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NOTE ON DIFFERENT KINDS OF SCHUR CONVEXITIES OF HEINZ TYPE MEAN

K. SRIDEVI, K. M. NAGARAJA, AND P. SIVA KOTA REDDY

ABSTRACT. The Schur convexity of functions relating to special means is a very significant research subject and has attracted the interest of many mathematicians. In this note, a new family of one parameterized Heinz type mean is introduced and we discuss the different kinds of Schur convexity and concavity of Heinz type mean.

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1. INTRODUCTION

The property of Schur convexity and Schur concavity has invoked the interest of many researchers and numerous papers have been dedicated to the investigation of it. The object of this paper is to present an overview of the results related to the study of Schur convexity of a new family of one parameterized Heinz type mean and we contribute to the subject with some new results.

Named after a German mathematician *Erhard Heinz*, for any two non-negative real numbers a and b, the Heinz mean was defined by Bhatia [1] as:

$$H_v(a,b) = \frac{a^{1-v}b^v + a^v b^{1-v}}{2}\ ; \qquad 0 \le v \le 1,\ a\ ,\ b > \ 0.$$

It is observed that

$$H_v(a,b) \ = \begin{cases} A(a,b) & , \text{v} = 0\ \ \&\ \ \text{v} = 1 \\ G(a,b) & , \text{v} = \frac{1}{2} \end{cases}$$

where in the usual notation A and G stand for arithmetic and geometric means of (a,b) respectively.

Therefore for different values of v, the Heinz mean interpolates between the arithmetic mean $(v = 0,1)$ and geometric mean $\left(v = \frac{1}{2}\right)$ such that for $\frac{1}{2} \le v \le 1$,

$$\left[G = H_{\frac{1}{2}}\right] \le \ H_v \le [\ H_1 \ = \ A].$$

[1]Corresponding author: pskreddy@jssstuniv.in

The fact that this Heinz mean can also be defined in the same way for positive semi-definite matrices and a similar interpolation formula being satisfied, evinces an interest to further study the applications of this mean and its properties in different fields.

Several authors have introduced and studied in depth about parameterized family of means such as: Stolarsky's mean, functional means, Heinz mean etc., and one can easily find the generous and interesting results (See [1, 4–11]). This has motivated us to introduce a new mean similar to Heinz mean. New family of Heinz type mean is denoted by $H_v^*(a,b)$ and is defined for two distinct positive real values a,b and the parameter $v \in [0,1]$:

$$H_v^* = H_v^*(a\ ,\ b) \ = \ \frac{a^{1\ +\ v}b^{1\ -\ v} \ + \ a^{1\ -\ v}b^{1\ +\ v}}{a^v b^{1\ -\ v} \ + \ a^{1\ -\ v}b^v} \tag{1}$$

The validity of definition of mean and properties can easily be verified for this Heinz type mean.

K. M. Nagaraja et al. [3] proved that $H_v^*(a,b)$ is a mean for all values of v, stated the symmetric and homogeneous properties, proved the monotonic condition with respect to the parameter v and some well known means are extracted from $H_v^*(a\ ,\ b)$. Also, log-convexity and log-concavity nature of $H_v^*(a,b)$ are discussed and these properties on double sequences are found in [5].

Also it is interesting to note that

$$H_v^*(a,b) \ = \ \begin{cases} H(a,b) = \frac{2ab}{a+b} & ,\ \mathrm{v} = 0 \\ A(a,b) = \frac{a+b}{2} & ,\ \mathrm{v} = \frac{1}{2} \\ C(a,b) = \frac{a^2+b^2}{a+b} & ,\ \mathrm{v} = 1 \end{cases}$$

where H, A and C are Harmonic, Arithmetic and Contra-harmonic means of $(a\ ,\ b)$ respectively.

That is, for $v \in [0,1]$ the Heinz type mean $H_v^*(a,b)$ interpolates between the harmonic mean $(v = 0)$ and contra-harmonic mean $(v = 1)$ such that for $0 \le v \le 1$,

$$[H = H_0^*] \ \le H_v^* \ \le \ [H_1^* = C]\,.$$

After understanding important properties satisfied by this Heinz type mean, it becomes essential to study some of the important convexity and concavity conditions of this mean. To prove such results it is essential recalling some important definitions and lemmas concerning them.

