

## Some Result on Analogous to Ramanujan's Remarkable Product of Theta Function

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Dedicated to Prof. C. Adiga on the occasion of his 62<sup>nd</sup> birthday.

### Abstract

In this paper, we study the analogous of Ramanujan's Remarkable product of theta-function of degree 9 and their explicit values.

**Mathematics Subject Classification (2000):** Primary14K25, 30B70,11J70,11A55

**Keywords:** Continued fraction, Theta functions.

### 1. INTRODUCTION

In Chapter 16 of his second notebook [2], Ramanujan develops the theory of theta-function and is defined by

$$(1.1) \quad f(a, b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, |ab| < 1,$$
$$= (-a; ab)_{\infty}(-b; ab)_{\infty}(ab; ab)_{\infty}$$

where  $(a; q)_0 = 1$  and  $(a; q)_{\infty} = (1 - a)(1 - aq)(1 - aq^2)\dots$ .

Following Ramanujan, we defined

$$(1.2) \quad \varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q; -q)_{\infty}}{(q; -q)_{\infty}},$$

$$(1.3) \quad \psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}},$$

$$(1.4) \quad f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q; q)_{\infty},$$

and

$$(1.5) \quad \chi(q) := (-q; q^2)_{\infty}.$$

On page 338 in his first notebook [11, p.338], Ramanujan defines

$$(1.6) \quad a_{m,n} = \frac{ne^{\frac{-(n-1)\pi}{4}\sqrt{\frac{m}{n}}}\psi^2(e^{-\pi\sqrt{mn}})\varphi^2(-e^{-2\pi\sqrt{mn}})}{\psi^2(e^{-\pi\sqrt{\frac{m}{n}}})\varphi^2(-e^{-2\pi\sqrt{\frac{m}{n}}})}.$$

He then, on pages 338 and 339, offers a list of eighteen particular values. All these eighteen values have been established by Berndt, Chan and Zhang [1]. An account of these can be found in Berndt's book [3], M. S. Mahadeva Naika and B. N. Dharmendra [4], also established some general theorems for explicit evaluations of the product of  $a_{m,n}$  and found some new explicit values therefrom. Further results on  $a_{m,n}$  can be found by Mahadeva Naika, Dharmendra and K. Shivashankar [5], and Mahadeva Naika and M. C. Mahesh Kumar [6].

In [7], Mahadeva Naika et al. defined the product

$$(1.7) \quad b_{m,n} = \frac{ne^{\frac{-(n-1)\pi}{4}\sqrt{\frac{m}{n}}}\psi^2(-e^{-\pi\sqrt{mn}})\varphi^2(-e^{-2\pi\sqrt{mn}})}{\psi^2(-e^{-\pi\sqrt{\frac{m}{n}}})\varphi^2(-e^{-2\pi\sqrt{\frac{m}{n}}})}.$$

They established general theorems for explicit evaluation of  $b_{m,n}$  and obtained some particular values. Mahadeva Naika et al. [8] established general formulas for explicit values of Ramanujan's cubic continued fraction  $V(q)$  in terms of the products  $a_{m,n}$  and  $b_{m,n}$  defined above, where

$$(1.8) \quad V(q) := \frac{q^{1/3}}{1} + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + \frac{q^3+q^6}{1} + \cdots, \quad |q| < 1,$$

and found some particular values of  $V(q)$ .

In [10], Nipen Saikia defined the product of theta-functions  $I_{m,n}$  as

$$(1.9) \quad I_{m,n} = \frac{\psi(-q)\varphi(q^m)}{q^{(m-1)/8}\psi(-q^m)\varphi(q)}; \quad q = e^{-\pi\sqrt{\frac{n}{m}}},$$

where  $m$  and  $n$  are positive real numbers. They established several properties of the product  $I_{m,n}$  and proved general formulas for explicit evaluation of  $I_{m,n}$ .

In this paper, we establish several General Theorems and explicit evaluation of  $I_{9,n}$ .

Now we define a modular equation in brief. The ordinary hypergeometric series  ${}_2F_1(a, b; c; x)$  is defined by

$${}_2F_1(a, b; c; x) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n,$$

where  $(a)_0 = 1$ ,  $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$  for any positive integer  $n$ , and  $|x| < 1$ .

Let

$$(1.10) \quad z := z(x) := {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)$$

and

$$(1.11) \quad q := q(x) := \exp\left(-\pi \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)}\right),$$

where  $0 < x < 1$ .

Let  $r$  denote a fixed natural number and assume that the following relation holds:

$$(1.12) \quad r \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \beta\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)}.$$

Then a modular equation of degree  $r$  in the classical theory is a relation between  $\alpha$  and  $\beta$  induced by (1.12). We often say that  $\beta$  is of degree  $r$  over  $\alpha$  and  $m := \frac{z(\alpha)}{z(\beta)}$  is called the multiplier. We also use the notations  $z_1 := z(\alpha)$  and  $z_r := z(\beta)$  to indicate that  $\beta$  has degree  $r$  over  $\alpha$ .

