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SOME COUPLED BEST PROXIMITY POINT RESULTS FOR WEAK GKT CYCLIC ϕ-CONTRACTION MAPPINGS ON METRIC SPACES

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Abstract. In this paper we define a generalization of GKT cyclic ϕ-contraction mappings [18] in a metric space. Some existence results of coupled best proximity point for such mappings are derived with suitable examples. An application for the existence of the solution to an initial value problem through fixed point formulation is also discussed.

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1. Introduction and preliminaries

In 1922, Banach introduced one important result in non-linear analysis which is known as the Banach contraction principle [4]. Since then many researchers have genearlized this result in different directions (see [8, 9, 11, 14, 15, 22]). In 1987, Guo et al [10] introduced coupled fixed point and derived coupled fixed point theorems for some operators. Thereafter, many authors have been working on coupled fixed point theory for different types of mappings(see [1, 12]).

In 1969, Fan [24] introduced the concept of best proximity point in metric spaces. In 2012, Sintunavarat et al [21] defined the concept of coupled best proximity point in metric spaces and established some coupled best proximity point results. Afterwards, work on generalization of some of these results have been going on by several researchers (see [2, 16]).

First, we present some basic definitions and results which are useful for our work.

Definition 1.1. [24] *For two non empty subsets A and B of a metric space (X, d), let $T : A \longrightarrow B$ be a mapping. A point $x \in A$ is said to be a best proximity point of T if and only if it satisfies the condition $d(Tx, x) = d(A, B)$.*

Example 1.2. *Let $X = \mathbb{R}$ with usual metric $d(x, y) =\mid x - y \mid$ for all $x, y \in \mathbb{R}$ and $A = [1, 2]$, $B = [0, \frac{1}{2}]$. Let $T : A \longrightarrow B$ be a mapping defined by $T(x) = \frac{1}{2}$ for all $x \in A$.*

Here, $d(A, B) = \frac{1}{2}$ and $d(1, T1) =\mid 1 - \frac{1}{2} \mid= \frac{1}{2}$. Therefore, 1 is a best proximity point of T.

Definition 1.3. [21] *For two non empty subsets A and B of a metric space (X, d), let $S : A \times A \longrightarrow B$ be a mapping. A point $(x, x') \in A \times A$ is called a coupled best proximity point of S if*

$$d(x, S(x, x')) = d(x', S(x', x)) = d(A, B).$$

Example 1.4. *Let* $X = \mathbb{R}$ *with usual metric* $d(x, y) =| x - y |$ *for all* $x, y \in \mathbb{R}$ *and* $A = [1, 2]$, $B = [-1, 0]$. *A mapping* $S : A \times A \longrightarrow B$ *be defined by*

$$S(x, x') = \frac{2 - x - x'}{2} \quad for\ all\ (x, x') \in A \times A.$$

Here $d(A, B) = 1$,

$$\begin{aligned} d(x, S(x, y)) &=| x - \frac{2 - x - y}{2} | \\ &=| \frac{3x + y - 2}{2} |, \end{aligned}$$

$$\begin{aligned} d(y, S(y, x)) &=| y - \frac{2 - y - x}{2} | \\ &=| \frac{3y + x - 2}{2} | . \end{aligned}$$

For $x = y = 1$, $d(x, S(x, y)) = d(y, S(y, x)) = 1 = d(A, B)$. Therefore, $(1, 1)$ is a coupled best proximity point of S.

In the study of best proximity point theory, recently we have defined GKT cyclic ϕ-contraction mappings in [18] and obtained some existence results of best proximity point in metric spaces.

Definition 1.5. [18] *For two non empty subsets A and B of a metric space (X,d), let* $\phi : [0, \infty) \longrightarrow [0, \infty)$ *be a strictly increasing mapping. A mapping* $T : A \cup B \longrightarrow A \cup B$ *is said to be a GKT cyclic* ϕ*-contraction if the following conditions are satisfied:*
(1) $T(A) \subseteq B$, $T(B) \subseteq A$,
(2) $d(x, y) \geq \phi(d(x, y)) - \phi(d(A, B))$,
(3) *for some* $k \in [0, 1)$,

$$d(Tx, Ty) \leq (1 - k)[d(x, y) - \phi(d(x, y)) + \phi(d(A, B))] + \tfrac{k}{2}[d(Tx, x) + d(Ty, y)], \quad for\ \ all\ x \in A, y \in B.$$

Definition 1.6. [23] *For two non empty subsets A and B of a metric space* (X, d), (A, B) *is said to satisfy UC property if for sequences* $\{x_n\}$, $\{x'_n\}$ *in A and* $\{y_n\}$ *in B such that*

$\lim_{n\to\infty} d(x_n, y_n) = d(A, B) = \lim_{n\to\infty} d(x'_n, y_n)$ *imply that* $\lim_{n\to\infty} d(x_n, x'_n) = 0$.

In [21], Sintunavarat defined cyclic contraction pair and obtained a coupled best proximity point theorem in metric space (X, d).

Definition 1.7. [21] *For two non empty subsets A and B of a metric space* (X, d), *let* $S : A \times A \longrightarrow B$, $T : B \times B \longrightarrow A$ *be two mappings. The ordered pair* (S, T) *is called a cyclic contraction if for some non-negative number* $\alpha < 1$ *the following condition is satisfied:*

$$d(S(x, x'), T(y, y')) \leq \frac{\alpha}{2}[d(x, y) + d(x', y')] + (1 - \alpha)d(A, B)$$

for all $(x, x') \in A \times A$ *and* $(y, y') \in B \times B$.

Theorem 1.8. [21] *For two non empty compact subsets A, B of a metric space (X, d), let (S, T) be a cyclic contraction. For $(x_0, y_0) \in A \times A$, define*

$$x_{2n+1} = S(x_{2n}, y_{2n}), \quad y_{2n+1} = S(y_{2n}, x_{2n}),$$
$$x_{2n+2} = T(x_{2n+1}, y_{2n+1}), \quad y_{2n+2} = T(y_{2n+1}, x_{2n+1})$$

for all $n \in \mathbb{N} \cup \{0\}$, then each of S and T has a coupled best proximity point.

2. Main results

First, we define the following type of mappings.

