Proceedings of the Jangjeon Mathematical Society
23 (2020), No. 3, pp. 433 – 443
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2020.23.3.433

Dedicated to Prof. Chandrashekar Adiga on his 62nd Birthday

ON ANDREWS' PARTITION FUNCTION $\overline{\mathcal{EO}}(n)$

FATHIMA S. N. AND VEENA V. S.

Abstract. Recently, Andrews introduced partition functions $\mathcal{EO}(n)$ and $\overline{\mathcal{EO}}(n)$ where the function $\mathcal{EO}(n)$ denotes the number of partitions of n in which every even part is less than each odd part and the function $\overline{\mathcal{EO}}(n)$ denotes the number of partitions enumerated by $\mathcal{EO}(n)$ in which only the largest even part appears an odd number of times. In this paper, we prove new congruences for $\overline{\mathcal{EO}}(n)$ and $p_D(n)$, the number of partitions into distinct (or, odd) parts. We further establish linear recurrence relations for $\overline{p}_D(n)$, which counts the number of partitions of n into distinct parts with 2 types of each part and $\overline{\mathcal{EO}}(n)$.

2000 Mathematics Subject Classification. 05A17, 11P83.

Keywords and phrases. Partitions, Generating functions, Congruences.

1. Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum equals n. The number of partitions of n is denoted by $p(n)$. The generating function for $p(n)$, is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} = \frac{1}{f_1}, \tag{1.1}$$

where as customary, we define

$$f_k := (q^k;q^k)_\infty = \prod_{m=1}^{\infty}(1-q^{mk}).$$

In [3], Andrews gave a detailed study of the partition function $\mathcal{EO}(n)$, where the function $\mathcal{EO}(n)$ counts the number of partitions of n in which every even part is less than each odd part. The generating function of $\mathcal{EO}(n)$, is given by

$$\sum_{n=0}^{\infty} \mathcal{EO}(n)q^n = \frac{1}{(1-q)(q^2;q^2)_\infty}. \tag{1.2}$$

For example, $\mathcal{EO}(10) = 19$ with the relevant partitions being $10, 9+1, 8+2, 7+3, 7+1+1+1, 6+4, 6+2+2, 5+5, 5+3+1+1, 5+2+2, 5+1+1+1+1+1, 4+4+2, 4+2+2+2, 3+3+3+1, 3+3+2+2, 3+3+1+1+1+1, 3+1+1+1+1+1+1+1, 2+2+2+2+2, 1+1+1+1+1+1+1+1+1+1$.
Andrews [3], also defined the partition function $\overline{\mathcal{EO}}(n)$ that counts the number of partitions enumerated by $\mathcal{EO}(n)$ in which only the largest even part appears an odd number of times. The generating function for $\overline{\mathcal{EO}}(n)$, is given by

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(n)q^n = \frac{f_4^3}{f_2^2}. \tag{1.3}$$

The first author's research is supported by Pondicherry University Fellowship,
Department of Mathematics, Pondicherry University, Puducherry-605014.

For example, $\overline{\mathcal{EO}}(10)$=6 with the relevant partitions being $10, 6+2+2, 5+5, 3+3+1+1+1+1, 2+2+2+2+2, 1+1+1+1+1+1+1+1+1+1$.
In [3], Andrews proved the congruence

$$\overline{\mathcal{EO}}(10n+8) \equiv 0 \mod (5). \tag{1.4}$$

In this paper, we give an alternate proof for this congruence by finding the exact generating formula for $\overline{\mathcal{EO}}(10n+8)$.
Let $p_D(n)$ be the partition function that counts the number of partitions of n into distinct parts, then

$$\sum_{n=0}^{\infty} p_D(n)q^n = (-q;q)_\infty = \frac{f_2}{f_1}. \tag{1.5}$$

For example, $p_D(3) = 2$, the two partitions of 3 are 3, 2+1. Using Ramanujan's theta function Baruah et.al., [5], have obtained exact generating function for $p_D(5n+1)$, $p_D(25n+1)$ and $p_D(125n+26)$. For more works on $p_D(n)$, see [1, 11, 12, 15].
Now, let $\overline{p}_D(n)$ be the number of partitions of n into distinct parts with 2 types of each part, then

