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Some Congruence Properties for Ramanujan's General Partitions

D. Anu Radha, B. R. Srivatsa Kumar[1], Shruthi
Department of Mathematics, Manipal Institute of Technology,
Manipal Academy of Higher Education, Manipal - 576 104, India.

E-mail: anurad13@gmail.com, sri_vatsabr@yahoo.com,
shruthikarranth@gmail.com

Dedicated to Prof. Chandrashekar Adiga on his 62^{nd} *birthday.*

Abstract

In this paper, we prove four new infinite families of congruences modulo 13 for the general partition function $p_r(n)$ for negative values of r. Our emphasis throughout this paper is to exhibit the use of q-identities to generate congruences for general partitions.

Keywords: q-identities, Partition congruence, Ramanujan's general partition function.
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1 Introduction

Throughout this paper, we assume $|q| < 1$ and as customary, define

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k).$$

For $|ab| < 1$, Ramanujan's general theta function $f(a,b)$ is given by

$$f(a,b) := \sum_{k=-\infty}^{\infty} a^{\frac{k(k+1)}{2}} b^{\frac{k(k-1)}{2}}.$$

By Jacobi's triple product identity [4, p.35], we have

$$f(a,b) := (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

One of the special case of $f(a,b)$ as defined by S. Ramanujan [4, p.36] is as follows:

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty.$$

For convenience, we write $f_n = f(-q^n)$. Due to Euler, we have

[1]Corresponding author

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{f_1},$$

where $p(n)$ is the number of partitions of n. S. Ramanujan initiated the general partition function $p_r(n)$ as

$$\sum_{n=0}^{\infty} p_r(n)q^n = \frac{1}{f_1^r}, \tag{1.1}$$

for non-zero integer r. For partition function $p(n)$, Ramanujan's so called "most beautiful identity" is given by

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{f_5^5}{f_1^6},$$

which readily implies

$$p(5n+4) \equiv 0 \pmod{5}.$$

Ramanujan's two more beautiful congruences are

$$\begin{aligned} p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

The generalization of the above congruences are as follows:

$$\begin{aligned} p(5^\mu n+\zeta_{5,\mu}) &\equiv 0 \pmod{5^\mu}, \\ p(7^\mu n+\zeta_{7,\mu}) &\equiv 0 \pmod{7^{\lfloor \mu/2 \rfloor+1}}, \\ p(11^\mu n+\zeta_{11,\mu}) &\equiv 0 \pmod{11^\mu}, \end{aligned}$$

where $\zeta_{j,\mu} = 1/24 \pmod{j^\mu}$. The generalization of the congruences modulo powers of 5 and 7 for all $p_r(n)$ was proved by K. G. Ramanathan [16]. Later A. O. L. Atkin [1] found that Ramanathan's proof is not correct. M. Newmann [13–15], studied the function $p_r(n)$ and obtained several interesting congruences and identities involving $p_r(n)$. The functions $p_r(n)$ have been studied by many mathematicians. For the wonderful work one can see [1–3, 5, 6, 8–10, 12, 17–20]. For $r = -2$, P. Hammond and R. Lewis [11] proved that

$$p_{-2}(5n+\ell) \equiv 0 \pmod{5},$$

where $\ell \in \{2, 3, 4\}$. Also in [7], W. Y. C. Chen et. al. proved

$$p_{-2}(25n+23) \equiv 0 \pmod{25}$$

by using modular forms. More recently D.Tang [21] for $p_r(n)$ proved some new congruences for $p_r(n)$, where $r \in \{-2,-6,-7\}$. For example,

$$p_{-2}\left(5^{2\delta-1}n+\frac{7\times 5^{2\delta-1}+1}{12}\right)\equiv 0 \pmod{5^\delta},$$
$$p_{-6}\left(5^{2\delta}n+\frac{3\times 5^{\delta}+1}{4}\right)\equiv 0 \pmod{5^\delta}$$

and

$$p_{-7}\left(5^{2\delta-1}n+\frac{13\times 5^{2\delta-1}+7}{24}\right)\equiv 0 \pmod{5^\delta}.$$

In the sequel, in this paper, we demonstrate four new infinite families of congruences modulo 13 by using q-identities, for the general partition function $p_r(n)$, where r being negative. In particular, for any non-negative integer λ we demonstrate the following congruences and more frequently, we use the below mentioned binomial theorem.

$$f_1^{13}\equiv f_{13} \pmod{13} \qquad \text{and} \qquad f_1^{p^2}\equiv f_{p^2} \pmod{13}. \tag{1.2}$$

