

## ON (2, 5)-REGULAR BIPARTITIONS WITH ODD PARTS DISTINCT

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Dedicated to Prof. C. Adiga on the occasion of his 62<sup>nd</sup> birthday.

**ABSTRACT.** In his work, K. Alladi studied the partition function  $pod(n)$ , the number of partitions of an integer  $n$  with odd parts distinct (the even parts are unrestricted). He obtained a series expansion for the product generating function of partitions in which the odd parts do not repeat. Later, Hirschhorn and Sellers obtained some internal congruences involving the infinite families and Ramanujan-type congruences for  $pod(n)$ . Let  $B(n)$  denote the number of (2, 5)-regular bipartitions of a positive integer  $n$  with odd parts distinct (even parts are unrestricted). In this paper, we establish many infinite families of congruences modulo powers of 2 for  $B(n)$ . For example, for modulo 16,

$$\sum_{n=0}^{\infty} B \left( 16 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 6 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1 \right) q^n \equiv 8f_1^9,$$

where  $\alpha, \beta, \gamma \geq 0$ .

### 1. INTRODUCTION

A partition of a positive integer  $n$  is a non-increasing sequence of positive integers  $\lambda_1 \geq \lambda_2 \dots \geq \lambda_k$  such that  $\lambda_1 + \lambda_2 + \dots + \lambda_k = n$ . An  $\ell$ -regular partition is a partition in which none of the parts are divisible by  $\ell$ . Let  $b_\ell(n)$  denote the number of  $\ell$ -regular partitions of  $n$  with  $b_\ell(0) = 1$ . The generating function for  $b_\ell(n)$  is

$$\sum_{n=0}^{\infty} b_\ell(n) q^n = \frac{f_\ell}{f_1},$$

where  $f_\ell := (q^\ell; q^\ell)_\infty = \prod_{n=1}^{\infty} (1 - q^{n\ell})$ .

Arithmetic properties of  $\ell$ -regular partition functions have been studied by a number of mathematicians. The congruences for 5-regular partitions modulo 2 and for 13-regular partitions modulo 2 and 3 obtained using the theory of modular forms by Calkin et al. [4]. For more details, one can see [5], [8] and [9].

In his work, K. Alladi [2] studied the partition function  $pod(n)$ , the number of partitions of an integer  $n$  with odd parts distinct (the even parts are unrestricted). He obtained a series expansion for the product generating function of partitions in which the odd parts do not repeat. Later, Hirschhorn and Sellers [7] obtained some internal congruences involving the infinite families and Ramanujan-type congruences for  $pod(n)$ . For more details, one can see [12], [13].

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2000 Mathematics Subject Classification. 11P83, 05A17.

Key words and phrases. Partition identities, Theta-functions, Partition Congruences, Regular bipartition.

A bipartition of a positive integer  $n$  is an ordered pair of partitions  $(\lambda, \mu)$  such that the sum of all the parts equals to  $n$ . Let  $B_\ell(n)$  denote the number of  $\ell$ -regular bipartitions of  $n$  and the generating function is given by

$$\sum_{n=0}^{\infty} B_\ell(n)q^n = \frac{(q^\ell; q^\ell)_\infty^2}{(q; q)_\infty^2}.$$

Suppose  $\ell, m > 0$ . A partition is an  $(\ell, m)$ -regular partitions of a positive integer  $n$  if none of the parts are divisible by  $\ell$  and  $m$ . For more details, we can see [1, 11].

Let  $B(n)$  denote the number of  $(2, 5)$ -regular bipartitions of  $n$  with odd parts distinct. The generating function is given by

$$\sum_{n=0}^{\infty} B(n)q^n = \frac{f_2^4 f_5^2 f_8^2 f_{20}^4}{f_1^2 f_4^4 f_{10}^4 f_{40}^2}. \quad (1.1)$$

We prove many congruences of the form, for all  $\alpha, \beta, \gamma \geq 0$ ,

$$\sum_{n=0}^{\infty} B \left( 16 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 6 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1 \right) q^n \equiv 8f_1^9 \pmod{16}.$$

## 2. PRELIMINARY RESULTS

In this section, we record several identities which are useful in proving our main results.

**Lemma 2.1.** *The following 2-dissections hold:*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q \frac{f_4^2 f_{16}^2}{f_2^5 f_8} \quad (2.1)$$

and

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q \frac{f_4^2 f_8^4}{f_2^{10}}. \quad (2.2)$$

For proofs, see [3, p.40].

