

## CONGRUENCES MODULO LARGE POWERS OF 3 FOR $3^\ell$ - REGULAR CUBIC BIPARTITION

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Dedicated to Prof. Chandrashekhar Adiga on his 62<sup>nd</sup> Birthday

**ABSTRACT.** Let  $b_{3^\ell}(n)$  denote the number of  $3^\ell$ - regular cubic bipartition of  $n$ , where  $\ell=1$  and 2. In this paper, we establish some congruences modulo large powers 3 for  $b_{3^\ell}(n)$ . For example, for each  $\alpha \geq 0$  and  $n \geq 0$ ,

$$b_3 \left( 3^{2\alpha+2} n + \frac{3^{2\alpha+3} + 1}{4} \right) \equiv 0 \pmod{3^{2\alpha+2}}.$$

### 1. INTRODUCTION

A partition of a positive integer  $n$  is a non-increasing sequence of positive integers whose sum is  $n$ . Let  $p(n)$  be the number of partitions of  $n$  and the generating function is defined by

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q; q)_\infty} = \frac{1}{E_1}, \quad (1.1)$$

where,

$$E_k := (q^k; q^k)_\infty = \lim_{n \rightarrow \infty} \prod_{l=1}^n (1 - q^{lk}).$$

H. C. Chan [1, 2, 3] studied the congruence properties of cubic partition function  $a(n)$ , which is arised from the study of Ramanujan's cubic continued fraction, which is defined by

$$\sum_{n=0}^{\infty} a(n) q^n = \frac{1}{(q; q)_\infty (q^2; q^2)_\infty} = \frac{1}{E_1 E_2}. \quad (1.2)$$

He also proved the congruence, for  $\alpha \geq 1$  and  $n \geq 0$ ,

$$a(3^\alpha n + \delta_\alpha) \equiv 0 \pmod{3^{1+2\lfloor \delta/2 \rfloor}}, \quad (1.3)$$

where  $\delta_\alpha$  is the reciprocal of 8 modulo  $3^\alpha$ .

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H. H. Chan and P. C. Toh [4] has found the different proof of (1.3). They also established the congruences for  $a(n)$  modulo powers of 5, for  $\alpha \geq 2$  and  $n \geq 0$ ,

$$a(5^\alpha n + \delta_\alpha) \equiv 0 \pmod{5^{\lfloor \delta/2 \rfloor}}, \quad (1.4)$$

where  $\delta_\alpha$  is the reciprocal of 8 modulo  $5^\alpha$ .

Zhao and Zhong [11] studied cubic partition pairs, which are denoted by  $b(n)$  and the generating function is

$$\sum_{n=0}^{\infty} b(n)q^n = \frac{1}{(q;q)_\infty^2 (q^2;q^2)_\infty^2} = \frac{1}{E_1^2 E_2^2}. \quad (1.5)$$

He also proved the Ramanujan-type congruences like

$$b(5n+4) \equiv 0 \pmod{5}, \quad (1.6)$$

$$b(7n+i) \equiv 0 \pmod{7}, \quad (1.7)$$

$$b(9n+7) \equiv 0 \pmod{9}, \quad (1.8)$$

where  $i \in \{2, 3, 4, 6\}$ .

In [9], B. L. S. Lin obtained some congruences for  $b(n)$  modulo 27. For example

$$b(27n+16) \equiv 0 \pmod{27}, \quad (1.9)$$

$$b(27n+25) \equiv 0 \pmod{27}, \quad (1.10)$$

$$b(81n+7) \equiv 0 \pmod{27}. \quad (1.11)$$

He also made a conjecture for  $b(n)$  in [[9], Conjecture 5.1]

$$b(81n+61) \equiv 0 \pmod{243}. \quad (1.12)$$

Later on M. D. Hirschhorn [8] proved the above conjecture. He also obtained a stronger result that is

$$b(81n+61) \equiv 0 \pmod{729}. \quad (1.13)$$

The elementary proof of congruences (1.3) and (1.4) can be established by Hirschhorn in [6] and [7] respectively.

