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SUMS OF BALANCING-LIKE SEQUENCES WITH BINOMIAL COEFFICIENTS

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Abstract

For $a, b \in \mathbb{R}$ we study some binomial sums of the form $\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} x_{jk+m}$, $\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} y_{jk+m}$, $\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} z_{jk+m}$ and $\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} w_{jk+m}$ where $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ are the balancing-like, lucas balancing-like, cobalancing-like and lucas-cobalancing-like sequences respectively. Moreover, we express these sums in a different combinatorial way and also provide some closed form solutions of many forms of these sums.

1. Introduction

Several generalizations and variations of balancing numbers is available in literature [**?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**]. For each $A > 2$, Panda and Rout [**?**] studied recurrence sequences $\{x_n\}$ defined by $x_{n+1} = Ax_n - x_{n-1}$ with initial terms $x_0 = 0$ and $x_1 = 1$ which are subsequently known as balancing-like sequences. They proved that if x is a balancing-like number with respect to a given A, then $Dx^2 + 1$, where $D = \frac{A^2-4}{4}$ is a perfect rational square (a perfect integral square only if A is even) and its square root is known as a Lucas-balancing-like number. The Lucas-balancing-like sequence corresponding to $BL(A,-1)$ is denoted by $\{y_n\}$ and satisfy an identical recurrence relation as that of balancing-like numbers with initial terms $y_0 = 1$ and $y_1 = \frac{A}{2}$. Further Panda and Pradhan [?] introduced two associates of the balancing-like sequences called the cobalancing-like sequences $\{z_n\}_{n=1}^{\infty}$ having the recurrence relation $z_{n+1} = Az_n - z_{n-1}$ with initial terms $z_0 = z_1 = 0$ and lucas-cobalancing-like sequences $\{w_n\}_{n=1}^{\infty}$ satisfying the recurrence relation $w_{n+1} = Aw_n - w_{n-1}$ with $w_0 = -1,\ w_1 = 1$ for a fixed A, the balancing-like and/or the Lucas-balancing-like numbers with their above associates satisfy all properties of balancing and Lucas-balancing numbers for $A = 6$.[?, ?, ?]

The Binet forms derived from the recurrence relations of balancing-like numbers, lucas balancing-like numbers, cobalancing-like sequences and lucas-cobalancing-like sequences for a fixed A are

$$x_n = \frac{\alpha^n - \beta^n}{\sqrt{A^2-4}}, \tag{1}$$

$$y_n = \frac{\alpha^n + \beta^n}{2},, \tag{2}$$

$$z_n = \Big(\frac{1}{A-2} - \frac{1}{\sqrt{A^2-4}}\Big)\alpha^n + \Big(\frac{1}{A-2} + \frac{1}{\sqrt{A^2-4}}\Big)\beta^n - \frac{2}{A-2} \tag{3}$$

and

$$w_n = \frac{\alpha^n - \beta^n + \alpha^{n-1} - \beta^{n-1}}{\sqrt{A^2-4}} \tag{4}$$

for $n = 1, 2, \cdots$ where $\alpha = \frac{A+\sqrt{A^2-4}}{2}$ and $\beta = \frac{A-\sqrt{A^2-4}}{2}$. Clearly, $\alpha + \beta = A$ and $\alpha\beta = 1$.

One can derive the following expression by the help of binet forms as

$$\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k}x_{2k+m} &= \sum_{k=0}^{n}\binom{n}{k}\Big(\frac{\alpha^{2k+m} - \beta^{2k+m}}{\alpha-\beta}\Big)\\
&= \frac{1}{\alpha-\beta}\big[\alpha^m\sum_{k=0}^{n}\binom{n}{k}\alpha^{2k} - \beta^m\sum_{k=0}^{n}\binom{n}{k}\beta^{2k}\big]\\
&= \frac{1}{\alpha-\beta}\big[\alpha^m(1+\alpha^2)^n - \beta^m(1+\beta^2)^n\big]\\
&= \frac{1}{\alpha-\beta}\big[\alpha^{m+n}(\alpha+\beta^n - \beta^{m+n}(\alpha+\beta)^n\big]\\
&= \frac{1}{\alpha-\beta}(\alpha+\beta)^n(\alpha^{m+n} - \beta^{m+n})\\
&= A^n x_{n+m}
\end{aligned}$$