**Lemma 2.2.** *We have*

$$f_1^3 = \frac{f_6 f_9^6}{f_3 f_{18}^3} - 3q f_9^3 + 4q^3 \frac{f_3^2 f_{18}^6}{f_6^2 f_9^3}. \quad (2.3)$$

For a proof, see [3, p.345].

**Lemma 2.3.** *The following 2-dissections hold:*

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q \frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}} \quad (2.4)$$

and

$$\frac{f_1}{f_5} = \frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q \frac{f_4^2 f_{40}}{f_8 f_{10}^2}. \quad (2.5)$$

For proofs, see [8].

**Lemma 2.4.** *We have*

$$\frac{1}{f_1^3 f_5} = \frac{f_4^4}{f_2^7 f_{10}} - 2q \frac{f_4^6 f_{20}^2}{f_2^9 f_{10}^3} + 5q \frac{f_4^3 f_{20}}{f_2^8} + 2q^2 \frac{f_4^9 f_{40}^2}{f_2^{10} f_8^2 f_{10}^2 f_{20}}, \quad (2.6)$$

$$f_1^3 f_5 = \frac{f_2^2 f_4 f_{10}^2}{f_{20}} + 2q f_4^3 f_{20} - 5q f_2 f_{10}^3 + 2q^2 \frac{f_4^6 f_{10} f_{40}^2}{f_2 f_8^2 f_{20}^2} \quad (2.7)$$

and

$$f_1 f_5^3 = f_2^3 f_{10} - q \frac{f_2^2 f_{10} f_{20}}{f_4} + 2q^2 f_4 f_{20}^3 - 2q^3 \frac{f_4^4 f_{10} f_{40}^2}{f_2 f_8^2}. \quad (2.8)$$

For proofs, see [10].

**Lemma 2.5.** [3, p.303, Entry 17(v)] *We have*

$$f_1 = f_{49} \left( \frac{B(q^7)}{C(q^7)} - q \frac{A(q^7)}{B(q^7)} - q^2 + q^5 \frac{C(q^7)}{A(q^7)} \right), \quad (2.9)$$

where  $A(q) = f(-q^3, -q^4)$ ,  $B(q) = f(-q^2, -q^5)$  and  $C(q) = f(-q, -q^6)$ .

We prove the following theorems:

**Theorem 2.1.** *Let  $r_1 \in \{248, 312\}$ ,  $r_2 \in \{40, 104, 232, 296, 360, 424\}$ ,  $r_3 \in \{56, 184, 248, 312\}$ ,  $r_4 \in \{56, 248, 632, 824\}$ ,  $r_5 \in \{184, 376, 568, 952\}$ ,  $r_6 \in \{124, 156\}$ ,  $r_7 \in \{20, 52, 116, 148, 180, 212\}$ ,  $r_8 \in \{28, 92, 124, 156\}$ ,  $r_9 \in \{28, 124, 316, 412\}$ ,  $r_{10} \in \{92, 188, 284, 476\}$ ,  $r_{11} \in \{62, 78\}$ ,  $r_{12} \in \{10, 26, 58, 74, 90, 106\}$ ,  $r_{13} \in \{14, 46, 62, 78\}$ ,  $r_{14} \in \{14, 62, 158, 206\}$  and  $r_{15} \in \{46, 94, 142, 238\}$ . Then for all  $n \geq 0$  and  $\alpha, \beta, \gamma \geq 0$ , we have for modulo 16,*

$$B(16 \cdot 2^\alpha n + 1) \equiv B(8n + 1), \quad (2.10)$$

$$\sum_{n=0}^{\infty} B(64 \cdot 5^{2\alpha} n + 56 \cdot 5^{2\alpha} + 1) q^n \equiv 8f_1 f_{20} + 8f_2^3 f_5^3, \quad (2.11)$$

$$\sum_{n=0}^{\infty} B(64 \cdot 5^{2\alpha+1} n + 24 \cdot 5^{2\alpha+1} + 1) q^n \equiv 8f_4 f_5 + 8q f_1^3 f_{10}^3, \quad (2.12)$$

$$B(64 \cdot 5^{2\alpha+1} n + r_1 \cdot 5^{2\alpha} + 1) \equiv 0, \quad (2.13)$$

$$\sum_{n=0}^{\infty} B(64 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 24 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1) q^n \equiv 8f_1^9, \quad (2.14)$$