Definition 2.1. *Let A and B be two non empty subsets of a metric space (X, d) and $\phi : [0, \infty) \longrightarrow [0, \infty)$ be a strictly non-decreasing mapping.*
The mapping $S : (A \times A) \cup (B \times B) \longrightarrow A \cup B$ is said to be a weak GKT cyclic ϕ-contraction if the following conditions are satisfied:

$(W1)$ *$S(A \times A) \subseteq B$ and $S(B \times B) \subseteq A$,*

$(W2)$ *$d(x, y) \geq \phi(d(x', y')) - \phi(d(A, B))$,*

$(W3)$ *for some $k \in [0, 1)$*

$$d(S(x, x'), S(y, y')) \leq (1 - k)[d(x, y) + \phi(d(A, B)) - \phi(d(x', y'))] + \frac{k}{2}[d(S(x, x'), x) + d(S(y, y'), y)]$$

for all (x, x'), $(y, y') \in (A \times A) \cup (B \times B)$

Remark 2.2. *If we consider a mapping $T : A \cup B \longrightarrow A \cup B$ such that $Tx = S(x, x)$ for all $x \in A \cup B$, then*

(i) *$T(A) \subseteq B$ and $T(B) \subseteq A$,*

(ii) *$d(x, y) \geq \phi(d(x, y)) - \phi(d(A, B))$,*

(iii) *$d(Tx, Ty) \leq (1-k)[d(x, y)+\phi(d(A, B))-\phi(d(x, y))]+ \frac{k}{2}[d(Tx, x)+d(Ty, y)]$*

for all $x, y \in A \cup B$.

Thus, T satisfies GKT cyclic ϕ-contraction conditions.

Example 2.3. *Let $X = \mathbb{R}$ with the discrete metric and $A = [-1 - \frac{1}{2}]$, $B = [\frac{1}{2}, 1]$ be subsets of X and $\phi(x) = x$. A mapping $S : (A \times A) \cup (B \times B) \longrightarrow A \cup B$ be defined by* $S(x, x') = \begin{cases} 1, & (x, x') \in A \times A, \\ -1, & (x, x') \in B \times B. \end{cases}$
Then S is a weak GKT cyclic ϕ-contraction.

Example 2.4. *Let $X = \mathbb{R}^2$ with the metric $d((x, x'), (y, y')) = max\{|x-y|, |x'-y'|\}$ and $A = \{(x, 1) : 1 \leq x \leq 2\}$, $B = \{(x, 2) : 1 \leq x \leq 2\}$. A mapping $S : (A \times A) \cup (B \times B) \longrightarrow A \cup B$ be defined by*

$$S((x,x'),(y,y')) = \begin{cases} (\frac{x+y}{2}, 2); & ((x,x'),(y,y')) \in A \times A, \\ (\frac{x+y}{2}, 1); & ((x,x'),(y,y')) \in B \times B, \end{cases}$$

with $\phi(x) = x$. Then S is a weak GKT cyclic ϕ-contraction.

The following Lemma will be needed to obtain the main results.

Lemma 2.5. *For two non empty subsets A, B of a metric space (X,d), let $S : (A\times A)\cup(B\times B) \longrightarrow A\cup B$ be a weak GKT cyclic ϕ-contraction. For $(x_0, y_0) \in A\times A$, define $x_{n+1} = S(x_n, y_n)$ and $y_{n+1} = S(y_n, x_n)$ for all $n \in \mathbb{N}\cup\{0\}$. Then $d(x_n, x_{n+1}) \longrightarrow d(A,B)$ and $d(y_n, y_{n+1}) \longrightarrow d(A,B)$.*

Proof:
Using the definition of sequences $\{x_n\}$ and $\{y_n\}$, we get,

$$\begin{aligned} d(x_n, x_{n+1}) &= d(S(x_{n-1}, y_{n-1}), S(x_n, y_n)) \\ &\leq (1-k)[d(x_{n-1}, x_n) + \phi(d(A,B)) - \phi(d(y_{n-1}, y_n))] \\ &\quad + \frac{k}{2}[d(S(x_{n-1}, y_{n-1}), x_{n-1}) + d(S(x_n, y_n), x_n)] \\ &\leq (1-k)d(x_{n-1}, x_n) + \frac{k}{2}[d(x_n, x_{n-1}) + d(x_n, x_{n+1})], \end{aligned}$$

i.e., $\quad d(x_n, x_{n+1}) \leq d(x_n, x_{n-1})$.

Therefore, $\{d(x_n, x_{n+1})\}$ is a decreasing sequence. Hence, it converges to some t_0, say, i.e., $\lim_{n\to\infty} d(x_n, x_{n+1}) = t_0$.

Again,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq (1-k)[d(x_{n-1}, x_n) + \phi(d(A,B)) - \phi(d(y_{n-1}, y_n))] \\ &\quad + \frac{k}{2}[d(x_n, x_{n-1}) + d(x_{n+1}, x_n)], \end{aligned}$$

$$\begin{aligned} \text{i.e., } (1-k)\phi(d(y_{n-1}, y_n)) &\leq (1-\frac{k}{2})d(x_{n-1}, x_n) + (1-k)\phi(d(A,B)) \\ &\quad - (1-\frac{k}{2})d(x_n, x_{n+1})]. \end{aligned} \tag{1}$$

Taking $n \to \infty$ on both sides of (1), we get,

$$\phi(d(y_{n-1}, y_n)) \to \phi(d(A,B)).$$

Also, $\{d(y_{n+1}, y_n)\}$ is a decreasing sequence and so, converges to some l, say.

Now,

$$\begin{aligned} d(A,B) \leq l &\leq d(y_{n+1}, y_n) \\ &\leq d(y_n, y_{n-1}) \\ \text{i.e., } \phi(d(A,B)) \leq \phi(l) &\leq \phi(d(y_n, y_{n-1})). \end{aligned}$$

Therefore, $\phi(l) = \phi(d(A,B))$, which implies that $l = d(A,B)$. Similarly, it is easy to show that $t_0 = d(A,B)$. □

In the following result, we show the existence of coupled best proximity point for weak GKT cyclic ϕ-contraction mapping in a metric space X.

