Proceedings of the Jangjeon Mathematical Society
23 (2020), No. 4, pp. 459 - 464

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2020.23.4.459

Level operators over index matrices. Part 1: Index matrices with real non-negative elements

Krassimir Atanassov

Department of Bioinformatics and Mathematical Modelling
Institute of Biophysics and Biomedical Engineering
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 105, Sofia–1113, Bulgaria
and
Intelligent Systems Laboratory, "Prof. Asen Zlatarov" University
Burgas–8010, Bulgaria
e-mail: *krat@bas.bg*

ABSTRACT: For a given index matrix with real non-negative elements, level operators are introduced, that modify the elements of the matrix. The basic properties of these operators are studied.
KEYWORDS: Index matrix, Level operation

AMS Classification: $11C20$

1 Introduction

The concept of an Index Matrix (IM) was introduced in [1] as an extension of the well-known concept of a matrix (see, e.g., [4, 5, 6]). The forms, modificatiuons and properties of the IMs are discussed in a series of papers and the book [3]. In the first (standard) form, the IM is defined as follows (see [1, 2, 3]).

Let $\mathcal{I}$ be a fixed set of indices and $\mathcal{R}^+ = \{x | x \geq 0\}$ - the set of the real non-negative numbers.

Let operation $\circ : \mathcal{R}^+ \times \mathcal{R}^+ \to \mathcal{R}^+$ be fixed.

An IM with index sets K and L ($K, L \subset \mathcal{I}$) and elements from $\mathcal{R}^+$ is called the object (see, [1, 2, 3]):

$$[K, L, \{a_{k_i,l_j}\}] \equiv \begin{array}{c|ccccc} & l_1 & \dots & l_j & \dots & l_n \\ \hline k_1 & a_{k_1,l_1} & \dots & a_{k_1,l_j} & \dots & a_{k_1,l_n} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ k_i & a_{k_i,l_1} & \dots & a_{k_i,l_j} & \dots & a_{k_i,l_n} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ k_m & a_{k_m,l_1} & \dots & a_{k_m,l_j} & \dots & a_{k_m,l_n} \end{array},$$

where $K = \{k_1, k_2, ..., k_m\}$, $L = \{l_1, l_2, ..., l_n\}$, and for $1 \leq i \leq m$, and $1 \leq j \leq n$: $a_{k_i,l_j} \in \mathcal{R}^+$.

In [3], these IM are mentioned as $\mathcal{R}$-IMs, but here we will use only the brief form "IM".

For the IMs

$$A = [K, L, \{a_{k_i,l_j}\}],$$
$$B = [P, Q, \{b_{p_r,q_s}\}],$$

operations that are analogous to the usual matrix operations of addition and termwise multiplication are defined, as well as other, specific ones:

Addition

$$A \otimes_{(\circ)} B = [K \cup P, L \cup Q, \{c_{t_u,v_w}\}],$$

where

$$c_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ b_{p_r,q_s}, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ a_{k_i,l_j} \circ b_{p_r,q_s}, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q \\ 0, & \text{otherwise} \end{cases}.$$

Of course, here and below, if "$\circ$" is substituted by "+", then

$$a_{k_i,l_j} \circ b_{p_r,q_s} = a_{k_i,l_j} + b_{p_r,q_s},$$

while, if "$\circ$" is "max" or "min", then

$$a_{k_i,l_j} \circ b_{p_r,q_s} = \max(a_{k_i,l_j}, b_{p_r,q_s})$$

or

$$a_{k_i,l_j} \circ b_{p_r,q_s} = \min(a_{k_i,l_j}, b_{p_r,q_s}),$$

respectively.

Termwise multiplication

$$A \otimes_{(\circ)} B = [K \cap P, L \cap Q, \{c_{t_u,v_w}\}],$$

where

$$c_{t_u,v_w} = a_{k_i,l_j} \circ b_{p_r,q_s},$$

for $t_u = k_i = p_r \in K \cap P$ and $v_w = l_j = q_s \in L \cap Q$.