Recently, M. S. M. Naika and S. S. Nayaka [10] studied congruence properties of  $b_\ell(n)$ , which denotes  $\ell$ -regular cubic partition pairs of  $n$ . In [5], Gireesh and M. S. M. Naika obtained the alternative proof (1.12).

In this paper, we define  $b_{3^\ell}(n)$ , denote the number of  $3^\ell$ -regular cubic bipartition of  $n$ , whose generating function is given by

$$\sum_{n=0}^{\infty} b_{3^\ell}(n)q^n = \frac{(q^{3^\ell}; q^{3^\ell})_\infty^2 (q^{2 \cdot 3^\ell}; q^{2 \cdot 3^\ell})_\infty^2}{(q; q)_\infty^2 (q^2; q^2)_\infty^2} = \frac{E_{3^\ell}^2 E_{2 \cdot 3^\ell}^2}{E_1^2 E_2^2}. \quad (1.14)$$

Our main aim is to obtain some congruences modulo large powers 3 for  $b_{3^\ell}(n)$ , where  $\ell=1$  and 2. The main theorem as follows:

**Theorem 1.1.** *For each  $\alpha \geq 0$  and  $n \geq 0$ ,*

$$b_3 \left( 3^{2\alpha+1} n + \frac{3^{2\alpha+1} + 1}{4} \right) \equiv 0 \pmod{3^{2\alpha}}, \quad (1.15)$$

$$b_3 \left( 3^{2\alpha+2} n + \frac{3^{2\alpha+3} + 1}{4} \right) \equiv 0 \pmod{3^{2\alpha+2}}, \quad (1.16)$$

$$b_{3^2} \left( 3^{2\alpha+1} n + \frac{3^{2\alpha+1} + 1}{4} \right) \equiv 0 \pmod{3^{2\alpha}}, \quad (1.17)$$

$$b_{3^2} \left( 3^{2\alpha+2} n + \frac{3^{2\alpha+3} + 1}{4} \right) \equiv 0 \pmod{3^{2\alpha+2}}. \quad (1.18)$$

## 2. PRELIMINARIES

Ramanujan's cubic continued fraction  $\omega$  is given by

$$\omega := \frac{q^{1/3}}{1} + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + \frac{q^3+q^6}{1} + \dots,$$

We define the function  $x(q)$ ,  $a(q)$ ,  $b(q)$  and  $c(q)$  as follows

$$\begin{aligned} x(q) &= q^{-1/3}\omega, \\ a(q) &= \frac{1}{x(q)^2} - 2qx(q), \\ b(q) &= \frac{1}{x(q)} + 4qx(q)^2, \\ c(q) &= \frac{1}{x(q)^3} - 8q^2x(q)^3. \end{aligned}$$

From the definition of  $a(q)$ ,  $b(q)$  and  $c(q)$ , we get the following result:

$$a(q)b(q) = c(q) + 2q, \quad (2.1)$$

$$a(q)^3 + qb(q)^3 = c(q)^2 - 5qc(q) + 40q^2. \quad (2.2)$$

**Lemma 2.1.** [1, Theorem 3] *We have*

$$E_1 E_2 = E_9 E_{18} \left( \frac{1}{x(q^3)} - q - 2q^2x(q^3) \right), \quad (2.3)$$

$$\frac{1}{E_1 E_2} = \frac{E_9^3 E_{18}^3}{E_3^4 E_6^4} (a(q^3) + qb(q^3) + 3q^2), \quad (2.4)$$

$$c(q) = \frac{E_1^4 E_2^4}{E_3^4 E_6^4} + 7q. \quad (2.5)$$

Consider

$$\eta = \frac{E_1 E_2}{q E_9 E_{18}}, \quad \tau = \frac{1}{qx(q^3)}, \quad S = \frac{E_3^4 E_6^4}{q^3 E_9^4 E_{18}^4}. \quad (2.6)$$