$$\sum_{n=0}^{\infty} \overline{p}_D(n)q^n = \frac{f_2^2}{f_1^2}. \tag{1.6}$$

Clearly, $\overline{p}_D(3) = 6$, where the six partitions of 3 are $3, \overline{3}, 2+1, \overline{2}+1, 2+\overline{1}, \overline{2}+\overline{1}$.
The aim of this paper is to prove new congruences for $\overline{\mathcal{EO}}(n)$ and $p_D(n)$. The following are our main results:

Theorem 1.1. *For any integer $n \geqslant 0$, we have*

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(10n+2)q^n = 2\frac{f_2^5 f(q^2,q^3)}{f_1^5}, \tag{1.7}$$

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(10n+4)q^n = 2\frac{f_2^5 f(q,q^4)}{f_1^5}, \tag{1.8}$$

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(10n+8)q^n = 5\frac{f_2^2 f_5^2 f_{10}}{f_1^4}, \tag{1.9}$$

where $f(a,b)$ is defined in (2.1).

Theorem 1.2. *For any integer $n \geqslant 0$, $\alpha \geqslant 1$ and prime $p \geqslant 5$, we have*

$$\overline{\mathcal{EO}}\left(10p^{2\alpha}n + \frac{5p^{2\alpha-1}(6j+5p)-1}{3}\right) \equiv 0 \pmod{10}, \tag{1.10}$$

where $j = 1, 2, ..., p-1$.

In section 3, we prove Andrews congruence [3, Eqn. 1.6] and few new congruences for $\overline{\mathcal{EO}}(n)$ modulo 2, 10 and 20 are deduced from Theorem 1.1. We also prove the following new congruences for $p_D(n)$.

Theorem 1.3. *For any integer $n \geqslant 0$, we have*

$$p_D\left(5^2n+r\right) \equiv 0 \pmod 4, \tag{1.11}$$

where r = 6, 11, 16 and 21.

In section 4, motivated by M Merca's paper [14], new recurrence relations for $\overline{p}_D(n)$ and $\overline{\mathcal{EO}}(n)$ are obtained. We conclude this paper by results relating to $\overline{p_D}(n)$ and $\overline{\mathcal{EO}}(n)$ with variants of the partition function.

2. Preliminaries

To prove the main results of this paper, we collect some definitions and lemmas in this section.
Ramanujan's general theta function $f(a,b)$ is defined by [6, Eqn. 18.1]

$$f(a,b)=\sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \quad |ab|<1. \tag{2.1}$$

By [6, p.34, Entry 18] we have

$$f(1,a)=2f(a,a^3). \tag{2.2}$$

Three special cases of $f(a,b)$ are [6, p.36, Entry 22]

$$\varphi(q):=f(q,q)=\sum_{n=-\infty}^{\infty} q^{n^2}=\frac{f_2^5}{f_1^2f_4^2}, \tag{2.3}$$

$$\psi(q):=f(q,q^3)=\sum_{n=0}^{\infty} q^{n(n+1)/2}=\frac{f_2^2}{f_1}, \tag{2.4}$$

and

$$f(-q):=f(-q,-q^2)=\sum_{n=-\infty}^{\infty}(-1)^n q^{(3n^2+n)/2}=(q;q)_\infty=f_1, \tag{2.5}$$

where the product representations arise from Jacobi triple product identity [p.36, Entry 19]berndt2012ramanujan,

$$f(a,b)=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.6}$$

The last equality of identity (2.5) is Euler's famous pentagonal number theorem [2, Cor. 1.7].