A lot of relations are defined over two IMs. Here, we use only three of them:

The strict relation "inclusion about value" is

$$A \subset_v B \text{ iff } (K = P) \& (L = Q) \& (\forall k \in K)(\forall l \in L)(a_{k,l} < b_{k,l}).$$

The non-strict relation "inclusion about value" is

$$A \subseteq_v B \text{ iff } (K = P) \& (L = Q) \& (\forall k \in K)(\forall l \in L)(a_{k,l} \leq b_{k,l}),$$

Relation "equality about value" is

$$A =_v B \text{ iff } (K = P) \& (L = Q) \& (\forall k \in K)(\forall l \in L)(a_{k,l} = b_{k,l}).$$

2 Definition and properties of the new operators

Let for $K, L \subset \mathcal{I}$:

$$O^*_{K,L} = [K, L, \{0\}],$$

denote the zero IM with index sets K and L.

Let us have the IM $A = [K, L, \{a_{k_i,l_j}\}]$, where $a_{k_i,l_j} \geq 0$ is a real number. Then we define

$$\mathcal{L}^{>}_{\alpha}(A) = [K, L, \{b_{k_i,l_j}\}],$$

where

$$b_{k_i,l_j} = \begin{cases} a_{k_i,l_j}, & \text{if } a_{k_i,l_j} > \alpha \\ 0, & \text{otherwise} \end{cases};$$

$$\mathcal{L}^{\geq}_{\alpha}(A) = [K, L, \{b_{k_i,l_j}\}],$$

where

$$b_{k_i,l_j} = \begin{cases} a_{k_i,l_j}, & \text{if } a_{k_i,l_j} \geq \alpha \\ 0, & \text{otherwise} \end{cases}.$$

Therefore,

$$\mathcal{L}^{>}_{\alpha}(O^*_{K,L}) = O^*_{K,L},$$

$$\mathcal{L}^{\geq}_{\alpha}(O^*_{K,L}) = O^*_{K,L}.$$

Theorem 1. For each IM A and for every two real numbers $\alpha, \beta \geq 0$:

$$\mathcal{L}^{>}_{\alpha}(\mathcal{L}^{>}_{\beta}(A)) = \mathcal{L}^{>}_{\max(\alpha,\beta)}(A),$$

$$\mathcal{L}^{\geq}_{\alpha}(\mathcal{L}^{\geq}_{\beta}(A)) = \mathcal{L}^{\geq}_{\max(\alpha,\beta)}(A).$$

Proof. Let the above IM A be given. Then

$$B = \mathcal{L}^{>}_{\alpha}(\mathcal{L}^{>}_{\beta}(A)) = \mathcal{L}^{>}_{\alpha}([K, L, \{b_{k_i,l_j}\}]) = [K, L, \{c_{k_i,l_j}\}],$$

where

$$b_{k_i,l_j} = \begin{cases} a_{k_i,l_j}, & \text{if } a_{k_i,l_j} > \beta \\ 0, & \text{otherwise} \end{cases}$$

and

$$c_{k_i,l_j} = \begin{cases} b_{k_i,l_j}, & \text{if } b_{k_i,l_j} > \alpha \\ 0, & \text{otherwise} \end{cases}.$$

If $a_{k_i,l_j} > \max(\alpha, \beta)$, then

$$c_{k_i,l_j} = b_{k_i,l_j} = a_{k_i,l_j};$$

if $\alpha \geq a_{k_i,l_j} > \beta$, then $b_{k_i,l_j} = a_{k_i,l_j}$, but $c_{k_i,l_j} = 0$; if $\beta \geq a_{k_i,l_j} > \alpha$, then $b_{k_i,l_j} = 0$ and hence, $c_{k_i,l_j} = 0$; if $\min(\alpha, \beta) \geq a_{k_i,l_j}$, then $b_{k_i,l_j} = 0$ and hence, $c_{k_i,l_j} = 0$.