## 2. PRELIMINARY RESULTS

**Lemma 2.1.** [9] If  $P := \frac{\psi(-q)}{q\psi(-q^9)}$  and  $Q := \frac{\varphi(q)}{\varphi(q^9)}$ , then

$$(2.1) \quad Q + PQ = 3 + P.$$

**Lemma 2.2.** [9] If  $P := \frac{\varphi(q)}{\varphi(q^9)}$  and  $Q := \frac{\varphi(q^2)}{\varphi(q^{18})}$ , then

$$(2.2) \quad \begin{aligned} & (-4Q^3 + 6Q^2 + Q^4 - 4Q + 1)P^4 + (8Q^3 - 8Q - 4Q^4 - 12)P^3 \\ & + (-12Q^2 + 54 + 6Q^4)P^2 + (72Q - 4Q^4 - 8Q^3 - 108)P \\ & - 108Q + 54Q^2 + Q^4 + 81 - 12Q^3. \end{aligned}$$

**Lemma 2.3.** [9] If  $P = \frac{\varphi(q)}{\varphi(q^9)}$  and  $Q = \frac{\varphi(q^3)}{\varphi(q^{27})}$ , then

$$(2.3) \quad 9P^2Q - 3P^2Q^2 - 9P^2 - 3P^3Q + P^3Q^2 + 3P^3 - 9PQ + 3PQ^2 + 9P - Q^3 = 0.$$

**Lemma 2.4.** [9] If  $P = \frac{\varphi(q)}{\varphi(q^9)}$  and  $Q = \frac{\varphi(q^5)}{\varphi(q^{45})}$ , then

$$(2.4) \quad \begin{aligned} & P^6 + (-Q^5 - 15Q - 10Q^3 + 5Q^4 + 15Q^2)P^5 \\ & + (-45Q^2 + 5Q^5 - 20Q^4 + 30Q^3 + 45Q)P^4 \\ & + (30Q^4 - 40Q^3 - 90Q + 90Q^2 - 10Q^5)P^3 \\ & + (90Q^3 + 15Q^5 - 45Q^4 - 180Q^2 + 135Q)P^2 \\ & + (45Q^4 - 81Q - 15Q^5 - 90Q^3 + 135Q^2)P + Q^6 = 0. \end{aligned}$$

**Lemma 2.5.** [9] If  $P := \frac{\varphi(q)}{\varphi(q^5)}$  and  $Q := \frac{\varphi(q^7)}{\varphi(q^{63})}$ , then

$$(2.5) \quad \begin{aligned} & P^8 + (-21Q^5 - 63Q^3 + 42Q^4 + 7Q^6 - 35Q - Q^7 + 63Q^2)P^7 \\ & + (147Q^5 - 49Q^6 + 189Q + 7Q^7 - 413Q^2 - 294Q^4 + 441Q^3)P^6 \\ & + (-1379Q^3 - 567Q - 441Q^5 - 21Q^7 + 147Q^6 + 1323Q^2 + 882Q^4)P^5 \\ & + (-294Q^6 + 2646Q^3 + 1134Q - 2646Q^2 + 882Q^5 - 1694Q^4 + 42Q^7)P^4 \\ & + (-3969Q^3 - 1701Q + 441Q^6 + 3969Q^2 + 2646Q^4 - 63Q^7 - 1379Q^5)P^3 \\ & + (1701Q - 2646Q^4 + 63Q^7 + 3969Q^3 - 413Q^6 - 3969Q^2 + 1323Q^5)P^2 \\ & + (1701Q^2 - 729Q + 1134Q^4 + 189Q^6 - 567Q^5 - 35Q^7 - 1701Q^3)P + Q^8 = 0. \end{aligned}$$

**Lemma 2.6.** [10] We have

$$(2.6) \quad I_{m,1} = 1.$$

**Lemma 2.7.** [10] We have

$$(2.7) \quad I_{m,n}I_{m,1/n} = 1.$$

### 3. GENERAL THEOREMS AND EXPLICIT EVALUATIONS OF $I_{9,n}$

**Theorem 3.1.** If  $X := I_{9,n}$  and  $Y := I_{9,4n}$  then

$$(3.1) \quad \begin{aligned} & x^8y^5 + (16y^4 - y^7 - 6y^6 + 12y^2 + y + 30y^3 + 12y^5)x^7 \\ & + (-30y^2 - 132y^4 - 19y^3 + 12y - 96y^5 - 79y^6 - 6y^7)x^6 \\ & + (60y^3 + 12y^7 - 96y^6 - 19y^2 + 30y + y^8 - 74y^5 - 78y^4)x^5 \\ & + (-132y^2 - 78y^3 + 16y^7 - 132y^6 + 16y - 172y^4 - 78y^5)x^4 \\ & + (12y + 30y^7 - 78y^4 - 74y^3 + 60y^5 + 1 - 19y^6 - 96y^2)x^3 \\ & + (-19y^5 - 79y^2 - 96y^3 + 12y^7 - 30y^6 - 6y - 132y^4)x^2 \\ & + (y^7 + 12y^6 + 30y^5 + 16y^4 + 12y^3 - y - 6y^2)x + y^3 = 0. \end{aligned}$$