Theorem 2.6. *Let A and B be two non empty subsets of a metric space (X,d), A be closed and S be a weak GKT cyclic ϕ-contraction. For $(x_0,y_0)\in A\times A$, generate sequences $\{x_n\}$ and $\{y_n\}$ as in Lemma 2.5. If $\{x_{2n}\}$ and $\{y_{2n}\}$ have convergent subsequences, then S has a coupled best proximity point.*

Proof:
Let $\{x_{2n_k}\}$, $\{y_{2n_k}\}$ be convergent subsequences of $\{x_{2n}\}$, $\{y_{2n}\}$ respectively such that

$$x_{2n_k}\to p\in A\ ,\ y_{2n_k}\to q\in A.$$

Applying the definition of weak GKT cyclic ϕ-contraction,

$$\begin{aligned} d(x_{2n_k},S(p,q)) &= d(S(x_{2n_k-1},y_{2n_k-1}),S(p,q))\\ &\le (1-k)d(x_{2n_k-1},p)+\frac{k}{2}[d(S(x_{2n_k-1},y_{2n_k-1}),x_{2n_k-1})+d(S(p,q),p)]\\ &\le (1-k)[d(x_{2n_k},x_{2n_k-1})+d(x_{2n_k},p)]+\frac{k}{2}[d(x_{2n_k},x_{2n_k-1})+d(S(p,q),p)]\\ &\le (1-\frac{k}{2})d(x_{2n_k},x_{2n_k-1})+(1-k)d(x_{2n_k},p)+\frac{k}{2}d(S(p,q),p). \end{aligned} \tag{2}$$

Taking $n\to\infty$ on both sides of (2) and using Lemma 2.5, we get,

$$d(p,S(p,q))=d(A,B).$$

In the same way, we can show that $d(q,S(q,p))=d(A,B)$. So, (p,q) is a coupled best proximity point of S. □

Example 2.7. *Consider (X,d), A, B, and S as in Example 2.3.*

Here $d(A,B)=1$. Let $(x_0,y_0)=(-1,-\frac{1}{2})\in A\times A$. Then,

$x_1=S(-1,-\frac{1}{2})=1=y_1$

$x_2=S(1,1)=-1=y_2$

$x_3=S(-1,-1)=1=y_3$

$x_4=S(1,1)=-1=y_4$

$\vdots$

Thus, $\{x_{2n}\}=\{-1\}$, $\{y_{2n}\}=\{-1\}$ and $d(-1,S(-1,-1))=d(-1,1)=1=d(A,B)$. Therefore, $(-1,-1)$ is a coupled best proximity point of S.

By the fact that every sequence in a sequentially compact metric space has a convergent subsequence, we get the following:

Corollary 2.8. *For two non empty subsets A, B of a metric space (X,d), let A be sequentially compact and S be a weak GKT cyclic ϕ-contraction. For $(x_0,y_0)\in A\times A$, define $\{x_n\}$, $\{y_n\}$ as in Lemma 2.5. Then S has a coupled best proximity point.*

Replacing the convergence condition of subsequences in Theorem 2.6 by the UC property of (A, B) and (B, A), we obtain the following result.

Theorem 2.9. *Let A and B be two non empty closed subsets of a complete metric space (X, d). Suppose that (A, B) and (B, A) satisfy UC property and S is a weak GKT cyclic ϕ-contraction with $k \neq 0$. For $(x_0, y_0) \in A \times A$, generate $\{x_n\}$ and $\{y_n\}$ as in Lemma 2.5. Then S has a coupled best proximity point.*

Proof:
From Lemma 2.5, we have,

$$d(x_{2n}, x_{2n+1}) \to d(A, B),$$

$$d(y_{2n}, y_{2n+1}) \to d(A, B).$$

Now we show that $\{x_{2n}\}$ and $\{y_{2n}\}$ are Cauchy sequences, i.e., for $\epsilon > 0$, there exist $N, N' \in \mathbb{N}$ such that

$$d(x_{2m}, x_{2n+1}) \leq d(A, B) + \epsilon \text{ for all } m > n \geq N,$$

$$d(y_{2m}, y_{2n+1}) \leq d(A, B) + \epsilon \text{ for all } m > n \geq N'.$$

We prove it by contradiction. If $\{x_{2n}\}$ and $\{y_{2n}\}$ are not Cauchy sequences, then there exists $\epsilon > 0$ such that for all $l, l' \in \mathbb{N}$ there exists $m_l > n_l \geq l$ and $m_{l'} > n_{l'} \geq l'$ such that

$$d(x_{2m_l}, x_{2n_l+1}) > d(A, B) + \epsilon,$$

$$d(y_{2m_{l'}}, y_{2n_{l'}+1}) > d(A, B) + \epsilon.$$

We can choose m_l and $m_{l'}$ in such a way that it is the smallest integer for which the above inequality satisfies. Then

$$d(x_{2m_l-1}, x_{2n_l+1}) \leq d(A, B) + \epsilon,$$

$$d(y_{2m_{l'}-1}, y_{2n_{l'}+1}) \leq d(A, B) + \epsilon.$$

Now,

$$\begin{aligned} d(A, B) + \epsilon &< d(x_{2m_l}, x_{2n_l+1}) \\ &\leq d(x_{2m_l}, x_{2m_l-2}) + d(x_{2m_l-2}, x_{2n_l+1}) \end{aligned} \tag{3}$$

Since (A, B) and (B, A) satisfy the UC property, so, for $d(x_{2m_l}, x_{2m_l-1}) \to d(A, B)$, $d(x_{2m_l-2}, x_{2m_l-1}) \to d(A, B)$ imply $d(x_{2m_l}, x_{2m_l-2}) \to 0$.