Therefore,

$$c_{k_i,l_j} = \begin{cases} a_{k_i,l_j}, & \text{if } a_{k_i,l_j} > \max(\alpha,\beta) \\ 0, & \text{otherwise} \end{cases},$$

i.e.,

$$B = \mathcal{L}^{>}_{\max(\alpha,\beta)}(A).$$

The second equality is proved in the same manner. □

Theorem 2. Let the two IMs A and B be given and let $\alpha \in \mathcal{R}^+$ be an arbitrary number. Then

(a) $\mathcal{L}^{>}_{\alpha}(A) \oplus_{(+)} \mathcal{L}^{>}_{\alpha}(B) \subseteq_v \mathcal{L}^{>}_{\alpha}(A \oplus_{(+)} B)$,

(b) $\mathcal{L}^{>}_{\alpha}(A) \oplus_{(.)} \mathcal{L}^{>}_{\alpha}(B) \subseteq_v \mathcal{L}^{>}_{\alpha}(A \oplus_{(.)} B)$ for IMs with positive real elements that are greater or equal to 1,

(c) $\mathcal{L}^{>}_{\alpha}(A) \oplus_{(.)} \mathcal{L}^{>}_{\alpha}(B) =_v \mathcal{L}^{>}_{\alpha}(A \oplus_{(.)} B)$ for IMs with elements from interval $[0, 1]$,

(d) $\mathcal{L}^{>}_{\alpha}(A) \oplus_{(\max)} \mathcal{L}^{>}_{\alpha}(B) =_v \mathcal{L}^{>}_{\alpha}(A \oplus_{(\max)} B)$,

(e) $\mathcal{L}^{>}_{\alpha}(A) \oplus_{(\min)} \mathcal{L}^{>}_{\alpha}(B) =_v \mathcal{L}^{>}_{\alpha}(A \oplus_{(\min)} B)$,

(f) $\mathcal{L}^{\geq}_{\alpha}(A) \oplus_{(+)} \mathcal{L}^{\geq}_{\alpha}(B) \subseteq_v \mathcal{L}^{\geq}_{\alpha}(A \oplus_{(+)} B)$,

(g) $\mathcal{L}^{\geq}_{\alpha}(A) \oplus_{(.)} \mathcal{L}^{\geq}_{\alpha}(B) \subseteq_v \mathcal{L}^{\geq}_{\alpha}(A \oplus_{(.)} B)$ for IMs with positive real elements that are greater or equal to 1,

(h) $\mathcal{L}^{\geq}_{\alpha}(A) \oplus_{(.)} \mathcal{L}^{\geq}_{\alpha}(B) =_v \mathcal{L}^{\geq}_{\alpha}(A \oplus_{(.)} B)$ for IMs with elements from interval $[0, 1]$,

(i) $\mathcal{L}^{\geq}_{\alpha}(A) \oplus_{(\max)} \mathcal{L}^{\geq}_{\alpha}(B) =_v \mathcal{L}^{\geq}_{\alpha}(A \oplus_{(\max)} B)$,

(j) $\mathcal{L}^{\geq}_{\alpha}(A) \oplus_{(\min)} \mathcal{L}^{\geq}_{\alpha}(B) =_v \mathcal{L}^{\geq}_{\alpha}(A \oplus_{(\min)} B)$.

Proof. (a) From definition of operation $\oplus_{(+)}$ we have:

$$\begin{aligned} &\mathcal{L}^{>}_{\alpha}(A \oplus_{(+)} B) \\ &= \mathcal{L}^{>}_{\alpha}[K \cup P, L \cup Q, \{c_{t_u,v_w}\}], \\ &= [K \cup P, L \cup Q, \{d_{t_u,v_w}\}], \end{aligned}$$

where

$$d_{t_u,v_w} = \begin{cases} c_{t_u,v_w}, & \text{if } c_{t_u,v_w} > \alpha \\ 0, & \text{otherwise} \end{cases}.$$

Sequentially, we will study the different cases for d_{t_u,v_w}.

1. If $c_{t_u,v_w} = a_{k_i,l_j}$, then

$$d_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, & \text{if } a_{k_i,l_j} > \alpha \\ 0, & \text{otherwise} \end{cases}$$

and therefore, the elements with indices $\langle k_i, l_j \rangle$ will coincide in the two IMs $\mathcal{L}^{>}_{\alpha}(A \oplus_{(+)} B)$ and $\mathcal{L}^{>}_{\alpha}(A) \oplus_{(+)} \mathcal{L}^{>}_{\alpha}(B)$.