2. Definitions and Lemmas

The present work requires to recall some necessary definitions and lemmas.

Definition 2.1. *[2], [12] Consider two arbitrary n-tuple elements u , v in $\boldsymbol{R}^n$, $n \geq 2$, given by $u = (u_1, u_2, ..., u_n)$, $v = (v_1, v_2, ..., v_n)$.*

(1) *For the arrangements of u and v in descending order of the form $u_{[1]} \geq \cdots \geq u_{[n]}$ and $v_{[1]} \geq \cdots \geq v_{[n]}$, v mazorizes u(in symbol $u \prec v$),*
if for $1 \leq k \leq n-1$,

$$\sum_{i=1}^{k} u_{[i]} \leq \sum_{i=1}^{k} v_{[i]}$$

and

$$\sum_{i=1} u_{[i]} \leq \sum_{i=1} v_{[i]}.$$

(2) *Let $T \subseteq \boldsymbol{R}^n$, $n \geq 2$. For all $i = 1$ to n, $u \geq v$ means $u_i \geq v_i$.*
Let $\zeta : T \to \boldsymbol{R}$. The function ζ is considered to be increasing only if $-\zeta$ is decreasing.

(3) *The set $T \subseteq \boldsymbol{R}^n$ is said to be convex set if the set of linear combinations*
$(\alpha u_1 + \beta v_1, \cdots, \alpha u_n + \beta v_n) \in T$ for all u and v where parameters α , $\beta \in [0,1]$, $\alpha + \beta = 1$.

(4) *For $n \geq 2$, let $T \subseteq \boldsymbol{R}^n$, be a set having nonempty interior. The function $\zeta : T \to \boldsymbol{R}$ is considered to be Schur-convex whenever $u \prec v$ implies $\zeta(u) \leq \zeta(v)$.*
The function ζ is Schur convex if $-\zeta$ is a Schur concave function.

Definition 2.2. *[13] Consider $u = (u_1, u_2, ..., u_n)$ and $v = (v_1, v_2, ..., v_n)$ to be arbitrary elements in $\boldsymbol{R}^n_+$. Let $T \subseteq \boldsymbol{R}^n$ and u, $v \in T$. For parameters α , $\beta \in [0,1]$, $\alpha + \beta = 1$, the set T is said to be harmonically convex set if $(u_1^{\alpha} v_1^{\beta}, ..., u_1^{\alpha} v_1^{\beta}) \in T$. The function $\zeta : T \to \boldsymbol{R}_+$ is considered to be Schur harmonically convex function if $(lnu_1, ..., lnu_n) \prec (lnv_1, ..., lnv_n)$ implies $\zeta(u) \leq \zeta(v)$. ζ is considered to be Schur harmonically concave if $-\zeta$ is Schur harmonically convex.*

Definition 2.3. *[2], [12] Let P be a $n \times n$ permutation matrix. The set $T \subseteq \boldsymbol{R}^n$ is said to be symmetric if $u \in T \Rightarrow Pu \in T$ for every permutation matrix P.*
Further the function $\zeta : T \to \boldsymbol{R}$ is said to be symmetric if $\zeta(Pu) = \zeta(u)$ holds for all $u \in T$ and every such matrix P.

Definition 2.4. *[2], [12] Let $T \subseteq \boldsymbol{R}^n$. The function $\zeta : T \to \boldsymbol{R}$ is said to be symmetric and convex function if ζ is Schur convex on T.*

Lemma 2.5. *[13] Let $T \subseteq \boldsymbol{R}^n$ be a symmetric set and T^0 be its nonempty interior harmonic convex set. Let $\zeta : T \to \boldsymbol{R}_+$ be continuous function on T and differentiable function in T^0.*

For any $u = (u_1, u_2, \cdots, u_n) \in T^0$, the function ζ is Schur convex (Schur concave) if

$$(u_1 - u_2)\left(\frac{\partial \zeta}{\partial u_1} - \frac{\partial \zeta}{\partial u_2}\right) \geq 0 \quad (\leq 0) \tag{2}$$