Lemma 2.1. *[6, p.262, Entry 10] The following identity holds.*

$$\psi^2(q)-q\psi^2(q^5)=f(q,q^4)f(q^2,q^3). \tag{2.7}$$

Lemma 2.2. *[9, p.11, Eqn.7.1] We have Jacobi's identity,*

$$f_1^3=\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{(n^2+n)/2}. \tag{2.8}$$

Lemma 2.3. *[9, p.104, Eqn. 10.7.3] We have Ramanujan's identity,*

$$\frac{f_1^5}{f_2^2}=\sum_{n=-\infty}^{\infty}(6n+1)q^{\frac{3n^2+n}{2}}. \tag{2.9}$$

Lemma 2.4. *[8, Theorem 2.2] If $p \geqslant 5$ is a prime and*

$$\frac{\pm p-1}{6} := \begin{cases} \frac{p-1}{6}, & p \equiv 1 \pmod 6, \\ \frac{-p-1}{6}, & p \equiv -1 \pmod 6. \end{cases}$$

then

$$\begin{aligned} f(-q) =&(-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^{p^2}) \\ &+ \sum_{\substack{k=-\frac{p-1}{2} \\ k\neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}). \end{aligned} \tag{2.10}$$

Furthermore, if $-\frac{(p-1)}{2} \leqslant k \leqslant \frac{(p-1)}{2}$, $k \neq \frac{\pm p-1}{6}$, *then* $\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod p$.

Lemma 2.5. *The following 2-dissections holds.*
[6, p.40, Entry 25](Consequence of 2-dissection of $\varphi(q)$),

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{2.11}$$

[10, Theorem 2.1],

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}, \tag{2.12}$$

[13, Lemma 2.3],

$$f_1^3 f_5 = 2q^2 \frac{f_4^6 f_{40}^2 f_{10}}{f_2 f_8^2 f_{20}^2} + \frac{f_2^2 f_4 f_{10}^2}{f_{20}} + 2q f_4^3 f_{20} - 5q f_2 f_{10}^3. \tag{2.13}$$

Lemma 2.6. *From the binomial theorem, for any positive integer k,*

$$f_1^{2^k} \equiv f_2^{2^{k-1}} \pmod{2^k}. \tag{2.14}$$

Lemma 2.7. *[6, p.49, Entry 31] The following 5-dissection holds.*

$$\psi(q) = f(q^{10}; q^{15}) + qf(q^5; q^{20}) + q^3\psi(q^{25}). \tag{2.15}$$

Lemma 2.8. *The following 5-dissection holds.*

$$\frac{f_2}{f_1} = \frac{f_{10}}{f_5^3}(a^2 + qab + q^2b^2 + 2q^3ac + 2q^4bc), \tag{2.16}$$

where a=$f(q^{10}, q^{15})$, b=$f(q^5, q^{20})$ and c=$\psi(q^{25})$.

Proof. We have

$$\frac{f_2}{f_1} = \frac{f_{10}}{f_5^3}(-q, -q^4, q^5; q^5)_\infty(-q^2, -q^3, q^5; q^5)_\infty. \tag{2.17}$$

Using (2.1) and (2.6) in (2.17), we have

$$\frac{f_2}{f_1} = \frac{f_{10}}{f_5^3} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} q^{\frac{5k^2-3k+5l^2-l}{2}}. \tag{2.18}$$

Now split $\displaystyle\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} q^{\frac{5k^2-3k+5l^2-l}{2}}$ according to the residue of $k+2l$ modulo 5.
If $k+2l \equiv 0 \pmod 5$, then $k+2l = 5m$, define $t = l-2m$, so that $k = m-2t$, $l = 2m+t$, and the contribution of these terms to the sum is

$$\sum_{m=-\infty}^{\infty}\sum_{t=-\infty}^{\infty} q^{\frac{25m^2-5m+25t^2+5t}{2}} = f(q^{10};q^{15})^2. \tag{2.19}$$