2. If $c_{t_u,v_w} = b_{p_r,q_s}$, then

$$d_{t_u,v_w} = \begin{cases} b_{p_r,q_s}, & \text{if } b_{p_r,q_s} > \alpha \\ 0, & \text{otherwise} \end{cases}$$

and therefore, the elements with indices $\langle k_i, l_j \rangle$ will coincide in the two IMs $\mathcal{L}^{>}_{\alpha}(A \oplus_{(+)} B)$ and $\mathcal{L}^{>}_{\alpha}(A) \oplus_{(+)} \mathcal{L}^{>}_{\alpha}(B)$.

3. If $c_{t_u,v_w} = a_{k_i,l_j} + b_{p_r,q_s}$, then

$$d_{t_u,v_w} = \begin{cases} a_{k_i,l_j} + b_{p_r,q_s}, & \text{if } a_{k_i,l_j} + b_{p_r,q_s} > \alpha \\ 0, & \text{otherwise.} \end{cases}$$

By condition, the elements of A and B are positive. Now, there are the following four subcases:

3.1. if $a_{k_i,l_j} > \alpha$ and $b_{p_r,q_s} > \alpha$, then these elements will keep their values in $\mathcal{L}^{>}_{\alpha}(A)$ and $\mathcal{L}^{>}_{\alpha}(B)$, respectively and their sum will coincide with the value $a_{k_i,l_j} + b_{p_r,q_s}$ in IM $\mathcal{L}^{>}_{\alpha}(A \oplus_{(+)} B)$.

3.2 if $a_{k_i,l_j} > \alpha$ and $b_{p_r,q_s} \leq \alpha$, then only the a-element will keep its value in $\mathcal{L}^{>}_{\alpha}(A)$, while the b-element will obtain value 0 in $\mathcal{L}^{>}_{\alpha}(B)$. Therefore, the value of $d_{t_u,v_w} = a_{k_i,l_j} + b_{p_r,q_s}$ in IM $\mathcal{L}^{>}_{\alpha}(A \oplus_{(+)} B)$ will be greater than its corresponding value in IM $\mathcal{L}^{>}_{\alpha}(A) \oplus_{(+)} \mathcal{L}^{>}_{\alpha}(B)$.

3.3 if $a_{k_i,l_j} \leq \alpha$ and $b_{p_r,q_s} > \alpha$, then only the b-element will keep its value in $\mathcal{L}^{>}_{\alpha}(B)$, while the a-element will obtain value 0 in $\mathcal{L}^{>}_{\alpha}(A)$. Therefore, the value of $d_{t_u,v_w} = a_{k_i,l_j} + b_{p_r,q_s}$ in IM $\mathcal{L}^{>}_{\alpha}(A \oplus_{(+)} B)$ will be greater than its corresponding value in IM $\mathcal{L}^{>}_{\alpha}(A) \oplus_{(+)} \mathcal{L}^{>}_{\alpha}(B)$.

3.4 if $a_{k_i,l_j} \leq \alpha$ and $b_{p_r,q_s} \leq \alpha$, then the a- and b-elements will obtain value 0 in IMs $\mathcal{L}^{>}_{\alpha}(A)$ and $\mathcal{L}^{>}_{\alpha}(B)$. Therefore, the value of $d_{t_u,v_w} = a_{k_i,l_j} + b_{p_r,q_s}$ in IM $\mathcal{L}^{>}_{\alpha}(A \oplus_{(+)} B)$ will be equal to 0 and to its corresponding value in IM $\mathcal{L}^{>}_{\alpha}(A) \oplus_{(+)} \mathcal{L}^{>}_{\alpha}(B)$.

4. If $c_{t_u,v_w} = 0$, then $d_{t_u,v_w} = 0$ and therefore, the elements with indices $\langle k_i, l_j \rangle$ will coincide in the two IMs $\mathcal{L}^{>}_{\alpha}(A \oplus_{(+)} B)$ and $\mathcal{L}^{>}_{\alpha}(A) \oplus_{(+)} \mathcal{L}^{>}_{\alpha}(B)$.

Hence, relation $\subseteq_v$ between the two IMs exists and when there is at least one pair of a- and b-elements for which Case 3.2 or Case 3.3 is valid, then the relation will be strict $\subset_v$. If some of a- or b-elements are not positive, then the relation $\subseteq_v$ between the two IMs $\mathcal{L}^{>}_{\alpha}(A \oplus_{(+)} B)$ and $\mathcal{L}^{>}_{\alpha}(A) \oplus_{(+)} \mathcal{L}^{>}_{\alpha}(B)$ can be opposite.