Theorem 1.1. *We have*

$$p_{-(13\lambda+1)}(13n+\nu)\equiv 0 \pmod{13},$$

for $\nu = 3,4,6,8,10,11,$

Theorem 1.2. *We have*

$$p_{-(13\lambda+3)}(13n+\nu)\equiv 0 \pmod{13},$$

for $\nu = 4,5,7,8,9,11,12.$

Theorem 1.3. *We have*

$$p_{-(169\lambda+1)}(169n+13\nu+7)\equiv 0 \pmod{13},$$

for $1\le \nu\le 12.$

Theorem 1.4. *We have*

$$p_{-(169\lambda+2)}(169n+13\nu+1)\equiv 0 \pmod{13},$$

for $1\le \nu\le 12.$

2 Proofs of Theorem 1.1–1.4

All the congruences in this section are considered under modulo 13.

Proof of Theorem 1.1. From [4, p. 372, Entry 8(i)], we have

$$f_{1/13}=f_{13}(a-q^{1/13}b-q^{2/13}c+q^{5/13}d+q^{7/13}-q^{12/13}e+q^{22/13}f), \tag{2.1}$$

where

$$
\begin{aligned}
a &= \frac{(q^4;q^{13})_\infty(q^9;q^{13})_\infty}{(q^2;q^{13})_\infty(q^{11};q^{13})_\infty}, & b &= \frac{(q^6;q^{13})_\infty(q^7;q^{13})_\infty}{(q^3;q^{13})_\infty(q^{10};q^{13})_\infty},\\
c &= \frac{(q^2;q^{13})_\infty(q^{11};q^{13})_\infty}{(q;q^{13})_\infty(q^{12};q^{13})_\infty}, & d &= \frac{(q^5;q^{13})_\infty(q^8;q^{13})_\infty}{(q^4;q^{13})_\infty(q^9;q^{13})_\infty},\\
e &= \frac{(q^3;q^{13})_\infty(q^{10};q^{13})_\infty}{(q^5;q^{13})_\infty(q^8;q^{13})_\infty}, & f &= \frac{(q;q^{13})_\infty(q^{12};q^{13})_\infty}{(q^6;q^{13})_\infty(q^7;q^{13})_\infty}.
\end{aligned}
$$

On letting q to q^{13} in (2.1), we obtain

$$f_1 = f_{169}(A - qB - q^2C + q^5D + q^7 - q^{12}E + q^{22}F), \tag{2.2}$$

where $A = a(q^{13}), B = b(q^{13}), C = c(q^{13}), D = d(q^{13}), E = e(q^{13})$ and $F = f(q^{13})$. In (1.1), set $r = -(13\lambda + 1)$, then we see that

$$\sum_{n=0}^{\infty} p_{-(13\lambda+1)}(n)q^n = f_1^{13\lambda+1} = f_1^{13\lambda} f_1.$$

Employing (2.2) and (1.2) in the above, we observe that

$$\sum_{n=0}^{\infty} p_{-(13\lambda+1)}(n)q^n \equiv f_{13}^{\lambda} f_{169}(A - qB - q^2C + q^5D + q^7 - q^{12}E + q^{22}F).$$

On picking the terms containing $q^{13n+\nu}$ on both sides of the above for $\nu = 3, 4, 6, 8, 10, 11$, we obtain the required congruence. $\square$

Proof of Theorem 1.2. From [4, p. 39, Entry 24(ii)], we have

$$f_1^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2},$$

which simplifies to

$$
\begin{aligned}
f_1^3 = I_0(q^{13}) - 3qI_1(q^{13}) + 5q^3I_2(q^{13}) - 7q^6I_3(q^{13}) + 9q^{10}I_4(q^{13})\\
- 11q^{15}I_5(q^{13}) + 13q^{21}I_6(q^{13}),
\end{aligned}
$$

equivalently

$$
\begin{aligned}
f_1^3 \equiv I_0(q^{13}) - 3qI_1(q^{13}) + 5q^3I_2(q^{13}) - 7q^6I_3(q^{13})\\
+ 9q^{10}I_4(q^{13}) - 11q^{15}I_5(q^{13}),
\end{aligned} \tag{2.3}
$$

where $I_0, I_1, I_2, I_3, I_4, I_5$ and I_6 are the series with integral powers of q^{13}. In (1.1), set $r = -(13\lambda + 3)$, we have

$$\sum_{n=0}^{\infty} p_{-(13\lambda+3)}(n)q^n = f_1^{13\lambda+3} = f_1^{13\lambda} f_1^3.$$

Utilizing (1.2) and (2.3) in the above, we obtain

$$\sum_{n=0}^{\infty} p_{-(13\lambda+1)}(n)q^n \equiv f_{13}^{\lambda}(I_0 - 3qI_1 + 5q^3 I_2 - 7q^6 I_3 + 9q^{10} I_4 - 11q^{15} I_5).$$

