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EVEN-ODD BALANCING AND COBALANCING NUMBERS

S.G. RAYAGURU, R.K. DAVALA, AND G.K. PANDA

Abstract. In this paper, we solve the Diophantine equation

$$2+4+\cdots+2k=(2n+1)+(2n+3)+\cdots+(2n+2r-1)$$

for $k\in\{n-1,\,n\}$. Further, we study some properties and identities involving these numbers. We also establish some relations of these numbers with Pell, associated Pell, balancing and cobalancing numbers.

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1. Introduction

The concept of balancing numbers was first introduced by Behera and Panda [1] in the year 1999. They call a natural number n a balancing number if

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r)$$

for some natural number r, which they call the balancer corresponding to n. Subsequently, Panda and Ray [9] defined cobalancing numbers with little modification to the defining equation of balancing numbers. A natural number n is called a cobalancing number if

$$1+2+\cdots+n=(n+1)+(n+2)+\cdots+(n+r)$$

for some natural number r, called the cobalancer corresponding to n. Panda [10], further generalized the concept of balancing and cobalancing number to define sequence balancing and cobalancing numbers. For any sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers, a_m is a sequence balancing or sequence cobalancing number according as $a_1+a_2+\cdots+a_{m-1}=a_{m+1}+a_{m+2}+\cdots+a_{m+r}$ or $a_1+a_2+\cdots+a_m=a_{m+1}+a_{m+2}+\cdots+a_{m+r}$ for some natural number r. If $a_n=2n$ then the sequence balancing numbers are $2B_n$, where B_n denotes the nth balancing number. Similarly, the sequence balancing numbers corresponding to the sequence $a_n=n/2$ are $B_n/2$. Panda also proved that in the sequence of odd natural numbers, the sequence of sequence balancing numbers is given by $\{B_{n+1}+B_n\}_{n=1}^{\infty}$ and there does not exist any sequence cobalancing number. Panda and Panda [12], defined almost balancing numbers as solution of the Diophantine equation

$$|\{(n+1)+(n+2)+\cdots+(n+r)\}-\{1+2+\cdots+(n-1)\}|=1.$$

Subsequently, Davala and Panda [3, 4] studied the Diophantine equations

$$|\{(n+1)+(n+2)+\cdots+(n+r)\}-\{1+2+\cdots+(n-1)\}|=D \tag{1.1}$$

and

$$|\{(n+1)+(n+2)+\cdots+(n+r)\}-\{1+2+\cdots+n\}|=D. \tag{1.2}$$

The solutions of (1.1) are called subbalancing and superbalancing numbers, where as the solutions of (1.2) are called subcobalancing and supercobalancing numbers respectively.

"Dedicated to Prof. Chandrashekar Adiga on his 62nd Birthday".

In this paper, we define even-odd balancing and even-odd cobalancing numbers and establish their existence. We also establish some relations with the sequence of Pell, associated Pell, balancing and cobalancing numbers. Furthermore, we examine some ratio formulas and identities involving these numbers.

2. Preliminaries

A brief study on balancing and cobalancing numbers have been done in [14]. Further, identities involving Pell, associated Pell, balancing, Lucas-balancing and cobalancing numbers have been established [2, 8, 11, 13]. The Binet formulas of these numbers are given by:

$$P_n = \frac{\alpha_1^n - \alpha_2^n}{2\sqrt{2}}, q_n = \frac{\alpha_1^n + \alpha_2^n}{2}, B_n = \frac{\alpha_1^{2n} - \alpha_2^{2n}}{4\sqrt{2}}, C_n = \frac{\alpha_1^{2n} + \alpha_2^{2n}}{2}, b_n = \frac{\alpha_1^{2n-1} - \alpha_2^{2n-1}}{4\sqrt{2}} - \frac{1}{2},$$

where $\alpha_1 = 1 + \sqrt{2}$ and $\alpha_2 = 1 - \sqrt{2}$.

The identities in the following lemmas are useful in the forthcoming sections.

Lemma 2.1. *If k and n are natural numbers, then*

(a) $P_{n+k}^2 - P_{n-k}^2 = 4B_kB_n$

(b) $2(P_{n+k}^2 + P_{n-k}^2) + (-1)^{n-k} = C_kC_n$

(c) $P_{n+k}P_{n+k-1} - P_{n-k}P_{n-k-1} = 2B_k(B_n - B_{n-1})$

(d) $2P_{n+k}P_{n+k-1} + 2P_{n-k}P_{n-k-1} - (-1)^{n-k} = C_k(B_n + B_{n-1})$.

Proof. Using the Binet formula of Pell and balancing numbers and using $\alpha_1 - \alpha_2 = 2\sqrt{2}$, $\alpha_1\alpha_2 = -1$, we have

$$\begin{aligned}(a)\ P_{n+k}^2 - P_{n-k}^2 &= \frac{1}{8}[(\alpha_1^{n+k} - \alpha_2^{n+k})^2 - (\alpha_1^{n-k} - \alpha_2^{n-k})^2] \\ &= \frac{1}{8}[\alpha_1^{2n+2k} + \alpha_2^{2n+2k} - \alpha_1^{2n-2k} - \alpha_2^{2n-2k} + 2(\alpha_1\alpha_2)^{n-k}(1 - (\alpha_1\alpha_2)^{2k})] \\ &= \frac{1}{8}[\alpha_1^{2n+2k} + \alpha_2^{2n+2k} - \alpha_1^{2n-2k}(\alpha_1\alpha_2)^{2k} - \alpha_2^{2n-2k}(\alpha_1\alpha_2)^{2k}] \\ &= \frac{1}{8}[\alpha_1^{2n+2k} + \alpha_2^{2n+2k} - \alpha_1^{2n}\alpha_2^{2k} - \alpha_2^{2n}\alpha_1^{2k}] \\ &= \frac{1}{8}[(\alpha_1^{2n} - \alpha_2^{2n})(\alpha_1^{2k} - \alpha_2^{2k})] = 4B_nB_k\end{aligned}$$