From (2.3), (2.4) and (2.6), it follows that

$$\eta = \frac{E_1 E_2}{q E_9 E_{18}} = \tau - 1 - \frac{2}{\tau} \quad (2.7)$$

and

$$S = \tau^3 - 7 - \frac{8}{\tau^3}. \quad (2.8)$$

From (2.7) and (2.8), we have

$$\begin{aligned} \eta^3 &= \tau^3 - 3\tau^2 - 3\tau + 11 + \frac{6}{\tau} - \frac{12}{\tau^2} - \frac{8}{\tau^3} \\ &= S + 18 - 3\tau^2 - 3\tau + \frac{6}{\tau} - \frac{12}{\tau^2} \\ &= S + 9 - 3\eta^2 - 9\tau + \frac{18}{\tau} \\ &= S - 9\eta - 3\eta^2. \end{aligned} \quad (2.9)$$

It follows from (2.9) that

$$\eta^3 + 3\eta^2 + 9\eta = S. \quad (2.10)$$

We can write (2.10)

$$\frac{1}{\eta} = \frac{1}{S}(9 + 3\eta + \eta^2), \quad (2.11)$$

so

$$\frac{1}{\eta^i} = \frac{1}{S} \left( \frac{9}{\eta^{i-1}} + \frac{3}{\eta^{i-2}} + \frac{1}{\eta^{i-3}} \right). \quad (2.12)$$

Now let  $\mathbf{H}$  be the “huffing” operator modulo 3, we have

$$\mathbf{H}(\Sigma a_n q^n) = \Sigma a_{3n} q^{3n}. \quad (2.13)$$

If we apply  $\mathbf{H}$  to (2.12), we find

$$\mathbf{H}\left(\frac{1}{\eta^i}\right) = \frac{1}{S} \left( 9\mathbf{H}\left(\frac{1}{\eta^{i-1}}\right) + 3\mathbf{H}\left(\frac{1}{\eta^{i-2}}\right) + \mathbf{H}\left(\frac{1}{\eta^{i-3}}\right) \right). \quad (2.14)$$

Now,

$$\mathbf{H}(\eta^2) = \mathbf{H}\left(\tau^2 - 2\tau - 3 + \frac{4}{\tau} + \frac{4}{\tau^2}\right) = -3, \quad (2.15)$$

$$\mathbf{H}(\eta) = \mathbf{H}\left(\tau - 1 - \frac{2}{\tau}\right) = -1, \quad (2.16)$$

$$\mathbf{H}(1) = 1. \quad (2.17)$$

From (2.14)–(2.17), we obtain

$$\mathbf{H}\left(\frac{1}{\eta}\right) = \frac{3}{S}, \quad (2.18)$$

$$\mathbf{H}\left(\frac{1}{\eta^2}\right) = \frac{2}{S} + \frac{3^3}{S^2}, \quad (2.19)$$

$$\mathbf{H}\left(\frac{1}{\eta^3}\right) = \frac{1}{S} + \frac{3^3}{S^2} + \frac{3^5}{S^3}, \quad (2.20)$$

and so on.

Indeed, for  $i \geq 1$  we have

$$\mathbf{H}\left(\frac{1}{\eta^i}\right) = \sum_{j=1}^i \frac{m_{i,j}}{S^j}, \quad (2.21)$$

where the  $m_{i,j}$  are defined in the following matrix.

The  $m_{i,j}$  form a matrix  $M$ , the first six rows of which are

$$M = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 27 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 27 & 243 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 18 & 85 & 2187 & 0 & 0 & 0 & \cdots \\ 0 & 5 & 270 & 3645 & 19683 & 0 & 0 & \cdots \\ 0 & 1 & 126 & 3645 & 39366 & 177147 & 0 & \cdots \end{pmatrix} \quad (2.22)$$

and for  $i \geq 4$ ,  $m_{i,1} = 0$ , and for  $j \geq 2$ ,

$$m_{i,j} = 9m_{i-1,j-1} + 3m_{i-2,j-1} + m_{i-3,j-1}. \quad (2.23)$$

In fact  $m_{4i,j} = 0$  for  $j \leq i$ , so we can write

$$\mathbf{H}\left(\frac{1}{\eta^{4i}}\right) = \sum_{j=i+1}^{4i} \frac{m_{4i,j}}{S^j} = \sum_{j=1}^{3i} \frac{m_{4i,i+j}}{S^{i+j}} = \sum_{j=1}^{3i} \frac{a_{i,j}}{S^{i+j}}, \quad (2.24)$$