Lemma 2.6. *[13] Let $T \subseteq \boldsymbol{R}^n$ be a symmetric set and T^0 be its nonempty interior harmonic convex set. Let $\zeta : T \to \boldsymbol{R}_+$ be continuous function on T and differentiable function in T^0. For any $u = (u_1, u_2, \cdots, u_n) \in T^0$, the function ζ is Schur geometric convex (Schur geometric concave) if*

$$(lnu_1 - lnu_2)\left(u_1\frac{\partial \zeta}{\partial u_1} - u_2\frac{\partial \zeta}{\partial u_2}\right) \geq 0 \quad (\leq 0). \tag{3}$$

Lemma 2.7. *[13] Let $T \subseteq \boldsymbol{R}^n$ be a symmetric set and T^0 be its nonempty interior harmonic convex set. Let $\zeta : T \to \boldsymbol{R}_+$ be continuous function on T and differentiable function in T^0. For any $u = (u_1, u_2, \cdots, u_n) \in T^0$, the function ζ is Schur harmonic convex (Schur harmonic concave) if*

$$(u_1 - u_2)\left(u_1^2\frac{\partial \zeta}{\partial u_1} - u_2^2\frac{\partial \zeta}{\partial u_2}\right) \geq 0 \quad (\leq 0). \tag{4}$$

3. Main Results

In this section, theorems related to the different kinds of Schur convexity and concavity of Heinz type mean are stated and proved.

Theorem 3.1. *For $a, b > 0$ and $a, b \in R$, v be a parameter, then H_v^* is Schur convex (concave) if $v \geq (\leq)\ \frac{1}{2}$.*

Proof. The Heinz type mean is given by

$$H_v^* = \frac{a^{1+v}b^{1-v} + a^{1-v}b^{1+v}}{a^v b^{1-v} + a^{1-v}b^v}.$$

Differentiating partially with respect to $'a'$, after applying logarithm on both sides to get

$$\frac{\partial H_v^*}{\partial a} = H_v^*\left(\frac{(1+v)a^v b^{1-v} + (1-v)a^{-v}b^{1+v}}{a^{1+v}b^{1-v} + a^{1-v}b^{1+v}} - \frac{(v)a^{v-1}b^{1-v} + (1-v)a^v b^{-v}}{a^v b^{1-v} + a^{1-v}b^v}\right)$$

Similarly,

$$\frac{\partial H_v^*}{\partial b} = H_v^*\left(\frac{(1-v)a^{1+v}b^{-v} + (1+v)a^{1-v}b^v}{a^{1+v}b^{1-v} + a^{1-v}b^{1+v}} - \frac{(v)a^{v-1}b^{1-v} + (1-v)a^{1-v}b^{v-1}}{a^v b^{1-v} + a^{1-v}b^v}\right)$$

Consider

$$(a-b)\left(\frac{\partial H_v^*}{\partial a} - \frac{\partial H_v^*}{\partial b}\right) = (a-b)H_v^*\ [\triangle] \tag{5}$$

where

$$\triangle = \left(\frac{(1+v)[a^v b^{1-v} - a^{1-v}b^v] + (1-v)[a^{-v}b^{1+v} - a^{1+v}b^{-v}]}{a^{1+v}b^{1-v} + a^{1-v}b^{1+v}}\right) - \left(\frac{(v)[a^{v-1}b^{1-v} - a^{1-v}b^{v-1}] + (1-v)[a^{-v}b^v - a^v b^{-v}]}{a^v b^{1-v} + a^{1-v}b^v}\right)$$

Simplification leads to

$$\triangle = \frac{(a^{2v}b^{2-2v} - b^{2v}a^{2-2v}) + (2v-1)(a^2-b^2)}{(a^v b^{1-v} + a^{1-v}b^v)(a^v b^{1-v} + a^{1-v}b^v)}.$$

Then, Eq. (5) becomes

$$(a-b)\left(\frac{\partial H_v^*}{\partial a} - \frac{\partial H_v^*}{\partial b}\right) = (a-b)H_v^*\ [\triangle] \geq (\leq)\ 0, \quad \text{if} \quad v\ \geq (\leq)\ \frac{1}{2}.$$

This proves that $H_v^*(a\ ,\ b)$ is Schur convex (concave) if $v \geq (\leq)\ \frac{1}{2}$. □