Similar to the above identity we can obtain the subsequent identities,

$$\sum_{k=0}^{n}\binom{n}{k}y_{2k+m}=A^n y_{n+m}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}z_{2k+m}=A^n z_{n+m}+\frac{A^n}{2}-2^{n-1},\quad \sum_{k=0}^{n}\binom{n}{k}w_{2k+m}=A^n w_{n+m}$$

In this paper we will discuss these four parameter sums

$$S_n(x)=S_n(x;a,b,j,m)=\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}x_{jk+m},$$

$$S_n(y)=S_n(y;a,b,j,m)=\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}y_{jk+m}$$

and

$$S_n(z)=S_n(z;a,b,j,m)=\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}z_{jk+m},$$

$$S_n(w)=S_n(w;a,b,j,m)=\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}w_{jk+m}$$

where $a,b\in\mathbb{R}$ and $j,m\in\mathbb{N}$.

2. Results

We establish some lemmas to support the question: For two integers p and q with $p\neq q$. For what value of a and b does the identity

$$\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}x_{qk+m}=x_{pn+m}$$

hold? We also find out the same for y_n, z_n and w_n.

Lemma 2.1. *The generating function of the sequences* $\{x_{jn+m}\}_{n\geq 0}$ *and* $\{y_{jn+m}\}_{n\geq 0}$ *are given by*

$$f_{x_{jn+m}}(r)=\sum_{n=0}^{\infty}x_{jn+m}r^n=\frac{x_m+x_{j-m}r}{1-2y_jr+r^2} \tag{5}$$

and

$$f_{y_{jn+m}}(r)=\sum_{n=0}^{\infty}y_{jn+m}r^n=\frac{y_m-y_{j-m}r}{1-2y_jr+r^2} \tag{6}$$

Proof.

$$
\begin{aligned}
f_{x_{jn+m}}(r) = \sum_{n=0}^{\infty} x_{jn+m} r^n &= \frac{1}{(\alpha-\beta)} \sum_{n=0}^{\infty} (\alpha^{jn+m} - \beta^{jn+m}) r^n \\
&= \frac{1}{(\alpha-\beta)} \Big[\alpha^m \sum_{n=0}^{\infty} \alpha^{jn} r^n - \beta^m \sum_{n=0}^{\infty} \beta^{jn} r^n \Big] \\
&= \frac{1}{(\alpha-\beta)} \Big[\frac{\alpha^m}{1-\alpha^j r} - \frac{\beta^m}{1-\beta^j r} \Big] \\
&= \frac{1}{(\alpha-\beta)} \Big[\frac{\alpha^m - \beta^m + \alpha^{j-m} r - \beta^{j-m}}{1-(\alpha^j+\beta^m)+r^2} \Big] \\
&= \frac{x_m + x_{j-m} r}{1 - 2y_j r + r^2}
\end{aligned} \tag{7}
$$

The similar result for the lucas balancing sequence can be acquired accordingly. □

Lemma 2.2. *The generating function of the sequences $\{z_{jn+m}\}_{n\geq 0}$ and $\{w_{jn+m}\}_{n\geq 0}$ are given by*

$$
f_{z_{jn+m}}(r) = \sum_{n=0}^{\infty} x_{jn+m} r^n = \frac{2y_m - 2y_{j-m} r}{(A-2)(1-2y_j r + r^2)} - \frac{x_m + x_{j-m} r}{1-2y_j r + r^2} - \frac{2}{(A-2)(1-r)} \tag{8}
$$

and

$$
f_{w_{jn+m}}(r) = \sum_{n=0}^{\infty} w_{jn+m} r^n = \frac{x_m + x_{m-1} + (x_{j-m} + x_{j-m+1}) r}{1 - 2y_j r + r^2} \tag{9}
$$