where

$$a_{i,j} = m_{4i,i+j}. \quad (2.25)$$

Similarly,  $m_{4i+2,j} = 0$  if  $j \leq i$ , so we can write

$$\mathbf{H}\left(\frac{1}{\eta^{4i+2}}\right) = \sum_{j=i+1}^{4i+2} \frac{m_{4i+2,j}}{S^j} = \sum_{j=1}^{3i+2} \frac{m_{4i+2,i+j}}{S^{i+j}} = \sum_{j=1}^{3i+2} \frac{b_{i,j}}{S^{i+j}}, \quad (2.26)$$

where

$$b_{i,j} = m_{4i+2,i+j}. \quad (2.27)$$

We can write (2.24)

$$\mathbf{H}\left(\left(q \frac{E_9 E_{18}}{E_1 E_2}\right)^{4i}\right) = \sum_{j=1}^{3i} a_{i,j} \left(q^3 \frac{E_9^4 E_{18}^4}{E_3^4 E_6^4}\right)^{i+j}, \quad (2.28)$$

this can be rearranged to

$$\mathbf{H} \left( q^i \left( \frac{E_3 E_6}{E_1 E_2} \right)^{4i-2} \right) = \sum_{j=1}^{3i} a_{i,j} q^{3j} \left( \frac{E_9 E_{18}}{E_3 E_6} \right)^{4j+2} \quad (2.29)$$

or

$$\mathbf{H} \left( q^{i-1} \left( \frac{E_3 E_6}{E_1 E_2} \right)^{4i} \right) = \sum_{j=1}^{3i} a_{i,j} q^{3j-3} \left( \frac{E_9 E_{18}}{E_3 E_6} \right)^{4j}. \quad (2.30)$$

Similarly, we can write (2.26)

$$\mathbf{H} \left( \left( q \frac{E_9 E_{18}}{E_1 E_2} \right)^{4i+2} \right) = \sum_{j=1}^{3i+2} b_{i,j} \left( q^3 \frac{E_9^4 E_{18}^4}{E_3^4 E_6^4} \right)^{i+j}, \quad (2.31)$$

this can be rearranged to

$$\mathbf{H} \left( q^{i-1} \left( \frac{E_3 E_6}{E_1 E_2} \right)^{4i+2} \right) = \sum_{j=1}^{3i+2} b_{i,j} q^{3j-3} \left( \frac{E_9 E_{18}}{E_3 E_6} \right)^{4j-2}. \quad (2.32)$$

### 3. GENERATING FUNCTIONS

In this section, we find some generating functions which are play vital role to prove the congruences.

**Theorem 3.1.** *For each  $\alpha \geq 0$ ,*

$$\sum_{n \geq 0} b_3 \left( 3^{2\alpha+1} n + \frac{3^{2\alpha+1} + 1}{4} \right) q^n = \sum_{i \geq 1} x_{2\alpha,i} q^{i-1} \left( \frac{E_3 E_6}{E_1 E_2} \right)^{4i-2} \quad (3.1)$$

and

$$\sum_{n \geq 0} b_3 \left( 3^{2\alpha+2} n + \frac{3^{2\alpha+3} + 1}{4} \right) q^n = \frac{E_3^2 E_6^2}{E_1^2 E_2^2} \sum_{i \geq 1} x_{2\alpha+1,i} q^{i-1} \left( \frac{E_3 E_6}{E_1 E_2} \right)^{4i}, \quad (3.2)$$

where the coefficient vectors  $\mathbf{x}_\alpha = (x_{\alpha,1}, x_{\alpha,2}, \dots)$  are given by

$$\mathbf{x}_0 = (x_{0,1}, x_{0,2}, x_{0,3}, \dots) = (2, 27, 0, \dots) \quad (3.3)$$

and

$$\mathbf{x}_{\alpha+1} = \mathbf{x}_\alpha A \text{ if } \alpha \text{ is even,} \quad (3.4)$$

$$\mathbf{x}_{\alpha+1} = \mathbf{x}_\alpha B \text{ if } \alpha \text{ is odd,} \quad (3.5)$$

where  $A = (a_{i,j})_{i,j \geq 1}$  and  $B = (b_{i,j})_{i,j \geq 1}$ .