If $k+2l \equiv 1 \pmod 5$, then $k+2l = 5m+1$, define $t = l-2m$, so that $k = m+1-2t$, $l = 2m+t$, and the contribution of these terms to the sum is

$$q\sum_{m=-\infty}^{\infty}\sum_{t=-\infty}^{\infty} q^{\frac{25m^2+5m+25t^2-15t}{2}} = qf(q^{10};q^{15})f(q^5;q^{20}). \tag{2.20}$$

If $k+2l \equiv 2 \pmod 5$, then $k+2l = 5m+2$, define $t = l-2m-1$, so that $k = m-2t$, $l = 2m+1+t$, and the contribution of these terms to the sum is

$$q^2\sum_{m=-\infty}^{\infty}\sum_{t=-\infty}^{\infty} q^{\frac{25m^2+15m+25t^2+15t}{2}} = q^2f(q^5;q^{20})^2. \tag{2.21}$$

If $k+2l \equiv 3 \pmod 5$, then $k+2l = 5m+3$, define $t = l-2m-1$, so that $k = m-2t+1$, $l = 2m+1+t$, and the contribution of these terms to the sum is

$$q^3\sum_{m=-\infty}^{\infty}\sum_{t=-\infty}^{\infty} q^{\frac{25m^2+25m+25t^2-5t}{2}} = 2q^3\psi(q^{25})f(q^{10};q^{15}). \tag{2.22}$$

If $k+2l \equiv 4 \pmod 5$, then $k+2l = 5m-1$, define $t = l-2m+1$, so that $k = m-2t+1$, $l = 2m-1+t$, and the contribution of these terms to the sum is

$$q^4\sum_{m=-\infty}^{\infty}\sum_{t=-\infty}^{\infty} q^{\frac{25m^2-15m+25t^2-25t}{2}} = 2q^4\psi(q^{25})f(q^5;q^{20}). \tag{2.23}$$

Using (2.19)-(2.23) in (2.18), we complete the proof of Lemma 2.8. □

3. Proof of Theorem 1.1-1.3

Proof of Theorem 1.1. Employing (2.4), we can write (1.3) as

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(2n)q^n = \frac{f_2^3}{f_1^2} = \psi(q)\frac{f_2}{f_1}. \tag{3.1}$$

By multiplying (2.15) and (2.16), we deduce

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(2n)q^n =& \frac{f_{10}}{f_5^3}\big[a^3+3q^5b^2c+2qa^2b+2q^6ac^2+2q^2ab^2+2q^7bc^2\\ &+3q^3a^2c+q^3b^3+5q^4abc\big].\end{aligned} \tag{3.2}$$

Extracting the terms involving q^{5n+1} from both sides of the identity (3.2) and then replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(10n+2)q^n = 2\frac{f_2}{f_1^3}f(q^2,q^3)\Big[f(q^2,q^3)f(q,q^4)+q\psi^2(q^5)\Big]. \tag{3.3}$$

Invoking Lemma 2.1 in (3.3), we obtain

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(10n+2)q^n = 2\frac{f_2}{f_1^3}\psi^2(q)f(q^2,q^3). \tag{3.4}$$

Substituting (2.4) in (3.4), we complete the proof of (1.7). Similarly, extracting the terms involving q^{5n+2} and q^{5n+4} from both sides of the identity (3.2), we obtain (1.8) and (1.9) respectively.

Corollary 3.1. For $n \geqslant 0$, we have

$$\overline{\mathcal{EO}}\,(10n+r) \equiv 0 \pmod{2}, \quad \text{where r = 2, 4.} \tag{3.5}$$

$$\overline{\mathcal{EO}}\,(10n+8) \equiv 0 \pmod{5}, \tag{3.6}$$

$$\overline{\mathcal{EO}}\,(200n+r) \equiv 0 \pmod{10}, \quad \text{where r = 18, 28, ..., 198.} \tag{3.7}$$

$$\overline{\mathcal{EO}}\,(50n+r) \equiv 0 \pmod{20}, \quad \text{where r = 18, 28, 38, 48.} \tag{3.8}$$