Taking $l \to \infty$ on both sides of (3), we get,

$$\lim_{l\to\infty} d(x_{2m_l}, x_{2n_l+1}) = d(A, B) + \epsilon.$$

Using triangle inequality,

$$\begin{aligned}
d(x_{2m_l}, x_{2n_l+1}) &\le d(x_{2m_l}, x_{2m_l+2}) + d(x_{2m_l+2}, x_{2n_l+3}) + d(x_{2n_l+3}, x_{2n_l+1}) \\
&= d(x_{2m_l}, x_{2m_l+2}) + d(S(x_{2m_l+1}, y_{2m_l+1}), S(x_{2n_l+2}, y_{2n_l+2})) \\
&\quad + d(x_{2n_l+3}, x_{2n_l+1}) \\
&\le (1-k)[d(x_{2m_l+1}, x_{2n_l+2}) + \phi(d(A,B)) - \phi(d(y_{2m_l+1}, y_{2n_l+2}))] \\
&+ \frac{k}{2}[d(x_{2m_l+2}, x_{2m_l+1}) + d(x_{2n_l+3}, x_{2n_l+2})] + d(x_{2m_l}, x_{2m_l+2}) \\
&\quad + d(x_{2n_l+3}, x_{2n_l+1}) \\
&\le (1-k)d(x_{2m_l+1}, x_{2n_l+2}) + \frac{k}{2}[d(x_{2m_l+2}, x_{2m_l+1}) + d(x_{2n_l+3}, x_{2n_l+2})] \\
&\quad + d(x_{2m_l}, x_{2m_l+2}) + d(x_{2n_l+3}, x_{2n_l+1}) \\
&= (1-k)d(S(x_{2m_l}, y_{2m_l}), S(x_{2n_l+1}, y_{2n_l+1})) \\
&+ \frac{k}{2}[d(x_{2m_l+2}, x_{2m_l+1}) + d(x_{2n_l+3}, x_{2n_l+2})] \\
&+ d(x_{2m_l}, x_{2m_l+2}) + d(x_{2n_l+3}, x_{2n_l+1}) \\
&\le (1-k)(1-k)[d(x_{2m_l}, x_{2n_l+1}) + \phi(d(A,B)) - \phi(d(y_{2m_l}, y_{2n_l+1}))] \\
&+ \frac{(1-k)k}{2}[d(x_{2m_l+1}, x_{2m_l}) + d(x_{2n_l+2}, x_{2n_l+1})] \\
&+ \frac{k}{2}[d(x_{2m_l+2}, x_{2m_l+1}) + d(x_{2n_l+3}, x_{2n_l+2})] \\
&+ d(x_{2m_l}, x_{2m_l+2}) + d(x_{2n_l+3}, x_{2n_l+1}) \\
&\le (1-k)^2 d(x_{2m_l}, x_{2n_l+1}) + \frac{(1-k)k}{2}[d(x_{2m_l+1}, x_{2m_l}) + d(x_{2n_l+2}, x_{2n_l+1})] \\
&+ \frac{k}{2}[d(x_{2m_l+2}, x_{2m_l+1}) + d(x_{2n_l+3}, x_{2n_l+2})] \\
&+ d(x_{2m_l}, x_{2m_l+2}) + d(x_{2n_l+3}, x_{2n_l+1}) \\
i.e.\ d(x_{2m_l}, x_{2n_l+1})(2k-k^2) &\le \frac{(1-k)k}{2}[d(x_{2m_l+1}, x_{2m_l}) + d(x_{2n_l+2}, x_{2n_l+1})] \\
&+ \frac{k}{2}[d(x_{2m_l+2}, x_{2m_l+1}) + d(x_{2n_l+3}, x_{2n_l+2})] + d(x_{2m_l}, x_{2m_l+2}) \\
&+ d(x_{2n_l+3}, x_{2n_l+1}) \qquad (4)
\end{aligned}$$

Taking limit as $l \to \infty$ on both sides of (4), we have,

$$\begin{aligned}
(d(A,B)+\epsilon)(2k-k^2) &\le (1-k)kd(A,B) + kd(A,B) \\
&= d(A,B)(k-k^2+k),
\end{aligned}$$

which is contradiction, since $k \neq 0$. Therefore, $\{x_{2n}\}$ and $\{y_{2n}\}$ are Cauchy sequences. Since X is complete, $x_{2n} \to p$, $y_{2n} \to q$.

Now, proceeding as in Theorem 2.6, we can show that (p,q) is a coupled best proximity point of S. □

For the case when $(x_0, y_0) \in B \times B$ in the Theorem 2.6, we take B as a closed subset of X instead of A.

Now, we define weak GKT cyclic ϕ-contraction in case of a pair of mappings.

Definition 2.10. *For two non empty subsets A and B of a metric space (X, d), let $S : A \times A \longrightarrow B$, $T : B \times B \longrightarrow A$ be two mappings. The pair (S, T) is said to be weak GKT cyclic ϕ-contraction pair if the following conditions are satisfied:*

(i) $d(x, y) \geq \phi(d(x', y')) - \phi(d(A, B))$,

(ii) for some $k \in [0, 1)$

$$d(S(x, x'), T(y, y')) \leq (1 - k)[d(x, y) + \phi(d(A, B)) - \phi(d(x', y'))]+$$
$$\frac{k}{2}[d(S(x, x'), x) + d(T(y, y'), y)]$$

for all $(x, x') \in A \times A$ and $(y, y') \in B \times B$.

Example 2.11. *Let $X = \mathbb{R}^2$ be a metric space with metric $d((x_1, y_1), (x_2, y_2)) = max\{|x_1 - x_2|, |y_1 - y_2|\}$ and $A = \{(x, 1); -1 \leq x \leq 1\}$, $B = \{(y, -1); -1 \leq y \leq 1\}$. Define $S : A \times A \longrightarrow B$, $T : B \times B \longrightarrow A$ and $\phi : [0, \infty) \longrightarrow [0, \infty)$ by*

$$S((x_1, 1), (x_2, 1)) = (1, -1)$$
$$T((y_1, -1), (y_2, -1)) = (1, 1)$$

and

$$\phi(x) = x.$$

It is obvious that $d(A, B) = 2$.
Now for all $((x_1, 1), (x_2, 1)) \in A \times A$ and $((y_1, -1), (y_2, -1)) \in B \times B$ we get,

$$d(S((x_1, 1), (x_2, 1)), T((y_1, -1), (y_2, -1))) = d((1, -1), (1, 1)) = 2. \tag{5}$$

and

$$(1 - k)[d((x_1, 1), (y_1, -1)) + \phi(d(A, B)) - \phi(d(x_2, 1), (y_2, -1)))]+$$
$$\frac{k}{2}[d(S((x_1, 1), (x_2, 1)), (x_1, 1)) + d(T((y_1, -1), (y_2, -1)), (y_1, -1))]$$
$$= (1 - k)[max\{|x_1 - y_1|, 2\} + 2 - max\{|x_2 - y_2|, 2\}] + \frac{k}{2}[d((1, -1), (x_1, 1))$$
$$+ d((1, 1), (y_1, -1))]$$
$$= 2(1 - k) + \frac{k}{2}[max\{|1 - x_1|, 2\} + max\{|1 - y_1|, 2\}]$$
$$= 2.$$

So, all the conditions of Definition 2.10 are satisfied. Therefore, the pair (S, T) is a weak GKT cyclic ϕ-contraction pair.