and

$$\begin{aligned}&(d)\ 2P_{n+k}P_{n+k-1} + 2P_{n-k}P_{n-k-1} - (-1)^{n-k} \\ &= \frac{1}{4}[(\alpha_1^{n+k} - \alpha_2^{n+k})(\alpha_1^{n+k-1} - \alpha_2^{n+k-1}) + (\alpha_1^{n-k} - \alpha_2^{n-k})(\alpha_1^{n-k-1} - \alpha_2^{n-k-1})] - (-1)^{n-k} \\ &= \frac{1}{4}[\alpha_1^{2n+2k-1} + \alpha_2^{2n+2k-1} + \alpha_1^{2n-2k-1} + \alpha_2^{2n-2k-1}] \\ &= \frac{1}{4}[\alpha_1^{2n+2k-1} + \alpha_2^{2n+2k-1} + \alpha_1^{2n-1}\alpha_2^{2k} + \alpha_2^{2n-1}\alpha_1^{2k}] \\ &= \frac{1}{4}[(\alpha_1^{2n-1} + \alpha_2^{2n-1})(\alpha_1^{2k} + \alpha_2^{2k})] \\ &= \frac{1}{8\sqrt{2}}[(\alpha_1 - \alpha_2)(\alpha_1^{2n-1} + \alpha_2^{2n-1})(\alpha_1^{2k} + \alpha_2^{2k})] \\ &= \frac{1}{8\sqrt{2}}[(\alpha_1^{2k} + \alpha_2^{2k})(\alpha_1^{2n} - \alpha_2^{2n} + \alpha_1^{2n-2} - \alpha_2^{2n-2}] = C_k(B_n + B_{n-1}).\end{aligned}$$

The proof of (b) and (c) follows similarly. □

Lemma 2.2. *If k and n are natural numbers, then*

(a) $P_n + P_{n-1} = q_n,\ P_n - P_{n-1} = q_{n-1}$
(b) $2B_n = P_{2n},\ C_n = q_{2n},\ B_n = P_n q_n$
(c) $4B_n(B_n \pm 1) + 1 = (P_{2n} \pm 1)^2$
(d) $q_n + q_{n-1} = 2P_n,\ q_n - q_{n-1} = 2P_{n-1}$
(e) $b_{2n} = P_{2n}q_{2n-1},\ b_{2n+1} = P_{2n}q_{2n+1}$
(f) $4b_n b_{n+1} + 1 = P_{2n-1}^2$.

These identities can be obtained by using the recurrence relations and Binet forms of Pell, associated Pell, balancing and cobalancing numbers.

3. Even-Odd Balancing Numbers and Even-Odd Balancers

Definition 3.1. *We call a natural number n, an even-odd balancing number, if it satisfies the Diophantine equation*

$$2 + 4 + \cdots + (2n-2) = (2n+1) + (2n+3) + \cdots + (2n+2r-1) \tag{3.1}$$

for some natural number r, which we call an even-odd balancer corresponding to n.

Example 3.2. *The following summations hold.*

(i) $2 + 4 + \cdots + (2\cdot 25 - 2) = (2\cdot 25 + 1) + (2\cdot 25 + 3) + \cdots + (2\cdot 25 + 2\cdot 10 - 1) = 600$
(ii) $2+4+\cdots+(2\cdot 841-2) = (2\cdot 841+1)+(2\cdot 841+3)+\cdots+(2\cdot 841+2\cdot 348-1) = 706440$

The above example suggests that 25 and 841 are even-odd balancing numbers with 10 and 348 as corresponding even-odd balancers.

3.1. **Even-Odd Balancing Numbers.** Solving Eq. (3.1) for n and r, we get

$$n = \frac{1}{2}[(1+2r) + \sqrt{8r^2+4r+1}\,] \quad \text{and} \quad r = -n + \sqrt{2n^2-n}. \tag{3.2}$$

Thus, n is an even-odd balancing number with even-odd balancer r if and only if $2n^2 - n$ and $8r^2 + 4r + 1$ are perfect squares.

We denote the kth even integer by $E_k = 2k$, kth odd integer by $O_k = 2k-1$, $k = 1, 2, \ldots$ and the sequence of even-odd balancing numbers by $\{EOB_n\}_{n\geq 1}$. In view of (2.1), $2n^2 - n = \frac{2n(2n-1)}{2}$ is a perfect square and hence, $\sqrt{2n^2-n}$ is a balancing number, say $2n^2 - n = B_m^2$, which imply $n = \frac{1\pm C_m}{4}$. Since n is a positive integer and $C_m \equiv (-1)^m \pmod 4$, it follows that m is odd. Letting $m = 2l-1$, $EOB_n = \frac{1+C_{2l-1}}{4}$. Using the recurrence $C_{n+1} = 6C_n - C_{n-1}$ of Lucas-balancing numbers, one can verify that $C_{2l+1} = 34C_{2l-1} - C_{2l-3}$. Thus, the sequence of even-odd balancing numbers $\{EOB_n\}_{n\geq 1}$ satisfy the non homogeneous binary recurrence

$$EOB_{n+1} = 34EOB_n - EOB_{n-1} - 8 \tag{3.3}$$

with $EOB_1 = 25$ and $EOB_2 = 841$.