$$\sum_{n=0}^{\infty} B(64 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+1} n + 8 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+1} + 1) q^n \equiv 8q^2 f_7^9, \quad (2.15)$$

$$B(64 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2} n + r_2 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+1} + 1) \equiv 0, \quad (2.16)$$

$$\sum_{n=0}^{\infty} B(64 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 8 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1) q^n \equiv 8q f_5^9, \quad (2.17)$$

$$B(64 \cdot 3^{4\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + r_3 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1) \equiv 0, \quad (2.18)$$

$$\sum_{n=0}^{\infty} B(64 \cdot 3^{4\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 88 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1) q^n \equiv 8f_2 f_3^3, \quad (2.19)$$

$$\sum_{n=0}^{\infty} B \left( 64 \cdot 3^{4\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 152 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1 \right) q^n \equiv 8f_1 f_6^3, \quad (2.20)$$

$$\begin{aligned} & \sum_{n=0}^{\infty} B \left( 64 \cdot 3^{4\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 8 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+1} + 1 \right) q^n \\ & \equiv 8q^2 f_{10} f_{15}^3, \end{aligned} \quad (2.21)$$

$$\begin{aligned} & \sum_{n=0}^{\infty} B \left( 64 \cdot 3^{4\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 184 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) q^n \\ & \equiv 8q^3 f_5 f_{30}^3, \end{aligned} \quad (2.22)$$

$$B \left( 64 \cdot 3^{4\alpha+1} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + r_4 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) \equiv 0, \quad (2.23)$$

$$B \left( 64 \cdot 3^{4\alpha+1} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + r_5 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) \equiv 0, \quad (2.24)$$

$$\sum_{n=0}^{\infty} B \left( 32 \cdot 5^{2\alpha} n + 28 \cdot 5^{2\alpha} + 1 \right) q^n \equiv 8f_1 f_{20} + 8f_2^3 f_5^3, \quad (2.25)$$

$$\sum_{n=0}^{\infty} B \left( 32 \cdot 5^{2\alpha+1} n + 12 \cdot 5^{2\alpha+1} + 1 \right) q^n \equiv 8f_4 f_5 + 8q f_1^3 f_{10}^3, \quad (2.26)$$

$$B \left( 32 \cdot 5^{2\alpha+1} n + r_6 \cdot 5^{2\alpha} + 1 \right) \equiv 0, \quad (2.27)$$

$$\sum_{n=0}^{\infty} B \left( 32 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 12 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1 \right) q^n \equiv 8f_1^9, \quad (2.28)$$

$$\sum_{n=0}^{\infty} B \left( 32 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+1} n + 4 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+1} + 1 \right) q^n \equiv 8q^2 f_7^9, \quad (2.29)$$

$$B \left( 32 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2} n + r_7 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+1} + 1 \right) \equiv 0, \quad (2.30)$$

$$\sum_{n=0}^{\infty} B \left( 32 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 4 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+1} + 1 \right) q^n \equiv 8q f_5^9, \quad (2.31)$$

$$B \left( 32 \cdot 3^{4\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + r_8 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) \equiv 0, \quad (2.32)$$

$$\sum_{n=0}^{\infty} B \left( 32 \cdot 3^{4\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 44 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1 \right) q^n \equiv 8f_2 f_3^3, \quad (2.33)$$

$$\sum_{n=0}^{\infty} B \left( 32 \cdot 3^{4\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 76 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1 \right) q^n \equiv 8f_1 f_6^3, \quad (2.34)$$

$$\begin{aligned} & \sum_{n=0}^{\infty} B \left( 32 \cdot 3^{4\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 4 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+1} + 1 \right) q^n \\ & \equiv 8q^2 f_{10} f_{15}^3, \end{aligned} \quad (2.35)$$

$$\sum_{n=0}^{\infty} B \left( 32 \cdot 3^{4\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 92 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) q^n \equiv 8q^3 f_5 f_{30}^3, \quad (2.36)$$

$$B \left( 32 \cdot 3^{4\alpha+1} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + r_9 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) \equiv 0, \quad (2.37)$$

$$B \left( 32 \cdot 3^{4\alpha+1} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + r_{10} \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) \equiv 0, \quad (2.38)$$

$$\sum_{n=0}^{\infty} B \left( 16 \cdot 5^{2\alpha} n + 14 \cdot 5^{2\alpha} + 1 \right) q^n \equiv 8f_1 f_{20} + 8f_2^3 f_5^3, \quad (2.39)$$