Proof. Congruences (3.5) and (3.6) are direct consequences of (1.7), (1.8) and (1.9) respectively.
Employing binomial theorem (2.14) in (1.9), we deduce that

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(10n+8)q^n \equiv 5f_{20} \pmod{10}. \tag{3.9}$$

By comparing the coefficients of q^{20n+1}, q^{20n+2},..., q^{20n+19}, we obtain (3.7).
Again by applying binomial theorem on(1.9), we obtain

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(10n+8)q^n \equiv 5f_5^2f_{10} \pmod{20}. \tag{3.10}$$

By comparing the coefficients of q^{5n+1}, q^{5n+2}, q^{5n+3} and q^{5n+4}, we complete the proof of (3.8).
Remark: An alternate proof of (3.8) is given by Barman and Ray [4], where they use arithmetic properties of modular forms to prove the congruence. ☐

Corollary 3.2.

$$\overline{\mathcal{EO}}(20n+18) \equiv 0 \pmod{10}, \tag{3.11}$$

$$\overline{\mathcal{EO}}(40n+28) \equiv 0 \pmod{10}. \tag{3.12}$$

Proof. In light of (2.11) and (2.12), we deduce the following 2-dissection.

$$\begin{aligned}\frac{f_5^2}{f_1^4} =& \frac{f_8^7f_{20}^4}{f_2^9f_{16}^2f_{40}^2} + q^2\frac{f_4^6f_8^3f_{10}^2f_{40}^2}{f_2^{11}f_{16}^2f_{20}^2} + 4q^2\frac{f_4^5f_{10}f_{16}^2f_{20}}{f_2^{10}f_8}\\ &+ 2q\frac{f_4^2f_8f_{16}^2f_{20}^4}{f_2^9f_{40}^2} + 2q\frac{f_4^3f_8^5f_{10}f_{20}}{f_2^{10}f_{16}^2} + 2q^3\frac{f_4^8f_{10}^2f_{16}^2f_{40}^2}{f_2^{11}f_8^3f_{20}^2}.\end{aligned} \tag{3.13}$$

Extracting terms involving q^{2n+1} from (1.9), then replacing q^2 by q, it follows that

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(20n+18)q^n = 10\left[\frac{f_2^2f_4f_5f_8^2f_{10}^4}{f_1^7f_{20}^2} + \frac{f_2^3f_4^5f_5^2f_{10}}{f_1^8f_8^2} + q\frac{f_2^8f_5^3f_8^2f_{20}^2}{f_1^9f_4^3f_{10}^2}\right] \tag{3.14}$$

This completes the proof of (3.11).
Again, extracting terms involving q^{2n} from (1.9), then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(20n+8)q^n = 5\left[\frac{f_4^7 f_5 f_{10}^4}{f_1^7 f_8^2 f_{20}^2} + q\frac{f_2^6 f_4^3 f_5^3 f_{20}^2}{f_1^9 f_8^2 f_{10}^2} + 4q\frac{f_2^5 f_5^2 f_8^2 f_{10}}{f_1^8 f_4}\right]$$

Invoking binomial theorem (2.14), we obtain

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(20n+8)q^n \equiv 5f_1^3 f_5\left[f_2 + q\frac{f_{10}^3}{f_4}\right] \pmod{10}. \tag{3.15}$$

Using (2.13) in (3.15) and employing binomial theorem, we obtain

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(20n+8)q^n \equiv 5\left[f_2^5 + q^2\frac{f_{10}^6}{f_2}\right] \pmod{10}. \tag{3.16}$$

On comparing the terms involving q^{2n+1}, we complete the proof of (3.12). □

Proof of Theorem 1.2.
From (3.9), we have

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(10n+8)q^n \equiv 5f_{20} \pmod{10}.$$

Invoking Lemma 2.4 and extracting terms involving $q^{pn+\frac{5(p^2-1)}{6}}$ from the both sides of the resulting identity and then changing q^p to q, we obtain