Note $2n^2 - n$ is a perfect square for $n = 1$, we accept $EOB_0 = 1$. With $\alpha_1 = 1 + \sqrt{2}$ and $\alpha_2 = 1 - \sqrt{2}$, the Binet formula for the even-odd balancing numbers can be written as

$$EOB_n = \frac{\alpha_1^{4n+2} + \alpha_2^{4n+2}}{8} + \frac{1}{4}.$$

The Binet form of even-odd balancing numbers imply

$$EOB_n = \frac{1}{4}(q_{4n+2} + 1) = P_{2n}P_{2n+2} + 1 = P_{2n+1}^2 = (B_{n+1} - B_n)^2 = 4B_nB_{n+1} + 1.$$

Theorem 3.3. *The generating function for the sequence EOB_n is given by*

$$g(s) = \frac{1 - 10s + s^2}{(1-s)(1-34s+s^2)}.$$

Proof. Let the generating function of EOB_n be $g(s) = \sum_{n=0}^{\infty} EOB_n s^n$. From (3.3), we have

$$\sum_{n=0}^{\infty} EOB_{n+2}s^{n+2} = 34\sum_{n=0}^{\infty} EOB_{n+1}s^{n+2} - \sum_{n=0}^{\infty} EOB_n s^{n+2} - 8\sum_{n=0}^{\infty} s^{n+2}$$
$$\sum_{n=0}^{\infty} EOB_{n+2}s^{n+2} = 34s\sum_{n=0}^{\infty} EOB_{n+1}s^{n+1} - s^2\sum_{n=0}^{\infty} EOB_n s^{n} - 8s^2\sum_{n=0}^{\infty} s^{n}$$
$$g(s) - EOB_1 s - EOB_0 = 34s(g(s) - EOB_0) - s^2 g(s) - \frac{8s^2}{1-s}$$
$$g(s)(1-34s+s^2) = 1 - 9s - \frac{8s^2}{1-s}$$
$$g(s) = \frac{1-10s+s^2}{(1-s)(1-34s+s^2)}.$$

□

Theorem 3.4. *If n is any non negative integer, then*

$$\sum_{j=1}^{n} EOB_j = \frac{1}{32}(EOB_{n+2} - 33EOB_{n+1} + 8n - 16).$$

Proof. The proof of this theorem follows from the recurrence relation of even-odd balancing numbers by using method of mathematical induction. Hence, we omit the proof. □

The folllowing theorem is the Catalan's Identity for EOB_n.

Theorem 3.5. *If n and r are positive integers, then the even-odd balancing numbers satisfy*

$$EOB_{n+r} \cdot EOB_{n-r} = (EOB_n + 4B_r^2)^2.$$

In particular, $EOB_{n+1} \cdot EOB_{n-1} = (EOB_n + 4)^2$.

The following theorem generates even-odd balancing numbers.

Theorem 3.6. *If x is an even-odd balancing number and m is any non negative integer, then*

$$f_m(x) = C_{2m}x + 2B_{2m}\sqrt{2x^2 - x} - 4B_m^2$$

is also an even-odd balancing number. Moreover, if $x = EOB_n$, then $f_m(x) = EOB_{n+m}$.

Proof. Using the Binet formula of even-odd balancing numbers along with the Binet formula of balancing and Lucas-balancing number, we have

$$
\begin{aligned}
&f_m(EOB_n)\\
&= C_{2m}EOB_n + 2B_{2m}\sqrt{2EOB_n^2 - EOB_n} - 4B_m^2\\
&= C_{2m}EOB_n + 2B_{2m}B_{2n+1} - 4B_m^2\\
&= \left(\frac{\alpha_1^{4m}+\alpha_2^{4m}}{2}\right)\left(\frac{\alpha_1^{4n+2}+\alpha_2^{4n+2}}{8}+\frac{1}{4}\right) + 2\left(\frac{\alpha_1^{4m}-\alpha_2^{4m}}{4\sqrt{2}}\right)\left(\frac{\alpha_1^{4n+2}-\alpha_2^{4n+2}}{4\sqrt{2}}\right) - 4\left(\frac{\alpha_1^{2m}-\alpha_2^{2m}}{4\sqrt{2}}\right)^2\\
&= \frac{1}{16}\Big((\alpha_1^{4m}+\alpha_2^{4m})(\alpha_1^{4n+2}+\alpha_2^{4n+2}+2) + (\alpha_1^{4m}-\alpha_2^{4m})(\alpha_1^{4n+2}-\alpha_2^{4n+2}) - 2(\alpha_1^{4m}+\alpha_2^{4m}-2)\Big)\\
&= \frac{1}{16}\Big(2\alpha_1^{4n+4m+2} + 2\alpha_2^{4n+4m+2} + 4\Big),
\end{aligned}
$$

this completes the proof. □

Theorem 3.7. *Even-odd balancing numbers are neither triangular nor perfect.*

Proof. If EOB_n is triangular, then $8EOB_n + 1$ is a perfect square and thus, $\sqrt{EOB_n} = B_{n+1} - B_n$ is a balancing number say B_t for some $t \geq 1$. Since $B_n < B_{n+1} - B_n < B_{n+1}$, the difference of two consecutive balancing number is not a balancing number and so $B_t = B_{n+1} - B_n$ has no solution for $t, n \geq 1$. Hence, no even-odd balancing number is a triangular number.

Furthermore, $EOB_n = (B_{n+1} - B_n)^2$ is odd perfect square and perfect squares are not perfect numbers. □

Theorem 3.8. *The even-odd balancing numbers satisfy*

(a) $\dfrac{EOB_{m+2n+1} - EOB_m}{EOB_{m+n+1} - EOB_{m+n}} = B_{2n+1}\ ;\ n \geq 1$

(b) $\dfrac{EOB_{m+3n} - EOB_m}{EOB_{m+2n} - EOB_{m+n}} = B_{2n+1} - B_{2n-1} + 1\ ;\ n \geq 1$

(c) $\dfrac{2EOB_{m+3n} + 2EOB_m - 1}{2EOB_{m+2n} + 2EOB_{m+n} - 1} = B_{2n+1} - B_{2n-1} - 1\ ;\ n \geq 1$

(d) $\dfrac{6EOB_{m+2n+1} + 6EOB_m - 3}{2EOB_{m+n+1} + 2EOB_{m+n} - 1} = C_{2n+1}\ ;\ n \geq 1.$

Proof. Since $EOB_n = P_{2n+1}^2$ and P_n satisfies $P_{n+k}^2 - P_{n-k}^2 = 4B_kB_n$ (see Lemma 2.1), we have

$$
\begin{aligned}
\frac{EOB_{m+2n+1} - EOB_m}{EOB_{m+n+1} - EOB_{m+n}} &= \frac{P^2_{(2m+2n+2)+(2n+1)} - P^2_{(2m+2n+2)-(2n+1)}}{P^2_{(2m+2n+2)+1} - P^2_{(2m+2n+2)-1}}\\
&= \frac{B_{2m+2n+2}B_{2n+1}}{B_{2m+2n+2}B_1} = B_{2n+1},
\end{aligned}
$$

this completes the proof of (a).