*Proof.* Setting  $\ell = 1$  in (1.14) and using (2.7), we can write

$$\sum_{n=0}^{\infty} b_3(n)q^{n-1} = \frac{E_3^2 E_6^2}{q^3 E_9^2 E_{18}^2} \times \frac{1}{\eta^2}. \quad (3.6)$$

Taking the operator  $\mathbf{H}$  on both sides

$$\sum_{n=0}^{\infty} b_3(3n+1)q^{3n} = \frac{E_3^2 E_6^2}{q^3 E_9^2 E_{18}^2} \times \mathbf{H}\left(\frac{1}{\eta^2}\right). \quad (3.7)$$

Using (2.19), we can rewrite (3.7) as

$$\begin{aligned} \sum_{n=0}^{\infty} b_3(3n+1)q^{3n} &= \frac{E_3^2 E_6^2}{q^3 E_9^2 E_{18}^2} \left( \frac{2}{S} + \frac{3^3}{S^2} \right) \\ &= \frac{E_3^2 E_6^2}{q^3 E_9^2 E_{18}^2} \left( 2q^3 \frac{E_9^4 E_{18}^4}{E_3^4 E_6^4} + 27q^6 \frac{E_9^8 E_{18}^8}{E_3^8 E_6^8} \right). \end{aligned} \quad (3.8)$$

If we now replace  $q^3$  by  $q$ , we arrive at

$$\sum_{n=0}^{\infty} b_3(3n+1)q^n = 2 \frac{E_3^2 E_6^2}{E_1^2 E_2^2} + 27q \frac{E_3^6 E_6^6}{E_1^6 E_2^6}. \quad (3.9)$$

The identity (3.9) is the  $\alpha = 0$  case of (3.1).

Suppose (3.1) holds for some  $\alpha \geq 0$ . Then

$$\sum_{n=0}^{\infty} b_3 \left( 3^{2\alpha+1} n + \frac{3^{2\alpha+1} + 1}{4} \right) q^n = \sum_{i \geq 1} x_{2\alpha,i} q^{i-1} \left( \frac{E_3 E_6}{E_1 E_2} \right)^{4i-2}, \quad (3.10)$$

which is equivalent to

$$\sum_{n=0}^{\infty} b_3 \left( 3^{2\alpha+1} n + \frac{3^{2\alpha+1} + 1}{4} \right) q^{n-2} = \sum_{i \geq 1} x_{2\alpha,i} q^{i-3} \left( \frac{E_3 E_6}{E_1 E_2} \right)^{4i-2}. \quad (3.11)$$

Applying the operator  $\mathbf{H}$  to (3.11), we find that

$$\begin{aligned}
& \sum_{n=0}^{\infty} b_3 \left( 3^{2\alpha+1}(3n+2) + \frac{3^{2\alpha+1}+1}{4} \right) q^{3n} \\
&= q^{-3} \sum_{i \geq 1} x_{2\alpha,i} \mathbf{H} \left( q^i \left( \frac{E_3 E_6}{E_1 E_2} \right)^{4i-2} \right) \\
&= q^{-3} \sum_{i \geq 1} x_{2\alpha,i} \sum_{j=1}^{3i} a_{i,j} q^{3j} \left( \frac{E_9 E_{18}}{E_3 E_6} \right)^{4j+2} \\
&= \frac{E_9^2 E_{18}^2}{E_3^2 E_6^2} \sum_{j \geq 1} \left( \sum_{i \geq 1} x_{2\alpha,i} a_{i,j} \right) q^{3j-3} \left( \frac{E_9 E_{18}}{E_3 E_6} \right)^{4j} \\
&= \frac{E_9^2 E_{18}^2}{E_3^2 E_6^2} \sum_{j \geq 1} x_{2\alpha+1,j} q^{3j-3} \left( \frac{E_9 E_{18}}{E_3 E_6} \right)^{4j}.
\end{aligned}$$