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}\left(10\big(pn+\frac{5(p^2-1)}{6}\big)+8\right)q^n \equiv 5(-1)^{\frac{\pm p-1}{6}}f(-q^{20p}) \pmod{10},$$

Again applying Lemma 2.4 to the identity (3.9) and extracting the terms involving $q^{p^2n+\frac{5(p^2-1)}{6}}$ from the resulting identity and then replacing q^{p^2} to q, we deduce

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}\left(10\big(p^2n+\frac{5(p^2-1)}{6}\big)+8\right)q^n \equiv 5(-1)^{\frac{\pm p-1}{6}}f(-q^{20}) \pmod{10}.$$

Now apply Lemma 2.4 to the above identity and extracting the terms involving of $q^{pn+\frac{5(p^2-1)}{6}}$ and then changing q^p to q, we obtain

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}\left(10p^3n+\frac{25p^4-1}{3}\right)q^n \equiv 5f(-q^{20p}) \pmod{10}.$$

Hence, by induction on α, we derive that, for $\alpha \geqslant 1$,

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}\left(10p^{2\alpha-1}n+\frac{25p^{2\alpha}-1}{3}\right)q^n \equiv 5(-1)^{\alpha\left(\frac{\pm p-1}{6}\right)}f(-q^{20p}) \pmod{10}.$$

On comparing the coefficients of q^{pn+j} where j = 1, 2..., $p-1$, from the above identity we obtain

$$\overline{\mathcal{EO}}\left(10p^{2\alpha-1}(pn+j)+\frac{25p^{2\alpha}-1}{3}\right)q^n \equiv 0 \pmod{10}.$$

Thus we complete the proof.

Proof of Theorem 1.3
Extracting the terms involving q^{5n+1} from the identity (2.16) and then replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} p_D(5n+1)q^n = \frac{f_2^2 f_5^3}{f_1^4 f_{10}}. \tag{3.17}$$

Applying (2.14) in (3.17) and comparing the coefficients of q^{5n+1}, q^{5n+2}, q^{5n+3} and q^{5n+4} we complete the proof of (1.11).

Theorem 3.3. *$\overline{\mathcal{EO}}(2n)$ is odd if and only if 6n+1 is a perfect square.*

Proof. From (1.3), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(2n)q^n &\equiv f_4 \pmod 2,\\ \sum_{n=0}^{\infty} \overline{\mathcal{EO}}(2n)q^{6n+1} &\equiv \sum_{n=0}^{\infty} q^{(6n+1)^2} \pmod 2.\end{aligned}$$

This completes the proof of Theorem 3.1. □

Theorem 3.4. *$\overline{p}_D(n)$ is odd if and only if 12n+1 is a perfect square.*

Proof. The proof follows on the same lines. □

4. Recurrence relations for $\overline{p}_D(n)$ and $\overline{\mathcal{EO}}(n)$

In this section, we provide linear recurrence relations for $\overline{p}_D(n)$ and $\overline{\mathcal{EO}}(n)$.

Theorem 4.1.

$$\sum_{k=-\infty}^{\infty} (6k+1)\overline{p}_D\left(n - \frac{(3k^2+k)}{2}\right) = \begin{cases} (-1)^k(2k+1) & \text{if } n = T_k, k \in \mathbb{N} \text{ or } n=0\\ 0 & \text{otherwise.}\end{cases} \tag{4.1}$$

where T_k is the triangular number.
It is easy to evaluate $\overline{p}_D(n)$ for small values of n, say

$$\begin{aligned}\overline{p}_D(0) &= 1,\\ \overline{p}_D(1) &= -3 + 5\overline{p}_D(0) = 2,\\ \overline{p}_D(2) &= 5\overline{p}_D(1) - 7\overline{p}_D(0) = 3,\\ \overline{p}_D(3) &= 5 - 7\overline{p}_D(1) + 5\overline{p}_D(2) = 6,\\ \overline{p}_D(4) &= 5\overline{p}_D(3) - 7\overline{p}_D(2) = 9.\end{aligned}$$