The proof of (b) is similar to that of (a) and hence omitted. Similarly, the proofs of (c) and (d) follows from the identity $2P_{n+k}^2 + 2P_{n-k}^2 + (-1)^{n-k} = C_kC_n$, $C_{n+k} + C_{n-k} = 2C_kC_n$, $B_{n+1} - B_{n-1} = 2C_n$. □

3.2. **Even-Odd Balancers.** The even-odd balancers satisfy many results similar to that of even-odd balancing numbers. Now, r is an even-odd balancer if and only if $8r^2+4r+1=z^2$. Equivalently, $(4r+1)^2-2z^2=-1$, which is a Pell's equation. With the fundamental solution $(1,1)$, the total solution of $(4r+1)^2-2z^2=-1$ is given by

$$4r_n+1=\frac{1}{2}(\alpha_1^{4n+1}+\alpha_2^{4n+1}),\ z_n=\frac{1}{2\sqrt{2}}(\alpha_1^{4n+1}-\alpha_2^{4n+1})$$

where $\alpha_1=1+\sqrt{2}$, $\alpha_2=1-\sqrt{2}$ and thus

$$r_n=EOR_n=\frac{1}{8}(\alpha_1^{4n+1}+\alpha_2^{4n+1}-2),$$

which is the Binet formula for the even-odd balancers.

As like $EOB_0=1$, we consider $EOR_0=0$. Further, the even-odd balancers satisfy the non homogeneous linear binary recurrence

$$EOR_{n+1}=34EOR_n-EOR_{n-1}+8 \tag{3.4}$$

with $EOR_0=0$ and $EOR_1=10$. Using the Binet formula of even-odd balancers, we have

$$EOR_n=\frac{1}{4}(q_{4n+1}-1)=P_{2n}P_{2n+1}.$$

Theorem 3.9. *The generating function for the sequence EOR_n is given by*

$$g(s)=\frac{2s(5-s)}{(1-s)(1-34s+s^2)}.$$

Theorem 3.10. *For every non negative integer n,*

$$\sum_{j=1}^{n}EOR_j=\frac{1}{32}(EOR_{n+2}-33EOR_{n+1}-8n-18).$$

Proof. The proof of this theorem follows from the recurrence relation of even-odd balancers by using method of mathematical induction. Hence, we omit the proof. □

Theorem 3.11. *For every positive integer n and r, the even-odd balancers satisfy*

$$EOR_{n+r}\cdot EOR_{n-r}+B_{2r}^2=(EOR_n-4B_r^2)^2.$$

Proof. Using the Binet formula of even-odd balancers, the proof is similar to that of Theorem 3.5. □

Theorem 3.12. *If x is an even-odd balancer and m is any non negative integer, then*

$$f_m(x)=C_{2m}x+B_{2m}\sqrt{8x^2+4x+1}+4B_m^2$$

is also an even-odd balancer. Moreover, if $x=EOR_n$, then $f_m(x)=EOR_{n+m}$.

Proof. Using the Binet formula of even-odd balancers, the proof is similar to that of Theorem 3.6. □

Theorem 3.13. *The even-odd balancers are not perfect.*

Proof. Since $EOR_n=P_{2n}P_{2n+1}$, every even-odd balancer is even. Let us assume that, EOR_n is perfect. Hence, $EOR_n=2^{p-1}(2^p-1)$ with p and (2^p-1) both primes. Since $EOR_1=10$ is not a perfect number, r must be greater than 1. Now,

$$EOR_n=P_{2n}P_{2n+1}=2P_nq_nP_{2n+1}\implies P_nq_nP_{2n+1}=2^{p-2}(2^p-1).$$

Since $\gcd(P_n, q_n) = 1$ and q_n, P_{2n+1} are odd, either $q_n = 1$ or $P_{2n+1} = 1$. Thus, $r = 1$, which contradicts the fact that $r > 1$. $\square$

Theorem 3.14. *The even-odd balancers satisfy*

$$\text{(a)}\ \frac{EOR_{m+2n+1} - EOR_m}{EOR_{m+n+1} - EOR_{m+n}} = B_{2n+1}\ ;\ n \geq 1$$

$$\text{(b)}\ \frac{EOR_{m+3n} - EOR_m}{EOR_{m+2n} - EOR_{m+n}} = B_{2n+1} - B_{2n-1} + 1\ ;\ n \geq 1$$

$$\text{(c)}\ \frac{2EOR_{m+3n} + 2EOR_m + 1}{2EOR_{m+2n} + 2EOR_{m+n} + 1} = B_{2n+1} - B_{2n-1} - 1\ ;\ n \geq 1$$

$$\text{(d)}\ \frac{6EOR_{m+2n+1} + 6EOR_m + 3}{2EOR_{m+n+1} + 2EOR_{m+n} + 1} = C_{2n+1}\ ;\ n \geq 1.$$

Proof. Since $EOR_n = P_{2n}P_{2n+1}$ and P_n satisfies $P_{n+k}P_{n+k-1} - P_{n-k}P_{n-k-1} = 2B_k(B_n - B_{n-1})$ (see Lemma 2.1), we have

$$\begin{aligned}\frac{EOR_{m+2n+1} - EOR_m}{EOR_{m+n+1} - EOR_{m+n}} &= \frac{P_{2m+4n+3}P_{2m+4n+2} - P_{2m+1}P_{2m}}{P_{2m+2n+3}P_{2m+2n+2} - P_{2m+2n+1}P_{2m+2n}} \\ &= \frac{(B_{2m+2n+2} - B_{2m+2n+1})B_{2n+1}}{(B_{2m+2n+2} - B_{2m+2n+1})B_1} = B_{2n+1},\end{aligned}$$

this completes the proof of (a).