*Proof.* Employing the definition of  $I_{m,n}$  (1.9) with  $m = 9$ , we obtain

$$(3.2) \quad I_{9,n} = \frac{\psi(-q)\varphi(q^9)}{q\psi(-q^9)\varphi(q)}; \quad q := e^{-\pi\sqrt{\frac{n}{9}}}.$$

By using lemma (2.1), we obtain

$$(3.3) \quad P = \frac{3 - Q}{Q - 1}.$$

Combining (3.2) and (3.3), we can be write

$$(3.4) \quad I_{9,n} = \frac{3 - Q}{Q^2 - Q},$$

where  $Q = \frac{\varphi(q)}{\varphi(q^9)}$ , solve (3.4) for  $Q$  to obtain

$$(3.5) \quad Q = \frac{(a-1) + \sqrt{a^2 + 10a + 1}}{2a},$$

where  $a = I_{9,n}$ .

Employing the equation (3.5) in (2.2), we obtain (3.1)  $\square$

**Corollary 3.1.** *We have*

$$(3.6) \quad I_{9,2} = \frac{(9 + 5\sqrt{3})\sqrt{2} - 2(5 + 3\sqrt{3})}{2},$$

$$(3.7) \quad I_{9,1/2} = \frac{(9 - 5\sqrt{3})\sqrt{2} - 2(5 - 3\sqrt{3})}{2},$$

$$(3.8) \quad I_{9,4} = \frac{(9 + 5\sqrt{3})\sqrt{2} - 2 \left( (16 + 9\sqrt{3}) + 2\sqrt{1197 + 693\sqrt{3} - (837 + 483\sqrt{3})\sqrt{2}} \right)}{4},$$

$$(3.9) \quad I_{9,1/4} = \frac{(9 + 5\sqrt{3})\sqrt{2} - 2 \left( (16 + 9\sqrt{3}) - 2\sqrt{1197 + 693\sqrt{3} - (837 + 483\sqrt{3})\sqrt{2}} \right)}{4}.$$

*Proof.* Setting  $n = 1/2$  in Theorem (3.1) and using the Lemma (2.7), we obtain

$$(3.10) \quad (I_{9,2}^4 + 20I_{9,2}^3 - 60I_{9,2}^2 + 20I_{9,2} + 1)(I_{9,2}^2 + I_{9,2} + 1)^4 = 0.$$

Since the root of the second factor is imaginary and  $I_{9,2} < 1$ , we deduce that

$$(3.11) \quad I_{9,2}^4 + 20I_{9,2}^3 - 60I_{9,2}^2 + 20I_{9,2} + 1 = 0.$$

On solving the above equation (3.11), we arrive at the equations (3.6) and (3.7).

Setting  $n = 1$  in Theorem (3.1) and using the Lemma (2.6), we obtain

$$(3.12) \quad 1 + 64I_{9,4} + 64I_{9,4}^7 - 350I_{9,4}^6 - 164I_{9,4}^3 - 560I_{9,4}^4 + I_{9,4}^8 - 164I_{9,4}^5 - 350I_{9,4}^2 = 0,$$

by above equation (3.12) can be written as

$$(3.13) \quad z^4 + 64z^3 - 354z^2 - 356z + 142 = 0, \quad z = I_{9,4} + I_{9,4}^{-1}.$$

On solving the above equation (3.13), we arrive at the equations (3.8) and (3.9).  $\square$

**Theorem 3.2.** *If  $X := I_{9,n}$  and  $Y := I_{9,9n}$  then*

$$(3.14) \quad \begin{aligned} & (y^2 + 1 + y)x^6 + (-17y^2 - 10y^4 - 19y^3 + 1 - y^5 - 8y)x^5 \\ & + (-10y^5 - 28y^4 - 46y^3 - 17y + 1 - 35y^2)x^4 + (-19y^5 \\ & - 19y - 46y^2 - 74y^3 - 46y^4)x^3 + (-28y^2 - 17y^5 - 46y^3 \\ & + y^6 - 10y - 35y^4)x^2 + (y^6 - 17y^4 - 8y^5 - y - 10y^2 - 19y^3)x \\ & + y^6 + y^4 + y^5 = 0. \end{aligned}$$