Proof. On multiplying $\dfrac{f_2^2}{f_1^2}$ to Ramanujan's identity (2.9) we deduce that

$$f_1^3 = \sum_{n=0}^{\infty} \overline{p}_D(n)q^n \sum_{n=-\infty}^{\infty} (6n+1)q^{\frac{3n^2+n}{2}}. \tag{4.2}$$

Comparing (4.2) with Jacobi's identity (2.8), we obtain

$$\sum_{n=0}^{\infty} (-1)^n(2n+1)q^{\frac{n^2+n}{2}} = \sum_{n=0}^{\infty} \overline{p}_D(n)q^n \sum_{n=-\infty}^{\infty} (6n+1)q^{\frac{3n^2+n}{2}}. \tag{4.3}$$

Applying Cauchy multiplication of two power series on identity (4.3), we obtain recurrence relation (4.1). □

Theorem 4.2. $\overline{\mathcal{EO}}(2n) = \sum_{k=-\infty}^{\infty} (-1)^k \overline{p}_D(n - (3k^2 + k))$.

Proof. Using (1.3), (1.6) and (2.5) we have

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(2n) q^n = \sum_{n=0}^{\infty} \overline{p}_D(n) q^n \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2+n}. \tag{4.4}$$

Applying Cauchy multiplication for power series on the identity (4.4). On comparing the coefficients of q^n in the resulting identity we obtain Theorem 4.2.

□

Using above values of $\overline{p}_D(n)$, we evaluate $\overline{\mathcal{EO}}(2n)$ for small values of n, say

$$\begin{aligned}
\overline{\mathcal{EO}}(0) &= \overline{p}_D(0) = 1, \\
\overline{\mathcal{EO}}(2) &= \overline{p}_D(1) = 2, \\
\overline{\mathcal{EO}}(4) &= -\overline{p}_D(0) + \overline{p}_D(2) = 2, \\
\overline{\mathcal{EO}}(6) &= -\overline{p}_D(1) + \overline{p}_D(3) = 4, \\
\overline{\mathcal{EO}}(8) &= -\overline{p}_D(0) + \overline{p}_D(4) - \overline{p}_D(2) = 5.
\end{aligned}$$

5. $\overline{p}_D(n)$, $\overline{\mathcal{EO}}(n)$ and variants of partition functions

In [7], Corteel and Lovejoy introduced overpartitions. An overpartition of a nonnegative integer n is a partition of n where the first occurrence of parts of each size may be overlined. Let $\overline{p}(n)$ denote the number of overpartitions of n, then

$$\sum_{n=0}^{\infty} \overline{p}(n) q^n = \frac{f_2}{f_1^2}. \tag{5.1}$$

Let $\overline{pp_0}(n)$ denote the number of over partition pairs of n into odd parts, then

$$\sum_{n=0}^{\infty} \overline{pp_0}(n) q^n = \frac{f_2^6}{f_1^4 f_4^2}. \tag{5.2}$$

Let $\overline{p_0}(n)$ denote the number of overpartitions of n into odd parts, then

$$\sum_{n=0}^{\infty} \overline{p_0}(n) q^n = \frac{f_2^3}{f_1^2 f_4}. \tag{5.3}$$

Theorem 5.1. *For any nonnegative integer n, we have*

$$\sum_{k=-\infty}^{\infty} \overline{p}_D(n - k^2) = \sum_{k=-\infty}^{\infty} (-1)^k \overline{pp_0}(n - (3k^2 + k)), \tag{5.4}$$

$$\overline{p}_D(n) = \sum_{k=-\infty}^{\infty} (-1)^k \overline{p}(n - (3k^2 + k)). \tag{5.5}$$

Proof. Using (1.6), (2.3), (2.5) and (5.2) we have

$$\sum_{n=0}^{\infty}\overline{p}_D(n)q^n\varphi(q)=\sum_{n=-\infty}^{\infty}(-1)^nq^{3n^2+n}\sum_{n=0}^{\infty}\overline{pp_0}(n)q^n. \tag{5.6}$$