The proof of (b) is similar to that of (a) and hence omitted. Similarly, the proofs of (c) and (d) follows from the identity $2P_{n+k}P_{n+k-1} + 2P_{n-k}P_{n-k-1} - (-1)^{n-k} = C_k(B_n + B_{n-1})$, $C_{n+k} + C_{n-k} = 2C_kC_n$, $B_{n+1} - B_{n-1} = 2C_n$. $\square$

4. Some Links Between Even-Odd Balancing(Cobalancing) Numbers with Even-Odd Balancers(Cobalancers)

Definition 3.1 can be modified slightly to define even-odd cobalancing number.

Definition 4.1. *We call a natural number n, an even-odd cobalancing number, if it satisfies the Diophantine equation*

$$2 + 4 + \cdots + 2n = (2n+1) + (2n+3) + \cdots + (2n+2r-1) \tag{4.1}$$

for some natural number r, which we call an even-odd cobalancer corresponding to n.

Many results similar to even-odd balancing numbers and even-odd balancers can be established. One can check that, the binary recurrence relation for even-odd cobalancing numbers and even-odd cobalancers are same as the recurrence relation for even-odd balancers and even-odd balancing numbers respectively with initial terms $EOb_0 = 0, EOb_1 = 4$ and $EOr_0 = 0, EOr_1 = 2$.

Now, we represent these numbers in terms of Pell, associated Pell, balancing and cobalancing numbers.

Theorem 4.2. *If n is any non negative integer, then*

(a) $EOB_n = (q_{4n+2} + 1)/4 = P_{2n+1}^2 = (B_{n+1} - B_n)^2 = (2b_{n+1} + 1)^2$

(b) $EOR_n = (q_{4n+1} - 1)/4 = P_{2n}P_{2n+1} = 2B_n(B_{n+1} - B_n) = 2B_n(2b_{n+1} + 1)$

(c) $EOb_n = (q_{4n} - 1)/4 = P_{2n}^2 = 4B_n^2 = (b_n - b_{n-1})^2$

(d) $EOr_n = (q_{4n-1} + 1)/4 = P_{2n}P_{2n-1} = 2B_n(B_n - B_{n-1}) = 2B_n(2b_n + 1)$.

By looking at the above representation, it is easy to state the following result involving these numbers obtained in the balancing process of even and odd natural numbers.

Corollary 4.3. *The square of a Pell number is either an even-odd balancing number or an even-odd cobalancing number. Further, the product of two consecutive Pell number is either an even-odd balancer or an even-odd cobalancer.*

Corollary 4.4. *The sequence of associated Pell numbers can be written as the union of four disjoint sequences involving even-odd balancing(cobalancing) numbers and even-odd balancers(cobalancers). In other words,*

$$\{q_n\} = \{4EOB_n - 1\} \cup \{4EOR_n + 1\} \cup \{4EOb_n + 1\} \cup \{4EOr_n - 1\}.$$

In the following theorems we establish some relations involving Pell and associated pell numbers with even-odd and odd-even balancing(cobalancing) numbers and even-odd and odd-even balancers(cobalancers).

Theorem 4.5. *If n is any natural number, then*

(a) $P_{2n}^2 = EOb_n$
(b) $q_{2n}^2 = 2EOb_n + 1$
(c) $P_{2n}q_{2n-1} = EOb_n - EOr_n$
(d) $P_{2n+1}q_{2n} = EOB_n - EOR_n$
(e) $P_{2n-1}q_{2n} = EOb_n - EOr_n + 1$
(f) $P_{2n}P_{2n-1} = EOR_n - 2EOb_n$
(g) $q_{2n}q_{2n-1} = EOb_n - EOB_{n-1}$
(h) $q_{2n}q_{2n-1} = EOR_n + EOr_n - 2EOb_n - 1$
(i) $q_{2n}q_{2n+1} = EOR_n + EOr_{n+1} - 2EOB_n + 1$
(j) $P_{2n+1}^2 = EOB_n$
(k) $q_{2n+1}^2 = 2EOB_n - 1$
(l) $P_{2n}q_{2n} = EOb_n + EOr_n$
(m) $P_{2n+1}q_{2n+1} = EOB_n + EOR_n$
(n) $P_{2n}q_{2n+1} = EOB_n - EOR_n - 1$
(o) $P_{2n}P_{2n+1} = EOr_n + 2EOb_n$
(p) $q_{2n}q_{2n+1} = EOB_n - EOb_n$

Proof. From Theorem 4.2, (a) and (b) can be viewed directly. Further, (c) and (d) follows from (a) and (b) by using the identities $2P_{2n}^2 + 1 = q_{2n}^2$ and $2P_{2n+1}^2 - 1 = q_{2n+1}^2$ respectively. In a similar fashion, the proof of other results follow from Theorem 4.2 and Lemma 2.2. □

Theorem 4.6. *If n is any natural number, then*

(a) $(EOB_n + EOb_n)^2 = EOB_{2n}$
(b) $(EOB_{n-1} + EOb_n)^2 = EOB_{2n-1}$
(c) $EOB_n \cdot EOb_n = EOR_n^2$
(d) $EOB_n \cdot EOr_n^2 = EOB_{n-1} \cdot EOR_n^2$
(e) $(EOR_n + EOr_n)^2 = EOb_{2n}$
(f) $(EOR_{n-1} + EOr_n)^2 = EOb_{2n-1}$
(g) $EOB_{n-1} \cdot EOb_n = EOr_n^2$
(h) $EOB_n \cdot EOR_{n-1}^2 = EOb_{n-1} \cdot EOr_n^2$.