*Proof.* Employing the equation (3.5) in (2.3), we obtain (3.14)  $\square$

**Corollary 3.2.** *We have*

$$(3.15) \quad I_{9,3} = 2^{2/3} + 2^{1/3} + 1,$$

$$(3.16) \quad I_{9,1/3} = 2^{1/3} - 1,$$

$$(3.17) \quad I_{9,9} = \frac{13 [(11 + 7\sqrt{3})x + 13(3 + 2\sqrt{3})] + (51 + 23\sqrt{3})x^2}{169},$$

$$(3.18) \quad I_{9,1/9} = \frac{13 [(15 - 7\sqrt{3})x + 13(3 - 2\sqrt{3})] + (43 - 27\sqrt{3})x^2}{169},$$

where  $x = (47 + 2\sqrt{3})^{1/3}$ .

*Proof.* Employing Theorem (3.2), Lemma (2.7) and (2.6), solving the resulting equation for  $I_{9,3}$ ,  $I_{9,9}$  and nothing that  $I_{9,3} < 1$  and  $I_{9,9} < 1$ , we arrive (3.15)-(3.18).  $\square$

**Theorem 3.3.** *If  $X := I_{9,n}$  and  $Y := I_{9,25n}$  then*

$$(3.19) \quad \begin{aligned} & x^6 + (-45y - 60y^3 - 65y^2 - 15y^4 - y^5)x^5 \\ & + (-65y + 75y^2 + 60y^3 - 15y^5 + 140y^4)x^4 \\ & + (-60y + 60y^2 - 20y^3 - 60y^5 + 60y^4)x^3 \\ & + (-65y^5 + 60y^3 + 140y^2 - 15y + 75y^4)x^2 \\ & + (-y - 60y^3 - 15y^2 - 65y^4 - 45y^5)x + y^6 = 0. \end{aligned}$$

*Proof.* Employing the equation (3.5) in (2.4), we obtain (3.19).  $\square$

**Corollary 3.3.** *We have*

$$(3.20) \quad I_{9,5} = 4 + \sqrt{15},$$

$$(3.21) \quad I_{9,1/5} = 4 - \sqrt{15},$$

$$(3.22) \quad I_{9,25} = 2(23 + 6\sqrt{5}) + \sqrt{3(1425 + 368\sqrt{5})},$$

$$(3.23) \quad I_{9,1/25} = 2(23 + 6\sqrt{5}) - \sqrt{3(1425 + 368\sqrt{5})}.$$

*Proof.* Employing Theorem (3.3), Lemma (2.7) and (2.6), solving the resulting equation for  $I_{9,5}$ ,  $I_{9,25}$  and nothing that  $I_{9,5} < 1$  and  $I_{9,25} < 1$ , we arrive (3.20)-(3.23).  $\square$

**Theorem 3.4.** *If  $X1 := I_{9,n}$  and  $Y := I_{9,49n}$  then*

$$(3.24) \quad \begin{aligned} & x^8 + (-406y^4 - 357y^2 - 21y^6 - 147y^5 - 483y^3 - 105y - y^7)x^7 \\ & + (-21y^7 - 1134y^4 - 2415y^3 + 903y^6 - 1757y^2 - 357y + 273y^5)x^6 \\ & + (-6930y^4 - 7889y^3 + 273y^6 - 5145y^5 - 483y - 2415y^2 - 147y^7)x^5 \\ & + (-406y - 1134y^6 - 7182y^4 - 6930y^5 - 406y^7 - 6930y^3 - 1134y^2)x^4 \\ & + (-6930y^4 + 273y^2 - 483y^7 - 2415y^6 - 5145y^3 - 147y - 7889y^5)x^3 \\ & + (273y^3 - 357y^7 - 21y - 1757y^6 + 903y^2 - 1134y^4 - 2415y^5)x^2 \\ & + (-y - 406y^4 - 483y^5 - 21y^2 - 357y^6 - 105y^7 - 147y^3)x + y^8 = 0. \end{aligned}$$

*Proof.* Employing the equation (3.5) in (2.5), we obtain (3.24).  $\square$

**Corollary 3.4.** *We have*

$$(3.25) \quad I_{9,7} = \frac{(13 + 3\sqrt{21}) + \sqrt{6(57 + 13\sqrt{21})}}{4},$$

$$(3.26) \quad I_{9,1/7} = \frac{(13 + 3\sqrt{21}) - \sqrt{6(57 + 13\sqrt{21})}}{4}.$$

*Proof.* Employing Theorem (3.4), Lemma (2.7), solving the resulting equation for  $I_{9,7}$  and noting that  $I_{9,7} < 1$ , we arrive (3.25)-(3.25).  $\square$

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