$$\sum_{n=0}^{\infty} B \left( 16 \cdot 5^{2\alpha+1} n + 6 \cdot 5^{2\alpha+1} + 1 \right) q^n \equiv 8f_4 f_5 + 8q f_1^3 f_{10}^3, \quad (2.40)$$

$$B \left( 16 \cdot 5^{2\alpha+1} n + r_{11} \cdot 5^{2\alpha} + 1 \right) \equiv 0, \quad (2.41)$$

$$\sum_{n=0}^{\infty} B \left( 16 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 6 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1 \right) q^n \equiv 8f_1^9, \quad (2.42)$$

$$\sum_{n=0}^{\infty} B \left( 16 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+1} n + 2 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+1} + 1 \right) q^n \equiv 8q^2 f_7^9, \quad (2.43)$$

$$B \left( 16 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2} n + r_{12} \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+1} + 1 \right) \equiv 0, \quad (2.44)$$

$$\sum_{n=0}^{\infty} B \left( 16 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 2 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+1} + 1 \right) q^n \equiv 8q f_5^9, \quad (2.45)$$

$$B \left( 16 \cdot 3^{4\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + r_{13} \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) \equiv 0, \quad (2.46)$$

$$\sum_{n=0}^{\infty} B \left( 16 \cdot 3^{4\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 22 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1 \right) q^n \equiv 8f_2 f_3^3, \quad (2.47)$$

$$\sum_{n=0}^{\infty} B \left( 16 \cdot 3^{4\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 38 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1 \right) q^n \equiv 8f_1 f_6^3, \quad (2.48)$$

$$\begin{aligned} & \sum_{n=0}^{\infty} B \left( 16 \cdot 3^{4\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 2 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+1} + 1 \right) q^n \\ & \equiv 8q^2 f_{10} f_{15}^3, \end{aligned} \quad (2.49)$$

$$\begin{aligned} & \sum_{n=0}^{\infty} B \left( 16 \cdot 3^{4\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 46 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) q^n \\ & \equiv 8q^3 f_5 f_{30}^3, \end{aligned} \quad (2.50)$$

$$B \left( 16 \cdot 3^{4\alpha+1} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + r_{14} \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) \equiv 0, \quad (2.51)$$

$$B \left( 16 \cdot 3^{4\alpha+1} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + r_{15} \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) \equiv 0. \quad (2.52)$$

**Theorem 2.2.** Let  $r_{16} \in \{22, 38\}$ ,  $r_{17} \in \{34, 66\}$ ,  $r_{18} \in \{26, 42, 58, 74\}$ ,  $r_{19} \in \{44, 76\}$ ,  $r_{20} \in \{68, 132\}$ ,  $r_{21} \in \{52, 84, 116, 148\}$ ,  $r_{22} \in \{88, 152\}$ ,  $r_{23} \in \{136, 264\}$  and  $r_{24} \in \{104, 168, 232, 296\}$ . Then for all  $n \geq 0$  and  $\alpha, \beta, \gamma \geq 0$ , we have for modulo 4,

$$\sum_{n=0}^{\infty} B \left( 16 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1 \right) q^n \equiv 2f_1^3, \quad (2.53)$$

$$\sum_{n=0}^{\infty} B \left( 16 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+1} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2} + 1 \right) q^n \equiv 2f_7^3, \quad (2.54)$$

$$\begin{aligned} & B \left( 16 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1 \right) \\ & \equiv \begin{cases} 2 & \text{if } n = k(3k+1)/2 \text{ for some } k \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (2.55)$$

$$\sum_{n=0}^{\infty} B \left( 16 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 2 \cdot 3^{2\alpha+2} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1 \right) q^n \equiv 2f_3^3, \quad (2.56)$$

$$B \left( 16 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 34 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1 \right) \equiv 0, \quad (2.57)$$

$$B \left( 16 \cdot 3^{2\alpha+2} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + r_{16} \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1 \right) \equiv 0, \quad (2.58)$$

$$\sum_{n=0}^{\infty} B \left( 16 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} + 1 \right) q^n \equiv 2f_5^3, \quad (2.59)$$

$$B \left( 16 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + r_{17} \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1 \right) \equiv 0, \quad (2.60)$$

$$B \left( 16 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + r_{18} \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) \equiv 0, \quad (2.61)$$

$$\sum_{n=0}^{\infty} B \left( 16 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) q^n \equiv 2f_1^3, \quad (2.62)$$