Proof. The proof is similar to that of Lemma**??**. □

Lemma 2.3. *The generating functions for $S_n(x)$, $S_n(y)$ and $S_n(z)$, $S_n(w)$ are*

$$
f_{S_n(x)}(r) = \sum_{n=0}^{\infty} S_n(x) r^n = \frac{x_m + (a x_{j-m} - b x_m) r}{1 - (2b + 2a y_j) r + (a^2 + 2 y_j ab + b^2) r^2}, \tag{10}
$$

$$
f_{S_n(y)}(r) = \sum_{n=0}^{\infty} S_n(y) r^n = \frac{y_m - (a y_{j-m} + b y_m) r}{1 - (2b + 2a y_j) r + (a^2 + 2 y_j ab + b^2)} \tag{11}
$$

and

$$
\begin{aligned}
f_{S_n(z)}(r) = \sum_{n=0}^{\infty} S_n(z) r^n =& \frac{2y_m - (2a y_{j-m} + 2b y_m) r}{(A-2)\{1 - (2b + 2a y_j) r + (a^2 + 2 y_j ab + b^2) r^2\}} \\
&- \frac{x_m - (a x_{j-m} - b x_m)}{1 - (2b + 2a y_j) r + (a^2 + 2 y_j ab + b^2)} - \frac{2}{(A-2)(1-(a+b)r)}
\end{aligned} \tag{12}
$$

$$
f_{S_n(w)}(r) = \sum_{n=0}^{\infty} S_n(w) r^n = \frac{x_m + x_{m-1} + (a x_{j-m} + a x_{j-m+1} - b x_m - b x_{m-1})}{1 - (2b + 2a y_j) r + (a^2 + 2 y_j ab + b^2) r^2} \tag{13}
$$

Proof. Applying Theorem 1 from [**?**] it follows that for a integer sequence $\{e_n\}_{n\geq 0}$,

$$f_{S_n(e)}(r) = \frac{1}{1-br} f_{e_{jn+m}}\Big(\frac{ar}{1-br}\Big).$$

So using the above relation and the equations from Lemma 1 and Lemma 2 this Lemma can be proved. $\square$

Theorem 2.4. *We have*

$$\sum_{k=0}^{n} a^k b^{n-k} x_{qk+m} = x_{pn+m} \tag{14}$$

$$\sum_{k=0}^{n} a^k b^{n-k} y_{qk+m} = y_{pn+m} \tag{15}$$

and

$$\sum_{k=0}^{n} a^k b^{n-k} w_{qk+m} = w_{pn+m} \tag{16}$$

if and only if $a = x_p/x_q$ and $b = x_{q-p}/x_q$.

Proof. Comparing (**??**) and (**??**) we get the following system of equations

$$\begin{aligned} y_p &= b + ay_q, \\ x_{p-m} &= ax_{q-m} - bx_m, \\ 1 &= a^2 + 2y_j ab + b^2. \end{aligned}$$

The above equations gives

$$a = \frac{x_{p-m} + y_p x_m}{x_{q-m} + y_q x_m} = \frac{x_p}{x_q}$$

and

$$b = y_p - y_q \frac{x_p}{x_q} = \frac{x_{q-p}}{x_q}.$$

The proofs of second and third parts are similar so it can be easily inferred. $\square$

The cobalancing-like sequences however have some different results regarding the Theorem **??** which we can find out through the following theorem

Theorem 2.5. *The sum*

$$\sum_{k=0}^{n} a^k b^{n-k} z_{qk+m} = z_{pn+m} \tag{17}$$

has no solutions.

Proof. Analogous to the Theorem **??** we compare the equations (**??**) and (**??**) and can produce $a = \frac{x_p}{x_q}$ and $b = \frac{x_{q-p}}{x_q}$. But we can also see that here $a + b = 1$ which gives $x_{q-p} = x_q - x_p$ which is only possible if $p = q$ which is trivial or $p = 0$ and q be any integer. □

Now, we look into some special binomial sums.

Theorem 2.6. *For n be any positive integer, the sums*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \left(\frac{1}{y_j}\right)^k x_{jk+m} = \begin{cases} \left(\frac{A^2-4}{4}\right)^{\frac{n}{2}} x_m \left(\frac{x_j}{y_j}\right)^n, & \text{if } n \text{ is even} \\ \\ -\left(\frac{A^2-4}{4}\right)^{\frac{n-1}{2}} y_m \left(\frac{x_j}{y_j}\right)^n, & \text{if } n \text{ is odd} \end{cases} \tag{18}$$

and

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \left(\frac{1}{y_j}\right)^k y_{jk+m} = \begin{cases} \left(\frac{A^2-4}{4}\right)^{\frac{n}{2}} y_m \left(\frac{x_j}{y_j}\right)^n, & \text{if } n \text{ is even} \\ \\ -\left(\frac{A^2-4}{4}\right)^{\frac{n+1}{2}} x_m \left(\frac{x_j}{y_j}\right)^n, & \text{if } n \text{ is odd} \end{cases} \tag{19}$$

holds.

