subseteq F(S) \cap F(T)$.
Let $v \in \omega_w(x_n)$.Then by definition of $\omega_w(x_n)$ that, we have a sub-sequence $\{v_n\}$ of $\{x_n\}$ with $A(\{v_n\}) = \{v\}$. Now, using Lemma 2, to obtain a sub-sequence $\{p_n\}$ of $\{v_n\}$ such that $\Delta-\lim_{n\to\infty} p_n = p \in C$. Appealing to Lemma 7, $p \in F(S) \cap F(T)$. Since the sequence $\{d(v_n, p)\}$ is convergent, Lemma 3 yields $v = p$. Therefore, $\omega_w(x_n) \subseteq F(S) \cap F(T)$.
To show that $\omega_w(x_n)$ consists of exactly one point, let $\{v_n\}$ be a sub-sequence of $\{x_n\}$ such that $A(\{v_n\}) = \{v\}$ and let $A(\{x_n\}) = \{x\}$. Since $v = p \in F(S) \cap F(T)$, $\{d(x_n, p)\}$ converges, by Lemma 4, we have $x = p \in F(S) \cap F(T)$, that is, $\omega_w(x_n) = x$. □

Conclusion 1. *We have proved results using a general type of mapping with a general type of iteration scheme for common fixed points. Our results , of course, contain the results for one mapping case. Our results are also true for SP-iteration, CR-iteration, Thianwan's iteration, Ishikawa iteration as corollaries in $CAT(0)$ space setting (and for the spaces contained in $CAT(0)$ spaces like Banach spaces, R-trees etc.) for such mappings. Moreover, our results are also more general than those proved for the mappings like nonexpansive mappings as they are contained in our class of mappings as shown by the examples.*

References

[1] M. Abbas and T. Nazir, A new faster iteration process applied to constrained minimization and feasibility problems, *Matematicki Vesnik* **66(2)** (2014), 223–234.

[2] A. Abkar and M. Rastgoo, A new iteration process for approximation of fixed points of mean nonexpansive mappings in CAT(0) spaces, *Cogent Mathematics* **4: 1396642** (2017), 1–11.

[3] R.P. Agarwal, D.O. Regan and D.R. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, *Journal of Nonlinear and Convex Analysis* **8(1)** (2007), 61–79.

[4] S. Dhompongsa, W.A. Kirk and B. Sims, Fixed points of uniformly Lipschitzian mappings, *Nonlinear Analysis: Theory, Methods and Applications* **65** (2006), 762–772.

[5] K. Goebel and W.A. Kirk, Topics in metric fixed point theory, *Cambridge University Press* (1990).
[6] K. Nakprasit, Mean nonexpansive mappings and Suzuki-generalized nonexpansive mappings, *Journal of Nonlinear Anal. Optim.* **1** (2010), 93–96.
[7] B. Nanjaras and B. Panyanak, Demiclosed principle for asymptotically nonexpansive mappings in $CAT(0)$ spaces, *Fixed Point Theory and Applications* **Article ID 268780** (2010), 1–14.
[8] Ritika, Common fixed points of generalized quasi–contractive type operators by a three step iterative process in CAT(0) spaces, *International Journal of Mathematics and its Applications* 6(2018),1-9.
[9] B.S. Thakur, D. Thakur and M. Postolache, A new iteration scheme for approximating fixed points of nonexpansive mappings, *Filomat* **30:10** (2016), 2711–2720.
[10] C. Wu and L.J. Zhang, Fixed points for mean nonexpansive mappings, *Acta Math. Appl. Sin. Engl. Ser.* **23(3)** (2007), 489–494.
[11] S.S. Zhang, About fixed point theorem for mean nonexpansive mapping in Banach spaces, *Journal of Sichuan University* **2** (1975), 67–68.
[12] J. Zhou and Y. Cui, Fixed point theorems for mean nonexpansive mappings in CAT(0) spaces, *Numerical Functional Analysis and Optimization* **36** (2015), 1224–1238.

Safeer Hussain Khan, Department of Mathematics, Statistics and Physics, Qatar University, Doha 2713, State of Qatar.
E-mail address: `safeerhussain5@yahoo.com; safeer@qu.edu.qa`

Ritika S., Department of Mathematics, Pt N. R. S. Government College, M. D. University, Rohtak 124001, Haryana, India.
E-mail address: `math.riti@gmail.com`