$$\sum_{n=0}^{\infty} B \left( 16 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+1} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+2} + 1 \right) q^n \equiv 2f_7^3, \quad (2.63)$$

$$\begin{aligned} & B \left( 16 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) \\ & \equiv \begin{cases} 2 & \text{if } n = k(3k+1)/2 \text{ for some } k \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (2.64)$$

$$\sum_{n=0}^{\infty} B \left( 16 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 2 \cdot 3^{2\alpha+2} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) q^n \equiv 2f_3^3, \quad (2.65)$$

$$B \left( 16 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 34 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) \equiv 0, \quad (2.66)$$

$$B \left( 16 \cdot 3^{2\alpha+2} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + r_{16} \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) \equiv 0, \quad (2.67)$$

$$\sum_{n=0}^{\infty} B \left( 16 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta+3} \cdot 7^{2\gamma} + 1 \right) q^n \equiv 2f_5^3, \quad (2.68)$$

$$B \left( 16 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + r_{17} \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) \equiv 0, \quad (2.69)$$

$$B \left( 16 \cdot 3^{2\alpha} \cdot 5^{2\beta+3} \cdot 7^{2\gamma} n + r_{18} \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} + 1 \right) \equiv 0, \quad (2.70)$$

$$\sum_{n=0}^{\infty} B \left( 32 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 4 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1 \right) q^n \equiv 2f_1^3, \quad (2.71)$$

$$\sum_{n=0}^{\infty} B \left( 32 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+1} n + 4 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2} + 1 \right) q^n \equiv 2f_7^3, \quad (2.72)$$

$$\begin{aligned} & B \left( 32 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 4 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1 \right) \\ & \equiv \begin{cases} 2 & \text{if } n = k(3k+1)/2 \text{ for some } k \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (2.73)$$

$$\sum_{n=0}^{\infty} B \left( 32 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 4 \cdot 3^{2\alpha+2} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1 \right) q^n \equiv 2f_3^3, \quad (2.74)$$

$$B \left( 32 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 68 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1 \right) \equiv 0, \quad (2.75)$$

$$B \left( 32 \cdot 3^{2\alpha+2} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + r_{19} \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1 \right) \equiv 0, \quad (2.76)$$

$$\sum_{n=0}^{\infty} B \left( 32 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 4 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} + 1 \right) q^n \equiv 2f_5^3, \quad (2.77)$$

$$B \left( 32 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + r_{20} \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1 \right) \equiv 0, \quad (2.78)$$

$$B \left( 32 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + r_{21} \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) \equiv 0, \quad (2.79)$$

$$\sum_{n=0}^{\infty} B \left( 32 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 4 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) q^n \equiv 2f_1^3, \quad (2.80)$$

$$\sum_{n=0}^{\infty} B \left( 32 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+1} n + 4 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+2} + 1 \right) q^n \equiv 2f_7^3, \quad (2.81)$$

$$\begin{aligned} & B \left( 32 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 4 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) \\ & \equiv \begin{cases} 2 & \text{if } n = k(3k+1)/2 \text{ for some } k \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (2.82)$$

$$\sum_{n=0}^{\infty} B \left( 32 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 4 \cdot 3^{2\alpha+2} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) q^n \equiv 2f_3^3, \quad (2.83)$$

$$B \left( 32 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 68 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) \equiv 0, \quad (2.84)$$

$$B \left( 32 \cdot 3^{2\alpha+2} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + r_{19} \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) \equiv 0, \quad (2.85)$$

$$\sum_{n=0}^{\infty} B \left( 32 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + 4 \cdot 3^{2\alpha} \cdot 5^{2\beta+3} \cdot 7^{2\gamma} + 1 \right) q^n \equiv 2f_5^3, \quad (2.86)$$

$$B \left( 32 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + r_{20} \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) \equiv 0, \quad (2.87)$$

$$B \left( 32 \cdot 3^{2\alpha} \cdot 5^{2\beta+3} \cdot 7^{2\gamma} n + r_{21} \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} + 1 \right) \equiv 0. \quad (2.88)$$

$$\sum_{n=0}^{\infty} B \left( 64 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 8 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1 \right) q^n \equiv 2f_1^3, \quad (2.89)$$

$$\sum_{n=0}^{\infty} B \left( 64 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+1} n + 8 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2} + 1 \right) q^n \equiv 2f_7^3, \quad (2.90)$$

$$\begin{aligned} & B \left( 64 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 8 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1 \right) \\ & \equiv \begin{cases} 2 & \text{if } n = k(3k+1)/2 \text{ for some } k \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (2.91)$$