Proof. Using Lemma 2.2 and Theorem 4.2, the results of this theorem can be proved easily and hence, we omit the proof. □

Corollary 4.7. *If n is any natural number, then $(2EOR_n, EOB_n - EOb_n, EOB_n + EOb_n)$ and $(2EOr_n, EOb_n - EOB_{n-1}, EOb_n + EOB_{n-1})$ forms Pythagorean triples.*

Proof. Since,

$$(EOB_n + EOb_n)^2 - (EOB_n - EOb_n)^2 = 4EOB_n \cdot EOb_n$$

and

$$(EOb_n + EOB_{n-1})^2 - (EOb_n - EOB_{n-1})^2 = 4EOb_n \cdot EOB_{n-1},$$

the result follows immediately from the above theorem. □

Theorem 4.8. *If n is any natural number, then*

(a) $\sum_{i=1}^{4n+1} P_i + 1 = 2EOB_n$

(b) $\sum_{i=1}^{8n+3} P_i + 1 = (4EOB_n - 1)^2$

(c) $\sum_{i=1}^{8n+1} P_i = (2EOB_n - 2EOb_n - 1)^2$

(d) $\sum_{i=1}^{4n-1} P_i = 2EOb_n$

(e) $\sum_{i=1}^{8n-1} P_i + 1 = (4EOb_n + 1)^2$

(f) $\sum_{i=1}^{8n-3} P_i = (2EOB_n - 2EOb_n + 1)^2$.

Proof. Using Theorem 4.2 and the properties of Pell numbers, it is easy to establish the desired summation results. Hence, we omit the proof. □

Theorem 4.9. *If n is any natural number, then $EOb_n + EOr_n$ and $EOB_n + EOR_n$ are balancing numbers. Similarly, $EOb_n - EOr_n$ and $EOB_n - EOR_n - 1$ are cobalancing numbers. In particular, the sequence of balancing and cobalancing numbers can be written as*

$$\{B_n\} = \{EOb_n + EOr_n\} \cup \{EOB_n + EOR_n\},\ \{b_n\} = \{EOb_n - EOr_n\} \cup \{EOB_n - EOR_n - 1\}.$$

Proof. Since,

$$[EOb_n + EOr_n]^2 = \frac{x(x-1)}{2} \text{ and } [EOB_n + EOR_n]^2 = \frac{y(y+1)}{2}$$

for

$$x = EOB_{n-1} + EOb_n + 2EOr_n \text{ and } y = EOB_n + EOb_n + 2EOR_n,$$

it follows that $EOb_n + EOr_n$ and $EOB_n + EOR_n$ are balancing numbers. Further,

$$[EOb_n - EOr_n][EOb_n - EOr_n + 1] = \frac{x(x+1)}{2}$$

and

$$[EOB_n - EOR_n][EOB_n - EOR_n - 1] = \frac{y(y-1)}{2}$$

for

$$x = EOB_{n-1} + EOb_n + EOR_{n-1} - EOr_n \text{ and } y = EOB_n + EOb_n - EOR_n + EOr_n$$

proves that $EOb_n - EOr_n$ and $EOB_n - EOR_n - 1$ are cobalancing numbers. □

Theorem 4.10. *If n is any natural number, then*

(a) $4[EOb_n + EOr_n][EOB_n + EOR_n] + 1 = P_{4n+1}^2$
(b) $4[EOb_n + EOr_n][EOb_n + EOr_n + 1] + 1 = (P_{4n} + 1)^2$
(c) $4[EOb_n - EOr_n][EOB_n - EOR_n] + 1 = (P_{4n} - 1)^2$
(d) $4[EOb_n - EOr_n][EOb_n - EOr_n + 1] + 1 = P_{4n-1}^2$.

Proof. The proof of (a) and (d) follows from the fact that $EOb_n - EOr_n = b_{2n}$ and $EOB_n - EOR_n - 1 = b_{2n+1}$ using Lemma 2.2, whereas the proof of (b) and (c) follows from the fact that $EOb_n + EOr_n = B_{2n}$ and $EOB_n + EOR_n = B_{2n+1}$ using the Binet form of Pell and balancing numbers. □

Now, we state few Diophantine equations for which the solutions are expressible in terms of numbers induced in the balancing process of even and odd natural numbers. Some of these Diophantine equations has been solved in [5, 6, 7, 14] and others can be solved in a similar fashion. We start with the following theorem.

Theorem 4.11. *The Diophantine equations $x^2 - 2xy - y^2 + x = 0$ and $x^2 - 2xy - y^2 - x = 0$ results in*

$$(x, y) = (EOb_n, EOr_n) \text{ and } (x, y) = (EOB_n, EOR_n)$$

respectively, whereas the Diophantine equations $x^2 - 2xy - y^2 + y = 0$ and $x^2 - 2xy - y^2 - y = 0$ results in

$$(x, y) = (EOr_n, EOB_{n-1}) \text{ and } (x, y) = (EOR_n, EOb_n)$$

respectively with $n = 1, 2, 3, \ldots$

In the following table, we give solution of the Diophantine equation $ax^2 + by^2 + cx + dy = 0$ for some particular values of a, b, c and d, expressible in terms of even-odd and odd-even balancing(cobalancing) numbers and even-odd and odd-even balancers(cobalancers).

(a,b,c,d)	(x,y)
$(2,-1,1,0)$	$(EOb_n, EOb_n + EOr_n)$
$(2,-1,-1,0)$	$(EOr_n, EOB_{n-1} + EOR_{n-1})$
$(2,-1,0,1)$	$(EOb_n + EOr_n, 2EOb_n + 1)$ and $(EOB_n + EOR_n, 2EOB_n)$
$(2,-1,0,-1)$	$(EOb_n + EOr_n, 2EOb_n)$ and $(EOB_n + EOR_n, 2EOB_n - 1)$
$(2,-1,-2,1)$	$(EOB_n - EOR_n, 2EOR_n + 1)$ and $(EOb_n - EOr_n + 1, 2EOr_n)$
$(2,-1,-2,-1)$	$(EOB_n - EOR_n, 2EOR_n)$ and $(EOb_n - EOr_n + 1, 2EOr_n - 1)$
$(2,-1,2,1)$	$(EOB_n - EOR_n - 1, 2EOR_n + 1)$ and $(EOb_n - EOr_n, 2EOr_n)$
$(2,-1,2,-1)$	$(EOB_n - EOR_n - 1, 2EOR_n)$ and $(EOb_n - EOr_n, 2EOr_n - 1)$
$(8,-1,-2,1)$	$([EOb_n - EOr_n + EOB_{n-1} + EOR_{n-1} + 1]/4, EOb_n - EOr_n + 1)$
$(8,-1,-2,-1)$	$([EOb_n - EOr_n + EOB_{n-1} + EOR_{n-1} + 1]/4, EOb_n - EOr_n)$
$(8,-1,2,1)$	$([EOB_n - EOR_n + EOb_n + EOr_n - 1]/4, EOB_n - EOR_n)$
$(8,-1,2,-1)$	$([EOB_n - EOR_n + EOb_n + EOr_n - 1]/4, EOB_n - EOR_n - 1)$

In the following table, we give solutions of the Diophantine equation

$$x^2 - 2kxy - ly^2 + ma^2 = 0$$

(a is any fixed integer) for some particular values of k, l and m, expressible in terms of even-odd and odd-even balancing(cobalancing) numbers and even-odd and odd-even balancers(cobalancers).