On picking the terms containing $q^{13n+\nu}$ for $\nu = 4, 5, 7, 8, 9, 11, 12$, on the both sides of the above, we obtain Theorem 1.2. □

Proof of Theorem 1.3. In (1.1), set $r = -(169\lambda + 1)$, it follows that

$$\sum_{n=0}^{\infty} p_{-(169\lambda+1)}(n)q^n = f_1^{169\lambda+1} = f_1^{169\lambda} f_1. \tag{2.4}$$

Utilizing (1.2) in (2.4), we obtain

$$\sum_{n=0}^{\infty} p_{-(169\lambda+1)}(n)q^n \equiv f_{169}^{\lambda} f_1. \tag{2.5}$$

Invoking (2.2) in (2.5), it is observed that

$$\sum_{n=0}^{\infty} p_{-(169\lambda+1)}(n)q^n \equiv f_{169}^{\lambda+1}(A - qB - q^2C + q^5D + q^7 - q^{12}E + q^{22}F). \tag{2.6}$$

On selecting the terms containing q^{13n+7} on both sides of (2.6), dividing by q^7 and letting q to q^{13}, we obtain

$$\sum_{n=0}^{\infty} p_{-(169\lambda+1)}(n)q^n \equiv f_{13}^{\lambda+1}.$$

Selecting the terms containing $q^{13n+\nu}$ in both sides of the above for $1 \le \nu \le 12$, we arrive at Theorem 1.3. □

Proof of Theorem 1.4. In (1.1), put $r = -(169\lambda + 2)$, we have

$$\sum_{n=0}^{\infty} p_{-(169\lambda+2)}(n)q^n = f_1^{169\lambda+2} = f_1^{169\lambda} f_1^2. \tag{2.7}$$

Utilizing (1.2) in (2.7), we obtain

$$\sum_{n=0}^{\infty} p_{-(169\lambda+2)}(n)q^n \equiv f_{169}^{\lambda} f_1^2. \tag{2.8}$$

On squaring (2.2), we obtain

$$\begin{aligned} f_1^2 = f_{169}^2(&A^2 + [q^{13} - 2(AB - CEq^{13} - DFq^{26})]q + (B^2 - 2AC)q^2 \\ &+ 2BCq^3 + C^2q^4 + 2ADq^5 - 2BDq^6 + 2(A - CD)q^7 - 2Bq^8 - 2Cq^9 \\ &+ D^2q^{10} + 2(D - AE)q^{12} + 2BEq^{13} - 2DEq^{17} - 2Eq^{19} + 2AFq^{22} \\ &- 2BFq^{23} + (E^2 - 2CF)q^{24} + 2Fq^{29} - 2EFq^{34} + F^2q^{44}). \end{aligned} \tag{2.9}$$

Using (2.9) in (2.8), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} p_{-(169\lambda+2)}(n)q^n \equiv f_{169}^{\lambda+2}(&A^2 + [q^{13} - 2(AB - CEq^{13} - DFq^{26})]q \\ &+ (B^2 - 2AC)q^2 + 2BCq^3 + C^2q^4 + 2ADq^5 - 2BDq^6 \\ &+ 2(A - CD)q^7 - 2Bq^8 - 2Cq^9 + D^2q^{10} + 2(D - AE)q^{12} \\ &+ 2BEq^{13} - 2DEq^{17} - 2Eq^{19} + 2AFq^{22} - 2BFq^{23} \\ &+ (E^2 - 2CF)q^{24} + 2Fq^{29} - 2EFq^{34} + F^2q^{44}). \end{aligned} \tag{2.10}$$

From [4, p.372 Entry 8(i)], we have

$$1 + \frac{f_1^2}{qf_{13}^2} = \frac{ab}{q} - ce - qdf,$$

where a, b, c, d, e, and f are as defined as in (2.1). On letting q to q^{13} in the above, we obtain

$$q^{13} + \frac{f_{13}^2}{f_{169}^2} = (AB - q^{13}CE - q^{26}DF), \tag{2.11}$$

where A, B, C, D, E and F are as defined as in (2.2). Using (2.11) in the second term of the right side of (2.10) and selecting the terms containing q^{13n+1} on both sides, dividing throughout by q and then letting q to $q^{1/13}$, we deduce that

$$\sum_{n=0}^{\infty} p_{-(169\lambda+2)}(13n+1)q^n \equiv (-1)f_{13}^{\lambda+2}.$$

Selecting the terms containing $q^{13n+\nu}$ on both sides of the above for $1 \leq \nu \leq 12$, we obtain the desired congruence. □

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