Example 2.12. *Let $X = \mathbb{R}$ be with the discrete metric and $A = [1, 2]$, $B = [-2, -1]$. The mappings $S : A \times A \longrightarrow B$ and $T : B \times B \longrightarrow A$ be defined by*

$$S(x, x') = \frac{-x-x'}{2} \text{ and } T(y, y') = \frac{-y-y'}{2}$$

for all $(x, x') \in A \times A$, $(y, y') \in B \times B$ and $\phi(x) = x$, then (S, T) is a weak GKT cyclic ϕ-contraction pair .

In [21], there is one convergence result for a type of Picard's iteration scheme considering cyclic contraction mappings in metric spaces. We extend this result for a pair of weak GKT cyclic ϕ-contraction pair.

Lemma 2.13. *For two non empty subsets A, B of a metric space (X, d), let (S, T) be a weak GKT cyclic ϕ-contraction pair. For $(x_0, y_0) \in A \times A$,*

$$x_{2n+1} = S(x_{2n}, y_{2n}), \qquad y_{2n+1} = S(y_{2n}, x_{2n}),$$

$$x_{2n+2} = T(x_{2n+1}, y_{2n+1}), \qquad y_{2n+2} = T(y_{2n+1}, x_{2n+1})$$

(6)

for all $n \in \mathbb{N} \cup \{0\}$. Then, $\{d(x_{2n}, x_{2n+1})\}$, $\{d(x_{2n+1}, x_{2n+2})\}$, $\{d(y_{2n}, y_{2n+1})\}$ and $\{d(y_{2n+1}, y_{2n+2})\}$ are decreasing sequences and each of these converges to d(A,B).

Proof:

Using iteration (6) and the condition of weak GKT cyclic ϕ-contraction of (S, T), for all $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &= d(T(x_{2n-1}, y_{2n-1}), S(x_{2n}, y_{2n})) \\ &\leq (1-k)[d(x_{2n-1}, x_{2n}) + \phi(d(A, B)) - \phi(d(y_{2n-1}, y_{2n})]+ \\ &\qquad \frac{k}{2}[d(T(x_{2n-1}, y_{2n-1}), x_{2n-1}) + d(S(x_{2n}, y_{2n}), x_{2n}] \\ &\leq (1-k)d(x_{2n-1}, x_{2n}) + \frac{k}{2}[d(x_{2n}, x_{2n-1}) + d(x_{2n+1}, x_{2n})] \end{aligned}$$

i.e., $d(x_{2n+1}, x_{2n}) \leq d(x_{2n}, x_{2n-1})$

Therefore, $\{d(x_{2n}, x_{2n+1})\}$ is a decreasing sequence. Similarly, it is easy to prove that $\{d(x_{2n+1}, x_{2n+2})\}$, $\{d(y_{2n}, y_{2n+1})\}$ and $\{d(y_{2n+1}, y_{2n+2})\}$ are decreasing sequences.

Let $\lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = t_0 \geq d(A, B)$.

Now,

$$\begin{aligned}
d(A,B) \leq t_0 \leq d(x_{2n}, x_{2n+1}) &\\
&= d(T(x_{2n-1}, y_{2n-1}), S(x_{2n}, y_{2n}))\\
&\leq (1-k)[d(x_{2n-1}, x_{2n}) + \phi(d(A,B)) - \phi(d(y_{2n-1}, y_{2n})]+\\
&\qquad \frac{k}{2}[d(T(x_{2n-1}, y_{2n-1}), x_{2n-1}) + d(S(x_{2n}, y_{2n}), x_{2n}]\\
&\leq (1-k)[d(x_{2n-1}, x_{2n}) + \phi(d(A,B)) - \phi(d(y_{2n-1}, y_{2n})]+\\
&\qquad \frac{k}{2}[d(x_{2n}, x_{2n-1}) + d(x_{2n+1}, x_{2n})]
\end{aligned}$$

$$\begin{aligned}
i.e., \quad (1-k)\phi(d(y_{2n-1}, y_{2n}) &\leq (1-k)[d(x_{2n-1}, x_{2n}) + \phi(d(A,B))]+\\
&\quad \frac{k}{2}d(x_{2n}, x_{2n-1}) - (1-\frac{k}{2})d(x_{2n+1}, x_{2n})\\
&= (1-\frac{k}{2})[d(x_{2n-1}, x_{2n}) - d(x_{2n+1}, x_{2n})]+\\
&\quad (1-k)\phi(d(A,B))\\
&\leq (1-\frac{k}{2})[d(x_{2n-1}, x_{2n-2}) - d(x_{2n+1}, x_{2n})]+\\
&\quad (1-k)\phi(d(A,B)) \qquad (7)
\end{aligned}$$

Taking limit as $n \to \infty$ on both sides of (7), we get,

$$\lim_{n\to\infty} \phi(d(y_{2n-1}, y_{2n}) = \phi(d(A,B)).$$

In the same way we get,

$$\lim_{n\to\infty} \phi(d(x_{2n-1}, x_{2n})) = \phi(d(A,B)).$$

Now, if $t_0 > d(A,B)$,

$$\begin{aligned}
&d(A,B) < t_0 \leqslant d(x_{2n+1}, x_{2n}) \leq d(x_{2n}, x_{2n-1})\\
i.e., \quad &d(A,B) < d(x_{2n}, x_{2n-1})\\
i.e., \quad &\phi(d(A,B)) < \phi(d(x_{2n}, x_{2n-1}))\\
i.e., \quad &\phi(d(A,B)) < \phi(d(A,B)),
\end{aligned}$$

which is contradiction. Therefore, $\lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = d(A,B)$.
Similarly, $\lim_{n\to\infty} d(x_{2n+1}, x_{2n+2}) = d(A,B)$, $\lim_{n\to\infty} d(y_{2n}, y_{2n+1}) = d(A,B)$ and $\lim_{n\to\infty} d(y_{2n+2}, y_{2n+1}) = d(A,B)$. □

In the the following theorem we show the existence of coupled best proximity point for each of the mappings S and T in metric space (X,d).