(k,l,m)	(x,y)	(k,l,m)	(x,y)
$(1,1,1)$	(ai_n, aI_{n-1})	$(1,1,-1)$	(aI_n, ai_n)
$(1,1,2)$	(aj_n, aJ_{n-1})	$(1,1,-2)$	(aJ_n, aj_n)
$(2,-2,1)$	(aj_n, aI_{n-1})	$(2,-2,-1)$	(aJ_n, ai_n)
$(3,-1,4)$	(aI_n, aI_{n-1}) or (aI_{n-1}, aI_n)	$(3,-1,-4)$	(ai_n, ai_{n-1}) or (ai_{n-1}, ai_n)
$(3,-1,8)$	(aj_n, aj_{n-1}) or (aj_{n-1}, aj_n)	$(3,-1,-8)$	(aJ_n, aJ_{n-1}) or (aJ_{n-1}, aJ_n)
$(4,2,9)$	(aJ_n, aI_{n-1})	$(4,2,-9)$	(aj_n, ai_{n-1})
$(-4,2,9)$	(aJ_{n-1}, aI_n)	$(-4,2,-9)$	(aj_n, ai_{n+1})

where

$$I_n = \sqrt{EOB_n}, J_n = \frac{I_n^2 + EOR_n}{I_n}, i_n = \sqrt{EOb_n}, j_n = \frac{i_n^2 + EOr_n}{i_n},\ n = 1, 2, 3, \dots$$

In the following table, we give solution of the Diophantine equation

$$px^2 + qy^2 + rz^2 + 2sxyz + t = 0$$

for some particular values of p, q, r, s and t, expressible in terms of even-odd balancing(cobalancing) numbers and even-odd balancers(cobalancers).

(p, q, r, s, t)	(x, y, z)	(p, q, r, s, t)	(x, y, z)
$(1, 1, 1, -1, -1)$	(j_m, j_n, j_{m+n})	$(1, 2, 2, -2, -1)$	(j_m, I_n, I_{m+n})
$(1, 1, -1, 1, 1)$	(J_m, J_n, j_{m+n+1})	$(1, 2, -2, 2, 1)$	(J_m, i_n, I_{m+n})
$(1, -1, 1, -1, 1)$	(j_m, J_n, J_{m+n})	$(2, 2, -1, 2, 1)$	(i_m, i_n, j_{m+n})
$(1, -1, 1, -1, 1)$	(J_m, j_n, J_{m+n})	$(2, 2, 1, -2, -1)$	(I_m, I_n, j_{m+n+1})
$(1, -2, -2, 2, -1)$	(j_m, i_n, i_{m+n})	$(2, -2, -1, 2, -1)$	(I_m, i_n, J_{m+n})
$(1, -2, 2, -2, 1)$	(J_m, I_n, i_{m+n+1})	$(2, -2, 1, -2, 1)$	(i_m, I_n, J_{m+n})

Substituting $x = z$ and $r = 0$ in the above Diophantine equation, we have

$$px^2 + qy^2 + 2sx^2y + t = 0.$$

Hence, for suitable choice of p, q, s and t, one can have the following result:

Corollary 4.12. *The Diophantine equation $2x^2 + y^2 - 2x^2y - 1 = 0$ has solution $(x, y) = (j_n, j_{2n})$, whereas $2x^2 - y^2 + 2x^2y + 1 = 0$ has solution $(x, y) = (J_n, j_{2n+1})$. Similarly, the Diophantine equations $4x^2 - y^2 + 4x^2y + 1 = 0$ and $4x^2 + y^2 - 4x^2y - 1 = 0$ has solutions $(x, y) = (i_n, j_{2n})$ and $(x, y) = (I_n, j_{2n+1})$ respectively.*

5. Some Links Between Balancing(Cobalancing) Numbers with Balancers(Cobalancers) Corresponding to Even and Odd Natural Numbers

Similar to even-odd balancing numbers, we can define odd-even balancing numbers as follows.

Definition 5.1. *We call a natural number n, an odd-even balancing number, if it satisfies the Diophantine equation*

$$1 + 3 + \cdots + (2n - 3) = (2n + 2) + (2n + 4) + \cdots + (2n + 2r)$$

for some natural number r, which we call an odd-even balancer corresponding to n.

Further, one can check that 31 and 1045 are odd-even balancing numbers with 12 and 432 as odd-even balancers respectively. We can also define odd-even cobalancing numbers just by replacing the summation in the left side of Definition 5.1 with $1 + 3 + \cdots + (2n - 1)$. One can establish many results about these balancing numbers, cobalancing numbers, balancers and cobalancers.

Though the binary recurrence relation for all of these eight numbers obtained in the balancing process of even and odd natural numbers are different, they satisfy the following ternary recurrence (linear and homogeneous) relation:

$$x_{n+2} = 35x_{n+1} - 35x_n + x_{n-1}\ ; n \geq 1$$

with the initial values: $EOB_{0,1,2} = 1, 25, 841$, $EOR_{0,1,2} = 0, 10, 348$, $EOb_{0,1,2} = 0, 4, 144$, $EOr_{0,1,2} = 0, 2, 60$, $OEB_{0,1,2} = 1, 31, 1045$, $OER_{0,1,2} = 0, 12, 432$, $OEb_{0,1,2} = 0, 10, 348$, $OEr_{0,1,2} = 0, 4, 144$ respectively.