Theorem 3.2. *For $a, b > 0$ and $a, b \in R$, v be a parameter, then H_v^* is Schur geometric convex (concave) if $v \geq (\leq)\ \frac{1}{4}$.*

Proof. Consider

$$(6) \qquad (\ln a - \ln b)\left(a\frac{\partial H_v^*}{\partial a} - b\frac{\partial H_v^*}{\partial b}\right) = (\ln a - \ln b)H_v^*\ [\triangle_1]$$

where

$$\triangle_1 = (2v)\left(\frac{a^{1+v}b^{1-v} - a^{1-v}b^{1+v}}{a^{1+v}b^{1-v} + a^{1-v}b^{1+v}}\right) - (2v-1)\left(\frac{a^v b^{1-v} + a^{1-v}b^v}{a^v b^{1-v} + a^{1-v}b^v}\right).$$

Simplification leads to

$$\triangle_1 = \frac{(a^{2v+1}b^{2-2v} - b^{2v+1}a^{2-2v}) + (4v-1)(a-b)}{(a^v b^{1-v} + a^{1-v}b^v)(a^v b^{1-v} + a^{1-v}b^v)}$$

Then, Eq. (6) becomes

(7)

$$(\ln a - \ln b)\left(a\frac{\partial H_v^*}{\partial a} - b\frac{\partial H_v^*}{\partial b}\right) = (\ln a - \ln b)H_v^*\ [\triangle_1] \geq (\leq)\ 0, \quad \text{if} \quad v \geq (\leq)\ \frac{1}{4}.$$

This proves that $H_v^*(a, b)$ is Schur geometric convex (concave) if $v \geq (\leq)\ \frac{1}{4}$. □

Theorem 3.3. *For $a, b > 0$ and $a, b \in R$, v be a parameter, then H_v^* is Schur harmonic convex (concave) if $v \geq (\leq)\ 0$*

Proof. Consider

$$(8) \qquad (a-b)\left(a^2\frac{\partial H_v^*}{\partial a} - b^2\frac{\partial H_v^*}{\partial b}\right) = (a-b)H_v^*\ [\triangle_2]$$

where

$$\triangle_2 = \left(\frac{(1+v)[a^{2+v}b^{1-v} - a^{1-v}b^{2+v}] + (1-v)[a^{2-v}b^{1+v} - a^{1+v}b^{2-v}]}{a^{1+v}b^{1-v} + a^{1-v}b^{1+v}}\right) - \left(\frac{(v)[a^{v+1}b^{1-v} - a^{1-v}b^{v+1}] + (1-v)[a^{2-v}b^v - a^v b^{2-v}]}{a^v b^{1-v} + a^{1-v}b^v}\right).$$

Simplification leads to

$$\triangle_2 = \frac{(a^{2+2v}b^{2-2v} - b^{2+2v}a^{2-2v}) + (2v)ab(a^2-b^2)}{(a^v b^{1-v} + a^{1-v}b^v)(a^v b^{1-v} + a^{1-v}b^v)}$$

Then, Eq. (8) becomes

$$(a-b)\left(a^2\frac{\partial H_v^*}{\partial a} - b^2\frac{\partial H_v^*}{\partial b}\right) = (a-b)H_v^*\ [\triangle_2] \geq (\leq)\ 0, \quad \text{if} \quad v \geq (\leq)\ 0.$$

This proves that $H_v^*(a,b)$ is Schur harmonic convex (concave) if $v \geq (\leq) \ 0$. □

4. Conclusion

Different kinds of Schur convexity and concavity conditions have become necessary conditions for any mean or symmetric function to be considered for other applications. A new mean similar to standard Heinz mean is defined and is called Heinz type mean. Different kinds of Schur convexity and concavity conditions are obtained for this particular Heinz type mean. Knowing these conditions, further applications can be considered for future study.

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Department of Mathematics, R.V. College of Engineering, Bangalore-560 059, India

Department of Mathematics, JSS Academy of Technical Education, Bangalore-560 060, India

Department of Mathematics, Sri Jayachamarajendra College of Engineering, JSS Science and Technology University, Mysuru-570 006, India
Email address: pskreddy@jssstuniv.in