Theorem 2.14. *For two non empty closed subsets A, B of a metric space (X,d), let (S,T) be a weak GKT cyclic ϕ-contraction pair. For $(x_0,y_0) \in A \times A$, $\{x_{2n}\}$, $\{x_{2n+1}\}$, $\{y_{2n}\}$ and $\{y_{2n+1}\}$ be generated by iteration (6). If $\{x_{2n}\}$, $\{x_{2n+1}\}$, $\{y_{2n}\}$*

and $\{y_{2n+1}\}$ *have convergent subsequences, then each of* S *and* T *has a coupled best proximity point.*

Proof:

Since $\{x_{2n}\}$, $\{x_{2n+1}\}$, $\{y_{2n}\}$ and $\{y_{2n+1}\}$ have convergent subsequences and A, B are closed subset of X, so let

$$x_{2n_l} \longrightarrow p,\ y_{2n_l} \longrightarrow q\ ;\ (p,q) \in A \times A.$$
$$x_{2n_l+1} \longrightarrow p',\ y_{2n_l+1} \longrightarrow q'\ ;\ (p',q') \in B \times B.$$

Using iteration (6) for all $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} d(x_{2n_l}, S(p,q)) &= d(T(x_{2n_l-1}, y_{2n_l-1}), S(p,q)) \\ &\le (1-k)[d(x_{2n_l-1}, p) + \phi(d(A,B)) - \phi(d(y_{2n_l-1}, q))] + \\ &\quad \frac{k}{2}[d(x_{2n_l}, x_{2n_l-1}) + d(S(p,q),p)] \\ &\le (1-k)d(x_{2n_l-1}, p) + \frac{k}{2}[d(x_{2n_l}, x_{2n_l-1}) + d(S(p,q),p)]. \end{aligned} \tag{8}$$

Now,

$$d(p, S(p,q)) \le d(p, x_{2n_l}) + d(x_{2n_l}, S(p,q)). \tag{9}$$

Substituting (8) in (9),

$$d(p,S(p,q)) \le d(p,x_{2n_l}) + (1-k)d(x_{2n_l-1},p) + \frac{k}{2}[d(x_{2n_l}, x_{2n_l-1}) + d(S(p,q),p)],$$

(10)

$$i.e., (1-\frac{k}{2})d(p,S(p,q)) \le d(p,x_{2n_l}) + (1-k)[d(x_{2n_l}, x_{2n_l-1}) + d(p,x_{2n_l})] + \frac{k}{2}d(x_{2n_l}, x_{2n_l-1})$$

Taking limit as $l \to \infty$ on the both sides of (10) and using Lemma 2.14, we get,

$$d(p, S(p,q)) = d(A,B). \tag{11}$$

Again,

$$\begin{aligned} d(q, S(q,p)) &\le d(q, y_{2n_l}) + d(y_{2n_l}, S(q,p)) \\ &= d(q, y_{2n_l}) + d(T(y_{2n_l-1}, x_{2n_l-1}), S(q,p))] \\ &\le d(q, y_{2n_l}) + (1-k)d(y_{2n_l-1}, q) + \\ &\quad \frac{k}{2}[d(T(y_{2n_l-1}, x_{2n_l-1}), y_{2n_l-1}) + d(S(q,p),q)] \\ &\le d(q, y_{2n_l}) + (1-k)[d(y_{2n_l-1}, y_{2n_l}) + d(y_{2n_k}, q)] + \\ &\quad \frac{k}{2}[d(y_{2n_l}, y_{2n_l-1}) + d(S(q,p),q)], \end{aligned}$$

$$i.e., (1-\frac{k}{2})d(q,S(q,p)) \le (2-k)d(q,y_{2n_l}) + (1-\frac{k}{2})d(y_{2n_l-1}, y_{2n_l}). \tag{12}$$

Taking $l \to \infty$ on both side of (12), we get,

$$(13) \qquad d(q, S(q,p)) = d(A,B)$$

From equation (11) and (13),

$$d(q, S(q,p)) = d(p, S(p,q)) = d(A,B).$$

Therefore, (p,q) is a coupled best proximity point of S.
Similarly, it is easy to prove that (p', q') is a coupled best proximity point of T. □

The convergence condition of the above result can be replaced by taking A and B as sequentially compact subsets and ϕ as a continuous mapping.

Theorem 2.15. *Let A and B be two non empty sequentially compact subsets of a metric space (X,d) and (S,T) be a weak GKT cyclic ϕ-contraction pair. For $(x_0, y_0) \in A \times A$, $\{x_{2n}\}$, $\{x_{2n+1}\}$, $\{y_{2n}\}$ and $\{y_{2n+1}\}$ be generated by iteration (6) . If ϕ is a continuous mapping, then each of S and T has a coupled best proximity point.*

Proof:

For $(x_0, y_0) \in A \times A$, $\{x_{2n}\}$, $\{y_{2n}\} \subset A$ and $\{x_{2n+1}\}$, $\{y_{2n+1}\} \subset B$. Since A, B are sequentially compact, $\{x_{2n}\}$ and $\{y_{2n}\}$ have convergent subsequences $\{x_{2n_l}\}$ and $\{y_{2n_l}\}$ respectively such that

$$x_{2n_l} \longrightarrow p \in A, y_{2n_l} \longrightarrow q \in A.$$

Similarly, $\{x_{2n+1}\}$, $\{y_{2n+1}\}$ have convergent subsequences $\{x_{2n_l+1}\}$, $\{y_{2n_l+1}\}$ such that

$$x_{2n_l+1} \longrightarrow p' \in B, y_{2n_l+1} \longrightarrow q' \in B.$$

Now, as in Theorem 2.14, we can show that (p,q) is a coupled best proximity point of S and (p', q') is a coupled best proximity point of T. □

Example 2.16. *Consider (X,d), A, B, (S,T) and ϕ as in Example 2.13.*

Here, $d(A,B) = 1$.

For $(x_0, y_0) = (1,2) \in A \times A$,

$x_1 = S(1,2) = -\frac{3}{2}$, $y_1 = -\frac{3}{2}$

$x_2 = T(-\frac{3}{2}, -\frac{3}{2}) = \frac{3}{2}$, $y_2 = \frac{3}{2}$

$x_3 = S(\frac{3}{2}, \frac{3}{2}) = -\frac{3}{2}$, $y_3 = -\frac{3}{2}$

$x_4 = T(-\frac{3}{2}, -\frac{3}{2}) = \frac{3}{2}$, $y_4 = \frac{3}{2}$

$\vdots$

Hence $\{x_{2n+1}\} = \{-\frac{3}{2}\} = \{y_{2n+1}\}$ and $\{x_{2n+2}\} = \{\frac{3}{2}\} = \{y_{2n+2}\}$.