In Theorem 4.2, we have already presented even-odd balancing(cobalancing) numbers and even-odd balancers(cobalancers) in terms of Pell, associated Pell, balancing and cobalancing numbers. In the following theorem, we present a similar result for the odd-even balancing(cobalancing) numbers and odd-even balancers(cobalancers).

Theorem 5.2. *If n is any non negative integer, then*

(a) $OEB_n = (3q_{4n+1}+1)/4 = 3P_{2n}P_{2n+1}+1 = 6B_n(B_{n+1}-B_n)+1 = 6B_n(2b_{n+1}+1)+1$

(b) $OER_n = (3q_{4n}-3)/4 = 3P_{2n}^2 = 12B_n^2 = 3(b_n-b_{n-1})^2$

(c) $OEb_n = (q_{4n+1}-1)/4 = P_{2n}P_{2n+1} = 2B_n(B_{n+1}-B_n) = 2B_n(2b_{n+1}+1)$

(d) $OEr_n = (q_{4n}-1)/4 = P_{2n}^2 = 4B_n^2 = (b_n-b_{n-1})^2$.

Note: The greatest common divisor of the nth terms of any two of the eight numbers induced in the balancing process by considering the sequence of even(odd) and odd(even) natural numbers is either 1 or P_{2n} or P_{2n+1} or P_{2n}^2 or $P_{2n}P_{2n+1}$.

Using the properties of Pell and associated Pell numbers, one can easily derive many more identities (e.g, the closed form expressions for the sum formulas, Catalan type identities, etc.) involving these numbers.

The following two corollaries are direct consequences of Theorem 4.2 and 5.2.

Corollary 5.3. *Every odd-even cobalancing number is an even-odd balancer and every even-odd cobalancing number is an odd-even cobalancer.*

Corollary 5.4. *Three times of even-odd cobalancing number or three times of odd-even cobalancer is an odd-even balancer. Further, three times of odd-even cobalancing number or three times of even-odd balancer is one less than an odd-even balancing number.*

In the following theorem we establish results similar to Theorem 4.5-4.8.

Theorem 5.5. *If n is any natural number, then*

(a) $P_{2n}q_{2n+1} = OEb_n + OEr_n = (OEB_n + OER_n - 1)/3$
(b) $P_{2n+1}q_{2n} = OEb_n - OEr_n + 1 = (OEB_n + OER_n + 2)/3$
(c) $OEb_n \cdot (OEB_n - 1) = 3OEb_n^2$
(d) $OEb_n \cdot OER_{n+1}^2 = (OEB_{n+1} - 1) \cdot OEr_n^2$
(e) $EOB_n \cdot EOb_n = OEb_n^2$
(f) $(OEB_n - 1) \cdot OEb_n = 3EOR_n^2$
(g) $3\sum_{i=1}^{4n} P_i = 2OEB_n = 6OEb_n$.

Proof. The proofs of (a)-(b) is similar to that of Theorem 4.5 using Lemma 2.2, where as the results (c)-(f) can be proved as in Theorem 4.6. The proof of (g) is similar to that of Theorem 4.8. Hence, we omit them. □

Theorem 5.6. *If n is any natural number, then*

(a) $EOB_n + EOR_n + OEb_n + OEr_n = EOB_n + EOb_n + OEB_n - OEb_n - 1 = q_{2n+1}^2$
(b) $EOb_n + EOr_n + OEb_n + OEr_n + 1 = EOB_n + EOb_n - OEB_n + OEb_n + 1 = q_{2n}^2$
(c) $EOR_n - EOr_n - OER_n + OEr_n$
$= EOR_n + EOb_n - OEb_n - OEr_n = EOb_n + EOr_n - OEb_n + OEr_n = 0.$

The following are some links between the sums and differences of balancing(cobalancing) numbers and the balancers(cobalancers) coressponding to even and odd natural numbers with the balancing and cobalancing numbers.

Theorem 5.7. *If n is any natural number, then*

$$\text{(a) } B_{2n+1} = \sum_{i=1}^{2n+1} P_{2i-1} = \frac{1}{2}\sum_{i=1}^{4n+1} q_i + \frac{1}{2} = EOB_n + EOR_n = EOB_n + \frac{1}{3}(OEB_n - 1)$$

$$\text{(b) } B_{2n} = \sum_{i=1}^{2n} P_{2i-1} = \frac{1}{2}\sum_{i=1}^{4n-1} q_i + \frac{1}{2} = EOb_n + EOr_n = OEb_n - OEr_n = EOR_n - \frac{1}{3}OER_n$$
$$= OEr_n + EOr_n = OEb_n - EOb_n = \frac{1}{3}(OEB_n - OER_n - 1)$$

$$\text{(c) } b_{2n} = \sum_{i=1}^{2n-1} P_{2i} = \frac{1}{2}\sum_{i=1}^{4n-2} q_i = EOb_n - EOr_n = OEr_n - EOr_n$$

$$\text{(d) } b_{2n+1} = \sum_{i=1}^{2n} P_{2i} = \frac{1}{2}\sum_{i=1}^{4n} q_i = OEb_n + EOb_n = OEb_n + OEr_n = \frac{1}{3}(OEB_n + OER_n - 1)$$
$$= EOB_n - EOR_n - 1 = EOR_n + \frac{1}{3}OER_n = EOB_n - \frac{1}{3}(OEB_n + 2).$$

Proof. Using respective Binet forms of balancing, cobalancing, Pell, associated Pell and the numbers obtained in the balancing process of even and odd natural numbers, it is easy to proof the stated results. Hence, we omit it. □

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Department of Mathematics, National Institute of Technology, Rourkela-769008, India
E-mail address: saigopalrs@gmail.com

Department of Mathematics, Gayatri Vidya Parishad College of Engineering (A), Visakhapatnam-530048, India
E-mail address: davalaravikumar@gmail.com

Department of Mathematics, National Institute of Technology, Rourkela-769008, India
E-mail address: gkpanda_nit@rediffmail.com