Proof. If we choose $a = -b/y_j$, then $(2b + 2ay_j)r = 0$ from (**??**). Again we can see that

$$a^2 + 2y_j ab + b^2 = \frac{b^2(1 - y_j^2)}{y_j^2} = -b^2 \frac{\left(\frac{A^2-4}{4}\right) x_j^2}{y_j^2}$$

and

$$a x_{j-m} - b x_m = -b \frac{x_j y_m}{y_j}.$$

Hence,

$$f_{S_n(x)}(r) = \sum_{n=0}^{\infty} x_m \left(\frac{\left(\frac{A^2-4}{4}\right) x_j^2}{y_j^2} \right)^n b^{2n} r^{2n} - \sum_{n=0}^{\infty} \frac{x_j y_m}{y_j} \left(\frac{\left(\frac{A^2-4}{4}\right) x_j^2}{y_j^2} \right)^n b^{2n+1} r^{2n+1}$$

comparing the coefficients of x^n from (**??**) proves the stated identity. Similarly equation (**??**) can be inferred without proof. □

The companion result for cobalancing-like numbers and lucas-cobalancing-like numbers are stated in the following theorem

Theorem 2.7. *For n be any positive integer, the sums*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\Big(\frac{1}{y_j}\Big)^k z_{jk+m}=\begin{cases}\Big(\frac{2y_m}{A-2}-x_m\Big)\big(\frac{A^2-4}{4}\big)^{\frac{n}{2}}\big(\frac{x_j}{y_j}\big)^n-\frac{2}{A-2}\big(\frac{y_j-1}{y_j}\big)^n, & \textit{if } n \textit{ is even}\\ \\ \big(y_m-x_m\big(\frac{A+2}{2}\big)\big)\big(\frac{A^2-4}{4}\big)^{\frac{n-1}{2}}\big(\frac{x_j}{y_j}\big)^n-\frac{2}{A-2}\big(\frac{y_j-1}{y_j}\big)^n, & \textit{if } n \textit{ is odd}\end{cases} \tag{20}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\Big(\frac{1}{y_j}\Big)^k w_{jk+m}=\begin{cases}(x_m+x_{m-1})\Big(\frac{A^2-4}{4}\Big)^{\frac{n}{2}}\Big(\frac{x_j}{y_j}\Big)^n, & \textit{if } n \textit{ is even}\\ \\ -(y_m+y_{m-1})\Big(\frac{A^2-4}{4}\Big)^{\frac{n-1}{2}}\Big(\frac{x_j}{y_j}\Big)^n, & \textit{if } n \textit{ is odd}\end{cases} \tag{21}$$

holds.

Proof. We have already seen that taking $a=-b/y_j$, then

$$a^2+2y_jab+b^2=-b^2\frac{\Big(\frac{A^2-4}{4}\Big)x_j^2}{y_j^2}$$

,

$$ax_{j-m}-bx_m=-b\frac{x_jy_m}{y_j}.$$

and

$$ay_{j-m}+by_m=b\Big(\frac{A^2-4}{4}\Big)\frac{x_mx_j}{y_j}.$$

So,

$$\begin{aligned}f_{S_n(z)}(r)&=\big(\frac{2y_m}{A-2}-x_m\big)\sum_{n=0}^{\infty}\Big(\frac{A^2-4}{4}\Big)^n\Big(\frac{x_j}{y_j}\Big)^{2n}b^{2n}r^{2n}\\&\quad+\big(y_m-x_m\big(\frac{A+2}{2}\big)\big)\sum_{n=0}^{\infty}\Big(\frac{A^2-4}{4}\Big)^n\Big(\frac{x_j}{y_j}\Big)^{2n+1}b^{2n+1}r^{2n+1}-\frac{2}{A-2}\sum_{n=0}^{\infty}\Big(\frac{y_j-1}{y_j}\Big)^n b^nr^n,\end{aligned}$$