If we now replace  $q^3$  by  $q$ , we obtain

$$\sum_{n=0}^{\infty} b_3 \left( 3^{2\alpha+2} n + \frac{3^{2\alpha+3}+1}{4} \right) q^n = \frac{E_9^2 E_{18}^2}{E_3^2 E_6^2} \sum_{j \geq 1} x_{2\alpha+1,j} q^{j-1} \left( \frac{E_9 E_{18}}{E_3 E_6} \right)^{4j}, \quad (3.12)$$

which is (3.2).

Now suppose (3.2) holds for some  $\alpha \geq 0$ . Then

$$\sum_{n=0}^{\infty} b_3 \left( 3^{2\alpha+2} n + \frac{3^{2\alpha+3}+1}{4} \right) q^n = \frac{E_9^2 E_{18}^2}{E_3^2 E_6^2} \sum_{i \geq 1} x_{2\alpha+1,i} q^{i-1} \left( \frac{E_9 E_{18}}{E_3 E_6} \right)^{4i}, \quad (3.13)$$

which is same as

$$\sum_{n=0}^{\infty} b_3 \left( 3^{2\alpha+2} n + \frac{3^{2\alpha+3}+1}{4} \right) q^n = \sum_{i \geq 1} x_{2\alpha+1,i} q^{i-1} \left( \frac{E_9 E_{18}}{E_3 E_6} \right)^{4i+2}. \quad (3.14)$$

Applying the operator  $\mathbf{H}$  to (3.14), we find that

$$\begin{aligned}
& \sum_{n=0}^{\infty} b_3 \left( 3^{2\alpha+2}(3n) + \frac{3^{2\alpha+3} + 1}{4} \right) q^{3n} \\
&= \sum_{i \geq 1} x_{2\alpha+1,i} \mathbf{H} \left( q^{i-1} \left( \frac{E_3 E_6}{E_1 E_2} \right)^{4i+2} \right) \\
&= \sum_{i \geq 1} x_{2\alpha+1,i} \sum_{j=1}^{3i+2} b_{i,j} q^{3j-3} \left( \frac{E_9 E_{18}}{E_3 E_6} \right)^{4j-2} \\
&= \sum_{j \geq 1} \left( \sum_{i \geq 1} x_{2\alpha+1,i} b_{i,j} \right) q^{3j-3} \left( \frac{E_9 E_{18}}{E_3 E_6} \right)^{4j-2} \\
&= \sum_{j \geq 1} x_{2\alpha+2,j} q^{3j-3} \left( \frac{E_9 E_{18}}{E_3 E_6} \right)^{4j-2}.
\end{aligned}$$

If we now replace  $q^3$  by  $q$ , we find

$$\sum_{n=0}^{\infty} b_3 \left( 3^{2\alpha+3} n + \frac{3^{2\alpha+3} + 1}{4} \right) q^n = \sum_{j \geq 1} x_{2\alpha+2,j} q^{j-1} \left( \frac{E_3 E_6}{E_1 E_2} \right)^{4j-2}$$

which is (3.1) with  $\alpha + 1$  in place of  $\alpha$ . This completes the proof by induction of (3.1) and (3.2).  $\square$

**Theorem 3.2.** *For each  $\alpha \geq 0$ ,*

$$\sum_{n \geq 0} b_{3^2} \left( 3^{2\alpha+1} n + \frac{3^{2\alpha+1} + 1}{4} \right) q^n = \sum_{i \geq 1} x_{2\alpha,i} q^{i-1} \left( \frac{E_3 E_6}{E_1 E_2} \right)^{4i} \quad (3.15)$$

and

$$\sum_{n \geq 0} b_{3^2} \left( 3^{2\alpha+2} n + \frac{3^{2\alpha+3} + 1}{4} \right) q^n = \sum_{i \geq 1} x_{2\alpha+1,i} q^{i-1} \left( \frac{E_3 E_6}{E_1 E_2} \right)^{4i}, \quad (3.16)$$