Applying Cauchy multiplication of two power series in identity (5.6), we obtain

$$\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty}\overline{p}_D(n-k^2)q^n=\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty}(-1)^k\overline{pp_0}(n-(3k^2+k))q^n. \tag{5.7}$$

Equating the coefficients of q^n on each side of identity (5.7), we complete the proof of (5.4).
Using (1.6), (2.5) and (5.1) we have

$$\sum_{n=0}^{\infty}\overline{p}_D(n)q^n=\sum_{n=0}^{\infty}\overline{p}(n)q^n\sum_{n=-\infty}^{\infty}(-1)^nq^{3n^2+n}, \tag{5.8}$$

Again, applying Cauchy multiplication for power series in identity (5.8) and then comparing the coefficients of q^n in the resulting identity, we complete the proof of (5.5). □

Theorem 5.2. *For any nonnegative integer n, we have*

$$\overline{\mathcal{EO}}(2n)=\sum_{n=0}^{\infty}p_D(n-T_k), \tag{5.9}$$

$$\overline{p_0}(n)=\sum_{k=0}^{\infty}\overline{\mathcal{EO}}(2n-8k)p(k). \tag{5.10}$$

Proof. We omit the proof since it is similar to Theorem 5.1. □

Acknowledgment: The authors would like to thank Professor M. D. Hirschhorn, for his valuable comments and helpful suggestions, which has substantially improved our paper.

References

[1] S. Ahlgren, J. Lovejoy, The arithemetic of partitions into distinct parts, Mathematica **48** (2001), 203-211.
[2] G. E. Andrews, The Theory of Partitions, Cambridge Univ. Press, Cambridge, 1998.
[3] G. E. Andrews, Integer partitions with even parts below odd parts and the mock theta functions, Ann. Comb. **22** (2018), 433-445 .
[4] R. Barman and C. Ray, On Andrews' integer partitions with even parts below odd parts, (2018), arXiv:1812.08702.
[5] N. D Baruah, N. M Begum, On exact generating functions for the number of partitions into distinct parts, Int. J. Number Theory **14** (2018), 1995-2011.
[6] B. C. Berndt, Ramanujans Notebooks, Part III, Springer, New York, 1991.
[7] S. Corteel, J. Lovejoy, Overpartitions, Trans. Amer. Math. Soc. **356** (2004), 1623-1635.
[8] S. P Cui, N. S. S Gu, Arithmetic properties of l-regular partitions, Adv. Appl. Math. **51** (2013), 507-523.
[9] M. D. Hirschhorn, The Power of q, Developments in Mathematics **49**, Springer, 2017.
[10] M. D. Hirschhorn, J.A Sellers, Elementary proofs of parity results for 5-regular partitions, Bull. Aust. Math. Soc. **81** (2010), 58-63.
[11] J. Lovejoy, The divisibility and distribution of partitons into distinct parts, Adv. Math. **158** (2001), 253-263.

[12] J. Lovejoy, The number of partitions into distinct parts modulo powers of 5, B. Lond. Math. Soc. **35** (2003), 41-46.
[13] M. S. Mahadeva Naika, B. Hemanthkumar, Arithmetic properties of 5-regular partitions, Int. J. Number Theory **165** (2017), 937-956.
[14] M. Merca, New relations for the number of partitions with distinct even parts, J. Number Theory **176** (2017), 1-12.
[15] Ø. Rødseth, Congruence properties of the partition functions $q(n)$ and $q_0(n)$, Arbok Univ. Bergen Mat.-Natur. Ser. **13** (1969), 3-27.

Assistant Professor, Department of Mathematics, Pondicherry University, Puducherry-605014
E-mail address: dr.fathima.sn@gmail.com

Research Scholar, Department of Mathematics, Pondicherry University, Puducherry-605014
E-mail address: veenavsmath@gmail.com