and thus comparing the coefficients we get the proof. In the case of lucas cobalancing-like sequences applying the similar conditions we have

$$ax_{j-m}+ax_{j-m+1}-bx_m-bx_{m-1}=-b\frac{x_j}{y_j}(y_m+y_{m-1})$$

which leads to

$$f_{S_n(w)}(r)=(x_m+x_{m-1})\sum_{n=0}^{\infty}\Big(\frac{A^2-4}{4}\Big)^n\Big(\frac{x_j}{y_j}\Big)^{2n}b^{2n}r^{2n}-(y_m+y_{m-1})\sum_{n=0}^{\infty}\Big(\frac{A^2-4}{4}\Big)^n\Big(\frac{x_j}{y_j}\Big)^{2n+1}b^{2n+1}r^{2n+1}$$

and the proof can be inferred. □

Theorem 2.8. *For $n \geq 1$ the following identities are valid:*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \alpha^{jk} x_{jk+m} = (-1)^n \left(\sqrt{A^2-4}\right)^{n-1} \alpha^{jn+m} x_j^n \tag{22}$$

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \beta^{jk} x_{jk+m} = -\left(\sqrt{A^2-4}\right)^{n-1} \beta^{jn+m} x_j^n \tag{23}$$

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \alpha^{jk} y_{jk+m} = \frac{(-1)^n}{2} \left(\sqrt{A^2-4}\right)^{n} \alpha^{jn+m} x_j^n \tag{24}$$

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \beta^{jk} y_{jk+m} = -\frac{1}{2} \left(\sqrt{A^2-4}\right)^{n} \beta^{jn+m} x_j^n \tag{25}$$

Proof.

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} (-1)^k \alpha^{jk} x_{jk+m} &= \frac{1}{(\alpha-\beta)} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \alpha^{jk} (\alpha^{jk+m} - \beta^{jk+m}) \\
&= \frac{1}{(\alpha-\beta)} \sum_{k=0}^{n} \binom{n}{k} (-1)^k (\alpha^{2jk+m} - \beta^{m}) \\
&= \frac{1}{(\alpha-\beta)} (\alpha^m (1-\alpha^{2j})^n) \\
&= \frac{(-1)^n \alpha^{m+jn} (\alpha^j - \beta^j)^n}{(\alpha-\beta)} \\
&= (-1)^n (\alpha-\beta)^{n-1} \alpha^{jn+m} x_j^n
\end{aligned}$$

which concludes the proof for the Equation(**??**). The proofs of Equations(**??**), (**??**) and (**??**) can similarly be obtained. □

Corollary 2.9. *For $n \geq 1$ we have*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k x_{jk} x_{jk+m} = \begin{cases} 2\left(\sqrt{A^2-4}\right)^{n-2} x_j^n y_{jn+m}, & n \text{ is even} \\ -\left(\sqrt{A^2-4}\right)^{n-2} x_j^n x_{jn+m}, & n \text{ is odd} \end{cases} \tag{26}$$

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k y_{jk} x_{jk+m} = \begin{cases} \frac{1}{2}\left(\sqrt{A^2-4}\right)^{n} x_j^n x_{jn+m}, & n \text{ is even} \\ -\left(\sqrt{A^2-4}\right)^{n-1} x_j^n y_{jn+m}, & n \text{ is odd} \end{cases} \tag{27}$$

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k x_{jk} y_{jk+m} = \begin{cases} \frac{1}{2}\left(\sqrt{A^2-4}\right)^{n} x_j^n x_{jn+m}, & n \text{ is even} \\ -\left(\sqrt{A^2-4}\right)^{n-1} x_j^n y_{jn+m}, & n \text{ is odd} \end{cases} \tag{28}$$

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k y_{jk} y_{jk+m} = \begin{cases} \frac{1}{2}\left(\sqrt{A^2-4}\right)^{n} x_j^n y_{jn+m}, & n \text{ is even} \\ -\frac{1}{4}\left(\sqrt{A^2-4}\right)^{n+1} x_j^n x_{jn+m}, & n \text{ is odd} \end{cases} \tag{29}$$