$d(-\frac{3}{2}, S(-\frac{3}{2}, -\frac{3}{2})) = d(-\frac{3}{2}, \frac{3}{2}) = 1 = d(A,B)$. Therefore, $(-\frac{3}{2}, -\frac{3}{2})$ is coupled best proximity point of S.

Again, $d(\frac{3}{2}, S(\frac{3}{2}, \frac{3}{2})) = d(\frac{3}{2}, -\frac{3}{2}) = 1 = d(A,B)$. Therefore, $(\frac{3}{2}, \frac{3}{2})$ is a coupled best proximity point of T.

In fact, the existence of a coupled best proximity point for one mapping implies the existence of the same to the other one, which can be seen in the following proposition.

Proposition 2.17. *For two non empty subsets A, B of a metric space (X,d), let (S,T) be a weak GKT cyclic ϕ-contraction pair. Then*

(i) if $(x,y) \in A \times A$ is a coupled best proximity point of S, then $(S(x,y), S(y,x))$ is a coupled best proximity point of T;

(ii) if $(u,v) \in B \times B$ is a coupled best proximity point of T, then $(T(u,v), T(v,u))$ is a coupled best proximity point of S.

Proof:
Let $(x,y) \in A \times A$ is a coupled best proximity point of S,

i.e., $d(x, S(x,y)) = d(y, S(y,x)) = d(A,B)$.

Now, using weak GKT cyclic ϕ-contraction condition,

$$\begin{aligned} d(S(x,y), T(S(x,y), S(y,x))) &\leq (1-k)[d(x, S(x,y)) + \phi(d(A,B)) - \phi(d(y, S(y,x)))] + \\ &\quad \frac{k}{2}[d(S(x,y), x) + d(T(S(x,y), S(y,x)), S(x,y))] \\ &\leq (1-k)d(A,B) + \frac{k}{2}d(A,B) + \frac{k}{2}d(T(S(x,y), S(y,x)), S(x,y)) \end{aligned}$$

$$(1 - \frac{k}{2})d(S(x,y), T(S(x,y), S(y,x))) \leq (1 - \frac{k}{2})d(A,B). \tag{14}$$

From (14), $d(S(x,y), T(S(x,y), S(y,x))) = d(A,B)$.

Similarly, $d(S(y,x), T(S(y,x), S(x,y))) = d(A,B)$, i.e., $(S(x,y), S(y,x))$ is a coupled best proximity point of T.

In the same way, it is easy to prove that if $(u,v) \in B \times B$ is a coupled best proximity point of T, then $(T(u,v), T(v,u))$ is a coupled best proximity point of S. □

For two non empty subsets A and B of a metric space (X,d), define

$$A_0 = \{x \in A : d(x,y) = d(A,B), \text{ for some } y \in B\},$$
$$B_0 = \{y \in B : d(x,y) = d(A,B), \text{ for some } x \in A\}.$$

(refer to [3])

Theorem 2.18. *For two non empty subsets A, B of a metric space (X,d), let (S,T) be a weak GKT cyclic ϕ-contraction pair. If the following conditions are satisfied:*

(i) $A_0 \times A_0 \neq \phi$, $B_0 \times B_0 \neq \phi$ and A_0, B_0 are compact subsets,

(ii) $S(A_0 \times A_0) \subseteq B_0$, $T(B_0 \times B_0) \subseteq A_0$,

then each of S and T has a coupled best proximity point.

Proof:
Suppose $(x_0, y_0) \in A_0 \times A_0$, then $x_1 = S(x_0, y_0) \in B_0$ and $y_1 = S(y_0, x_0) \in B_0$. Therefore, $(x_1, y_1) \in B_0 \times B_0$ imply $x_2 = T(x_1, y_1) \in A_0$ and $y_2 = T(y_1, x_1) \in A_0$.

If we continue same process upto $2n + 1$ times then,

$$x_{2n+1} = S(x_{2n}, y_{2n}) \in B_0 \text{ and } y_{2n+1} = S(y_{2n}, x_{2n}) \in B_0.$$

$$x_{2n} = T(x_{2n-1}, y_{2n-1}) \in A_0 \text{ and } y_{2n} = T(y_{2n-1}, x_{2n-1}) \in A_0.$$

A_0 and B_0 are compact subsets of metric space X. So, $\{x_{2n}\}$, $\{y_{2n}\}$, $\{x_{2n+1}\}$ and $\{y_{2n+1}\}$ have convergent subsequences $\{x_{2n_l}\}$, $\{y_{2n_l}\}$, $\{x_{2n_l+1}\}$ and $\{y_{2n_l+1}\}$ respectively such that

$$x_{2n_l} \longrightarrow p \in A_0 \ , \ y_{2n_k} \longrightarrow q \in A_0,$$
$$x_{2n_l+1} \longrightarrow p' \in B_0 \ , \ y_{2n_k+1} \longrightarrow q' \in B_0.$$

Now,

$$\begin{aligned} d(x_{2n_l}, S(p, q)) &= d(T(x_{2n_l-1}, y_{2n_l-1})S(p, q)) \\ &\leq (1-k)d(x_{2n_l-1}, p) + \frac{k}{2}[d(x_{2n_l-1}, x_{2n_l}) + d(p, S(p, q)] \\ &\leq (1-k)[d(x_{2n_l-1}, x_{2n_l}) + d(x_{2n_l}, p)] + \\ &\quad \frac{k}{2}[d(x_{2n_l-1}, x_{2n_l}) + d(p, S(p, q)] \end{aligned} \tag{15}$$

It is obvious that $d(A, B) = d(x_{2n_l}, x_{2n_l-1})$.

Taking limit as $l \longrightarrow \infty$ on both sides of (15), we get,

$$d(p, S(p, q)) \leq d(A, B).$$

Therefore, $d(A, B) = d(p, S(p, q))$. Similarly, $d(A, B) = d(q, S(q, p))$.