(b) – (j) are proved in the same manner. □

The proofs of the next assertions are analogously.

Theorem 3. Let the two IMs A and B be given and let $\alpha \in \mathcal{R}^+$ be an arbitrary number. Then

(a) $\mathcal{L}^{>}_{\alpha}(A) \otimes_{(+)} \mathcal{L}^{>}_{\alpha}(B) \subseteq_v \mathcal{L}^{>}_{\alpha}(A \otimes_{(+)} B)$,

(b) $\mathcal{L}^{>}_{\alpha}(A) \otimes_{(.)} \mathcal{L}^{>}_{\alpha}(B) \subseteq_v \mathcal{L}^{>}_{\alpha}(A \otimes_{(.)} B)$ for IMs with positive real elements that are greater or equal to 1,

(c) $\mathcal{L}^{>}_{\alpha}(A) \otimes_{(.)} \mathcal{L}^{>}_{\alpha}(B) =_v \mathcal{L}^{>}_{\alpha}(A \otimes_{(.)} B)$ for IMs with elements from interval $[0, 1]$,

(d) $\mathcal{L}^{>}_{\alpha}(A) \otimes_{(\max)} \mathcal{L}^{>}_{\alpha}(B) =_v \mathcal{L}^{>}_{\alpha}(A \otimes_{(\max)} B)$,

(e) $\mathcal{L}^{>}_{\alpha}(A) \otimes_{(\min)} \mathcal{L}^{>}_{\alpha}(B) =_v \mathcal{L}^{>}_{\alpha}(A \otimes_{(\min)} B)$,

(f) $\mathcal{L}^{\geq}_{\alpha}(A) \otimes_{(+)} \mathcal{L}^{\geq}_{\alpha}(B) \subseteq_v \mathcal{L}^{\geq}_{\alpha}(A \otimes_{(+)} B)$,

(g) $\mathcal{L}^{\geq}_{\alpha}(A) \otimes_{(.)} \mathcal{L}^{\geq}_{\alpha}(B) \subseteq_v \mathcal{L}^{\geq}_{\alpha}(A \otimes_{(.)} B)$ for IMs with positive real elements that are greater or equal to 1,

(h) $\mathcal{L}^{\geq}_{\alpha}(A) \otimes_{(.)} \mathcal{L}^{\geq}_{\alpha}(B) =_v \mathcal{L}^{\geq}_{\alpha}(A \otimes_{(.)} B)$ for IMs with elements from interval $[0, 1]$,

(i) $\mathcal{L}^{\geq}_{\alpha}(A) \otimes_{(\max)} \mathcal{L}^{\geq}_{\alpha}(B) =_v \mathcal{L}^{\geq}_{\alpha}(A \otimes_{(\max)} B)$,

(j) $\mathcal{L}^{\geq}_{\alpha}(A) \otimes_{(\min)} \mathcal{L}^{\geq}_{\alpha}(B) =_v \mathcal{L}^{\geq}_{\alpha}(A \otimes_{(\min)} B)$.

3 Conclusion

In a next research new properties of the two operators defined over IMs will be studied. These and other operators will be defined over the IM-extensions like extended IMs, intuitionistic fuzzy IMs, extended intuitionistic fuzzy IMs, 3- and n-dimensional IMs and others.

Acknowledgements

This research was funded by Bulgarian National Science Fund, grant number KP-06-N22/1/2018 “Theoretical research and applications of InterCriteria Analysis”.

References

[1] Atanassov K., Generalized index matrices, Comptes rendus de l’Academie Bulgare des Sciences, vol.40, 1987, No.11, 15-18.

[2] Atanassov, K. On index matrices, Part 1: Standard cases. Advanced Studies in Contemporary Mathematics, Vol. 20, 2010, No. 2, 291-302.

[3] Atanassov, K., Index Matrices: Towards an Augmented matrix Calculus, Springer, Cham, 2014.

[4] Brown, W. C., Matrices and Vector Spaces. Marcel Dekker, New York, 1991.

[5] Lankaster, P. Theory of Matrices. Academic Press, New York, 1969.

[6] https://en.wikipedia.org/wiki/Matrix_(mathematics), 6 August 2020.