where the coefficient vectors  $\mathbf{x}_\alpha = (x_{\alpha,1}, x_{\alpha,2}, \dots)$  are given by

$$\mathbf{x}_0 = (x_{0,1}, x_{0,2}, x_{0,3}, \dots) = (2, 27, 0, \dots) \quad (3.17)$$

and

$$\mathbf{x}_{\alpha+1} = \mathbf{x}_\alpha A \text{ if } \alpha \text{ is even,} \quad (3.18)$$

$$\mathbf{x}_{\alpha+1} = \mathbf{x}_\alpha B \text{ if } \alpha \text{ is odd,} \quad (3.19)$$

where  $A = (a_{i,j})_{i,j \geq 1}$  and  $B = (b_{i,j})_{i,j \geq 1}$ .

*Proof.* Setting  $\ell = 2$  in identity (1.14) is the  $\alpha = 0$  case of (3.15).

Suppose (3.15) holds for some  $\alpha \geq 0$ . Then

$$\sum_{n=0}^{\infty} b_{3^2} \left( 3^{2\alpha+1} n + \frac{3^{2\alpha+1} + 1}{4} \right) q^n = \sum_{i \geq 1} x_{2\alpha,i} q^{i-1} \left( \frac{E_3 E_6}{E_1 E_2} \right)^{4i}, \quad (3.20)$$

which is equivalent to

$$\sum_{n=0}^{\infty} b_{3^2} \left( 3^{2\alpha+1} n + \frac{3^{2\alpha+1} + 1}{4} \right) q^{n-2} = \sum_{i \geq 1} x_{2\alpha,i} q^{i-3} \left( \frac{E_3 E_6}{E_1 E_2} \right)^{4i}. \quad (3.21)$$

Applying the operator  $\mathbf{H}$  to (3.11), we find that

$$\begin{aligned} & \sum_{n=0}^{\infty} b_{3^2} \left( 3^{2\alpha+1} (3n+2) + \frac{3^{2\alpha+1} + 1}{4} \right) q^{3n} \\ &= \sum_{i \geq 1} x_{2\alpha,i} \mathbf{H} \left( q^{i-3} \left( \frac{E_3 E_6}{E_1 E_2} \right)^{4i} \right) \\ &= \sum_{i \geq 1} x_{2\alpha,i} \sum_{j=1}^{3i} a_{i,j} q^{3j-3} \left( \frac{E_9 E_{18}}{E_3 E_6} \right)^{4j} \\ &= \sum_{j \geq 1} \left( \sum_{i \geq 1} x_{2\alpha,i} a_{i,j} \right) q^{3j-3} \left( \frac{E_9 E_{18}}{E_3 E_6} \right)^{4j} \\ &= \sum_{j \geq 1} x_{2\alpha+1,j} q^{3j-3} \left( \frac{E_9 E_{18}}{E_3 E_6} \right)^{4j}. \end{aligned}$$

If we now replace  $q^3$  by  $q$ , we obtain

$$\sum_{n=0}^{\infty} b_{3^2} \left( 3^{2\alpha+2} n + \frac{3^{2\alpha+3} + 1}{4} \right) q^n = \sum_{j \geq 1} x_{2\alpha+1,j} q^{j-1} \left( \frac{E_3 E_6}{E_1 E_2} \right)^{4j}, \quad (3.22)$$

which is (3.16).

Now suppose (3.16) holds for some  $\alpha \geq 0$ . Then

$$\sum_{n=0}^{\infty} b_{3^2} \left( 3^{2\alpha+2} n + \frac{3^{2\alpha+3} + 1}{4} \right) q^n = \sum_{i \geq 1} x_{2\alpha+1,i} q^{i-1} \left( \frac{E_3 E_6}{E_1 E_2} \right)^{4i}. \quad (3.23)$$