3. Combinatorial Identities for $S_n(x)$, $S_n(y)$, $S_n(z)$ and $S_n(w)$

Theorem 3.1. *The following combinatorial identity is valid*

$$S_n(x)=\delta(n)+\sum_{l=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-l-1}{l}(-1)^l(v+bu)^l u^{n-2l-1}\Big(\frac{n}{n-2l}(b+ay_j)x_m+ay_mx_j\Big),\tag{30}$$

where $u=2(b+ay_j)$, $v=a^2-b^2$ *and*

$$\delta(n)=\begin{cases}x_m(-1)^{\lfloor\frac{n}{2}\rfloor}(v+bu)^{\lfloor\frac{n}{2}\rfloor}, & \text{if } n \text{ is even}\\ 0, & \text{if } n \text{ is odd.}\end{cases}\tag{31}$$

Proof. We set $h=ax_{j-m}-bx_m$. Then, again from (**??**) we have

$$\begin{aligned}f_{S_n(x)}(r)&=(x_m+hr)\sum_{n=0}^{\infty}r^n(u-(v-bu)r)^n\\&=x_m\sum_{n=0}^{\infty}\sum_{s=0}^{n}\binom{n}{s}(-1)^s(v+bu)^k u^{n-s}r^{n+s}\\&+h\sum_{n=0}^{\infty}\sum_{s=0}^{n}\binom{n}{s}(-1)^s(v+bu)^s u^{n-s}r^{n+s+1}\\&=x_m\sum_{k=0}^{\infty}\sum_{l=0}^{\left[\frac{k}{2}\right]}\binom{k-l}{l}(-1)^l(v+bu)^l u^{k-2l}r^k\\&+h\sum_{k=1}^{\infty}\sum_{l=0}^{\left[\frac{k-1}{2}\right]}\binom{k-l-1}{l}(-1)^l(v+bu)^l u^{k-2l-1}r^k.\end{aligned}$$

Comparing the coefficients gives the relation

$$S_n(x)=\delta(n)+\sum_{l=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^l(v+bu)^l u^{n-2l-1}\left(ux_m\binom{n-l}{l}+h\binom{n-l-1}{l}\right),$$

where $\delta(n)$ is defined above. We have $h=-\frac{1}{2}ux_m+ay_mx_j$. The statement now follows since

$$\binom{n-l}{l}=\frac{n-l}{n-2l}\binom{n-l-1}{l},$$

and

$$\binom{n-l}{l}-\frac{1}{2}\binom{n-l-1}{l}=\frac{n}{2(n-2l)}\binom{n-l-1}{l}$$

The analogue result for $S_n(y)$ is stated without proof □

Theorem 3.2. *The following combinatorial identity is valid*

$$S_n(y) = \delta^*(n) + \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-l-1}{l} (-1)^l (v+bu)^l u^{n-2l-1} \Big(\frac{n}{n-2l}(b+ay_j)y_m + \Big(\frac{A^2-4}{4}\Big) a x_m x_j \Big), \tag{32}$$

where $u = 2(b+ay_j)$, $v = a^2 - b^2$ *and*

$$\delta^*(n) = \begin{cases} y_m (-1)^{\lfloor \frac{n}{2} \rfloor} (v+bu)^{\lfloor \frac{n}{2} \rfloor}, & if\ n\ is\ even \\ 0, & if\ n\ is\ odd. \end{cases} \tag{33}$$

Similarly, in the case of cobalancing-like numbers we have the following result

Theorem 3.3. *The following combinatorial identity is valid*

$$S_n(z) = \gamma(n) + \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-l-1}{l} (-1)^l (v+bu)^l u^{n-2l-1}$$
$$\Bigg(u\Big(\frac{n-l}{n-2l}\Big)\Big(\frac{2}{A-2}y_m - x_m\Big) - \Big(\frac{2g}{A-2} + h\Big) \Bigg) - \frac{2}{A-2}(a+b)^n$$

where $g = ay_{j-m} + by_m$, $h = ax_{j-m} - bx_m$ *and*

$$\gamma(n) = \begin{cases} \Big(\frac{2}{A-2}y_m - x_m - \frac{2}{A-2}\Big) + \\ \Big(\frac{2}{A-2}y_m - x_m\Big)(-1)^{\lfloor \frac{n}{2} \rfloor}(v+bu)^{\lfloor \frac{n}{2} \rfloor}, & if\ n\ is\ even \\ 0, & if\ n\ is\ odd. \end{cases} \tag{34}$$