So, (p, q) is a coupled proximity point of S. In the same way, (p', q') is a coupled proximity point of T. □

3. Application to an initial value problem

In this section, we discuss an application of our obtained results for showing the existence of solution to an initial value problem using the fixed point formulation. If we consider $d(A, B) = 0$ in the Theorem 2.9 then we get a coupled fixed point Theorem as follows:

Theorem 3.1. *Let (X, d) be a complete metric space and S be a weak GKT cyclic ϕ-contraction with $\phi(t) \leq t$. For $(x_0, y_0) \in X \times X$, generate $\{x_n\}$ and $\{y_n\}$ as in Lemma (2.5). Then S has a unique coupled fixed point.*

Proof:
Since $d(A, B) = 0$, so by Theorem 2.9, $\{x_{2n}\}$ and $\{y_{2n}\}$ are Cauchy sequences and converges to p and q respectively. Now using Theorem 2.6, we can easily show that $d(p, S(p, q)) = 0 = d(q, S(q, p))$, so (p, q) is a coupled fixed point of S.

To check uniqueness, we assume that (p, q) and (p', q') are two coupled fixed points of S,

$$i.e., \quad S(p,q)=p, \quad S(q,p)=q;$$
$$and \quad S(p',q')=p', \quad S(q',p')=q'.$$

Since S is a weak GKT cyclic ϕ-contraction,

$$\begin{aligned} d(S(p,q),S(p',q')) &\leq (1-k)[d(p,p')-\phi(d(q,q'))]+\frac{k}{2}[d(S(p,q),p)+d(S(p',q'),p')] \\ &= (1-k)[d(p,p')-\phi(d(q,q'))]. \end{aligned} \tag{16}$$

Now,

$$\begin{aligned} d(p,p') &\leq d(p,S(p,q))+d(S(p,q),S(p',q'))+d(S(p',q'),p') \\ &= d(S(p,q),S(p',q')) \end{aligned} \tag{17}$$

Comparing (16) and (17) we get,

$$d(p,p') \leq (1-k)[d(p,p')-\phi(d(q,q'))],$$

$$i.e., \ (1-k)\phi(d(q,q'))+kd(p,p') \leq 0. \tag{18}$$

Since both the terms in the left hand side of (18) is positive, the only possibility is $(1-k)\phi(d(q,q'))+kd(p,p')=0$, i.e., $(1-k)\phi(d(q,q'))=0$ and $kd(p,p')=0$.

Now, for $k=0$,
$\phi(d(q,q'))=0=\phi(0)$ i.e., $d(q,q')=0$.

Similarly, we can find out $d(p,p')=0$.

For $k \neq 0$,
$d(p,p')=0$ and $\phi(d(q,q'))=0$ i.e., $d(q,q')=0$.
Therefore, $p=p'$ and $q=q'$. □

Now, we take the following initial value problem:

$$\begin{aligned} x'(t) &= f(t,x(t),x(t)), \quad t \in I=[0,1], \\ x(0) &= x_0, \end{aligned} \tag{19}$$

where $f : I \times [0,1] \times [0,1] \longrightarrow [0,1]$ and $x_0 \in \mathbb{R}$.

Theorem 3.2. *Consider the initial value problem* (19) *with* $f \in C(I \times [0,1] \times [0,1])$ *and* $\int_0^t f(s,x(s),y(s))ds \leq \frac{1}{4}\int_0^t f(s,x(s),x(s))ds - x_0$ *if* $x(t) \geq y(t)$ *for all* $t \in [0,1]$. *Then the problem has a unique solution.*

Proof:
Integral equation corresponding to (19) is

$$x(t) = x_0 + \int_0^t f(s, x(s), x(s))ds. \tag{20}$$

Let $X = C[0,1]$, $d(x(t), y(t)) = max\{x(t), y(t)\}$. Define, $S : X \times X \longrightarrow X$ by $S(x(t), y(t)) = x_0 + \int_0^t f(s, x(s), y(s))ds$ and $\phi(x) = x$.
Now, for $x(t) \geq y(t)$ and $u(t) \geq v(t)$ for all $t \in [0,1]$,

$$\begin{aligned} d(S(x(t), y(t)), S(u(t), v(t))) &= max\{S(x(t), y(t)), S(u(t), v(t))\} \\ &= max\{x_0 + \int_0^t f(s, x(s), y(s))ds, x_0 + \int_0^t f(s, u(s), v(s))ds\} \\ &\leq max\{x_0 + \frac{1}{4}\int_0^t f(s, x(s), x(s))ds - x_0, \\ &\quad x_0 + \frac{1}{4}\int_0^t f(s, u(s), u(s))ds - x_0\} \\ &= max\{\frac{x(t)}{4} - \frac{x_0}{4}, \frac{u(t)}{4} - \frac{x_0}{4}\}. \end{aligned} \tag{21}$$

Here, $max\{x(t), y(t)\} = x(t)$ and $max\{u(t), v(t)\} = u(t)$. Therefore, it is clear that $[max\{x(t), u(t)\} - max\{y(t), v(t)\}] \geq 0$.
Also
$\frac{k}{2}[d(S(x(t), y(t)), x(t)) + d(S(u(t), v(t)), u(t))]$

$= \frac{k}{2}[max\{S(x(t), y(t)), x(t)\} + max\{S(u(t), v(t)), u(t)\}]$

$= \frac{k}{2}[max\{x_0 + \int_0^t f(s, x(s), y(s))ds, x(t)\} + max\{x_0 + \int_0^t f(s, u(s), v(s))ds, u(t)\}]$

$= \frac{k}{2}[x(t) + u(t)]$

$= \frac{1}{4}[x(t) + u(t)]$, for $k = \frac{1}{2}$. (25)

From equation (24) and (25), we get,

$$d(S(x(t), y(t)), S(u(t), v(t))) \leq (1-k)[d(x(t), u(t)) - \phi(d(y(t), v(t)))] +$$
$$\frac{k}{2}[d(S(x(t), y(t)), x(t)) + d(S(u, v), u)].$$

Thus using Theorem 3.1, we can conclude that the initial value problem has a unique solution. □

4. Conclusion

We have proved some existence theorems of coupled best proximity point for weak GKT cyclic ϕ-contraction mappings in a metric space, with an application to first order initial value problem. It is also interesting to investigate the application of the established results to some higher order initial value problems as well as some boundary value problems.

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