Applying the operator  $\mathbf{H}$  to (3.23), we find that

$$\begin{aligned} & \sum_{n=0}^{\infty} b_{3^2} \left( 3^{2\alpha+2}(3n) + \frac{3^{2\alpha+3} + 1}{4} \right) q^{3n} \\ &= \sum_{i \geq 1} x_{2\alpha+1,i} \mathbf{H} \left( q^{i-1} \left( \frac{E_3 E_6}{E_1 E_2} \right)^{4i} \right) \\ &= \sum_{i \geq 1} x_{2\alpha+1,i} \sum_{j=1}^{3i+2} b_{i,j} q^{3j-3} \left( \frac{E_9 E_{18}}{E_3 E_6} \right)^{4j} \\ &= \sum_{j \geq 1} \left( \sum_{i \geq 1} x_{2\alpha+1,i} b_{i,j} \right) q^{3j-3} \left( \frac{E_9 E_{18}}{E_3 E_6} \right)^{4j} \\ &= \sum_{j \geq 1} x_{2\alpha+2,j} q^{3j-3} \left( \frac{E_9 E_{18}}{E_3 E_6} \right)^{4j}. \end{aligned}$$

If we now replace  $q^3$  by  $q$ , we have

$$\sum_{n=0}^{\infty} b_9 \left( 3^{2\alpha+3} n + \frac{3^{2\alpha+3} + 1}{4} \right) q^n = \sum_{j \geq 1} x_{2\alpha+2,j} q^{j-1} \left( \frac{E_3 E_6}{E_1 E_2} \right)^{4j}$$

which is (3.15) with  $\alpha + 1$  in place of  $\alpha$ . This completes the proof by induction of (3.15) and (3.16).  $\square$

#### 4. CONGRUENCES

Let  $\nu(N)$  be the largest power of 3 that divides  $N$ . Note that  $\nu(0) = +\infty$ .

**Proof of the Theorem 1.1.** It follows from (2.22) and (2.23) that

$$\nu(m_{i,j}) \geq 3j - i - 1, \quad (4.1)$$

and then follows from (2.25), (2.27) and (4.1) that

$$\nu(a_{i,j}) \geq 3(i + j) - 4i - 1 = 3j - i - 1 \quad (4.2)$$

and

$$\nu(b_{i,j}) \geq 3(i + j) - (4i + 2) - 1 = 3j - i - 3. \quad (4.3)$$

But note that

$$\nu(a_{1,j}) = \nu(m_{4,j+1}) \geq 2j \quad (4.4)$$

and

$$\nu(b_{1,j}) = \nu(m_{6,j+1}) \geq 2(j-1) \quad (4.5)$$

It is not hard to show that

$$\nu(x_{2\alpha,j}) \geq 2\alpha + 2(j-1) \quad (4.6)$$

and

$$\nu(x_{2\alpha+1,j}) \geq 2\alpha + 2j. \quad (4.7)$$

The identity (4.6) is true for  $\alpha = 0$ , by (3.3).

Suppose (4.6) is true for some  $\alpha \geq 0$ . Then

$$\begin{aligned} \nu(x_{2\alpha+1,j}) &\geq \min_{i \geq 1} (\nu(x_{2\alpha,i}) + \nu(a_{i,j})) \\ &= \nu(x_{2\alpha,1}) + \nu(a_{1,j}) \\ &\geq 2\alpha + 2j, \end{aligned}$$

which is (4.7).

Now suppose (4.7) is true for all  $\alpha \geq 0$ . Then

$$\begin{aligned} \nu(x_{2\alpha+2,j}) &\geq \min_{i \geq 1} (\nu(x_{2\alpha+1,i}) + \nu(b_{i,j})) \\ &= \nu(x_{2\alpha+1,1}) + \nu(b_{1,j}) \\ &\geq 2\alpha + 2 + 2(j-1) \\ &\geq 2\alpha + 2j, \end{aligned}$$

which is (4.6) with  $\alpha + 1$  in place of  $\alpha$ . This completes the proof by induction of (4.6) and (4.7).

The congruence (1.15) follows from (3.1) together with (4.6), and the congruence (1.16) follows from (3.2) together with (4.7). Similarly the congruence (1.17) follows from (3.15) together with (4.6), and the congruence (1.18) follows from (3.16) together with (4.7).

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