Proof. Let $h = ax_{j-m} - bx_m$ and $g = ay_{j-m} + by_m$, and then by (**??**), we have

$$f_{S_n(z)}(r) = \sum_{n=1}^{\infty} \Bigg(\Big(\frac{2}{A-2}y_m - x_m\Big) + \Big(\frac{2}{A-2}y_m - x_m\Big) \binom{n - \lfloor \frac{n}{2} \rfloor}{\lfloor \frac{n}{2} \rfloor} (-1)^{\lfloor \frac{n}{2} \rfloor} (v+bu)^{\lfloor \frac{n}{2} \rfloor}$$
$$+ \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-l-1}{l} (-1)^l (v+bu)^l u^{n-2l-1} \Big\{ u\Big(\frac{n-l}{n-2l}\Big)\Big(\frac{2}{A-2}y_m - x_m\Big) - \Big(\frac{2g}{A-2} + h\Big)\Big\}$$
$$- \frac{2}{A-2} - \frac{2}{A-2}(a+b)^n \Bigg) r^n$$

Thus, the proof follows □

Likewise, for the lucas cobalancing-like numbers we can state the following theorem without proof.

Theorem 3.4.

$$S_n(w) = \gamma^*(n) + \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-l-1}{l} (-1)^l (v+bu)^l u^{n-2l-1} \left(u(x_m + x_{m-1}) \frac{n-l}{n-2l} + t \right) \tag{35}$$

where $t = ax_{j-m} + ax_{j-m+1} - bx_m - bx_{m-1}$ *and*

$$\gamma^*(n) = \begin{cases} (x_m + x_{m-1})(-1)^{\lfloor \frac{n}{2} \rfloor} (v+bu)^{\lfloor \frac{n}{2} \rfloor}, & if\ n\ is\ even \\ 0, & if\ n\ is\ odd. \end{cases} \tag{36}$$

The following examples demonstrate some identities which can be derived from the above results.

From $S_n(x; 1, 1, 2, 0)$ we get the identity

$$x_n = \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-l-1}{l} (-1)^l A^{n-2l-1}$$

. Likewise, $S_n(y; 1, 1, 2, 0)$ gives

$$y_n = d(n) + \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-l-1}{l} (-1)^l A^{n-2l} \frac{n}{2(n-2l)}$$

where

$$d(n) = \begin{cases} (-1)^{\lfloor \frac{n}{2} \rfloor}, & if\ n\ is\ even \\ 0, & if\ n\ is\ odd \end{cases}$$

From, $S_{2n}(z; 1, 1, 2, 0)$ we have the identity

$$z_{2n} = \frac{2}{A-2}((-1)^n - 1) + \sum_{l=0}^{n-1} \binom{2n-l-1}{l} (-1)^l A^{2n-2l-2} \left(\frac{2A^2}{A-2} \Big(\frac{2n-l}{2n-2l} \Big) - \frac{A^2}{A-2} - A \right)$$

and similarly from $S_{2n+1}(z; 1, 1, 2, 0)$ we have

$$z_{2n+1} = \sum_{l=0}^{n} \binom{2n-l}{l} (-1)^l A^{2n-2l-1} \left(\frac{2A^2}{A-2} \Big(\frac{2n+1-l}{2n+1-2l} \Big) - \frac{A^2}{A-2} - A \right) - \Big(\frac{2}{A-2} \Big)$$

. Now from $S_{2n}(w; 1, 1, 2, 0)$ we have

$$w_{2n} = (-1)^{n+1} + \sum_{l=0}^{n-1} \binom{2n-l-1}{l} (-1)^l A^{2n-2l-2} \left(-A^2 \Big(\frac{2n-l}{2n-2l} \Big) + A^2 + A \right)$$

and from $S_{2n+1}(w; 1, 1, 2, 0)$, we get

$$w_{2n+1} = \sum_{l=0}^{n} \binom{2n-l}{l} (-1)^l A^{2n-2l-1} \left(-A^2 \Big(\frac{2n+1-l}{2n+1-2l} \Big) + A^2 + A \right).$$

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