$$\sum_{n=0}^{\infty} B \left( 64 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 8 \cdot 3^{2\alpha+2} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1 \right) q^n \equiv 2f_3^3, \quad (2.92)$$

$$B \left( 64 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 136 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1 \right) \equiv 0, \quad (2.93)$$

$$B \left( 64 \cdot 3^{2\alpha+2} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + r_{22} \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1 \right) \equiv 0, \quad (2.94)$$

$$\sum_{n=0}^{\infty} B \left( 64 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 8 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} + 1 \right) q^n \equiv 2f_5^3, \quad (2.95)$$

$$B \left( 64 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + r_{23} \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1 \right) \equiv 0, \quad (2.96)$$

$$B \left( 64 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + r_{24} \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) \equiv 0, \quad (2.97)$$

$$\sum_{n=0}^{\infty} B \left( 64 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 8 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) q^n \equiv 2f_1^3, \quad (2.98)$$

$$\sum_{n=0}^{\infty} B \left( 64 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+1} n + 8 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+2} + 1 \right) q^n \equiv 2f_7^3, \quad (2.99)$$

$$\begin{aligned} & B \left( 64 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 8 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) \\ & \equiv \begin{cases} 2 & \text{if } n = k(3k+1)/2 \text{ for some } k \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (2.100)$$

$$\sum_{n=0}^{\infty} B \left( 64 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 8 \cdot 3^{2\alpha+2} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) q^n \equiv 2f_3^3, \quad (2.101)$$

$$B \left( 64 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 136 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) \equiv 0, \quad (2.102)$$

$$B \left( 64 \cdot 3^{2\alpha+2} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + r_{22} \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) \equiv 0, \quad (2.103)$$

$$\sum_{n=0}^{\infty} B \left( 64 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + 8 \cdot 3^{2\alpha} \cdot 5^{2\beta+3} \cdot 7^{2\gamma} + 1 \right) q^n \equiv 2f_5^3, \quad (2.104)$$

$$B \left( 64 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + r_{23} \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1 \right) \equiv 0, \quad (2.105)$$

$$B \left( 64 \cdot 3^{2\alpha} \cdot 5^{2\beta+3} \cdot 7^{2\gamma} n + r_{24} \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} + 1 \right) \equiv 0. \quad (2.106)$$

### 3. PROOF OF THE THEOREM (2.1)

Using (2.4) in (1.1) and then collecting the terms involving  $q^{2n+1}$  from both sides, we arrive at

$$\sum_{n=0}^{\infty} B(2n+1) q^n = 2 \frac{f_4^2 f_{10}^5}{f_2 f_{20}^2 f_1 f_5^3}. \quad (3.1)$$

From the binomial theorem, it is easy to see that for any positive integers  $k$  and  $m$ ,

$$f_k^{2m} \equiv f_{2k}^m \pmod{2}, \quad (3.2)$$

$$f_k^{4m} \equiv f_{2k}^{2m} \pmod{4}, \quad (3.3)$$

$$f_k^{8m} \equiv f_{2k}^{4m} \pmod{8}. \quad (3.4)$$

Employing (2.2) and (2.4) along with (3.2) and (3.4) in (3.1), we get, for modulo 16,

$$\sum_{n=0}^{\infty} B(4n+1) q^n \equiv 2 \frac{f_2^2 f_4 f_{10}^2}{f_{20} f_1^3 f_5} + 8q^3 f_2 f_{10}^7 \quad (3.5)$$

and

$$\sum_{n=0}^{\infty} B(4n+3) q^n \equiv 2 \frac{f_2^5 f_{20}}{f_4 f_{10} f_1^4} + 8q^2 f_2^2 f_{20}^2 f_1 f_5^3. \quad (3.6)$$

Utilizing (2.2) and (2.5) in (3.5), we arrive at

$$\sum_{n=0}^{\infty} B(8n+1) q^n \equiv 2 \frac{f_2^2 f_4 f_{10}^2}{f_{20} f_1^3 f_5} + 8q f_2^7 f_{10} \quad (3.7)$$

and

$$\sum_{n=0}^{\infty} B(8n+5) q^n \equiv 8 \frac{f_4^5}{f_1^3 f_5} - 2 \frac{f_2^5 f_{20}}{f_4 f_{10} f_1^4} + 8q f_{10}^2 f_1 f_5^3. \quad (3.8)$$

Using (2.2) and (2.5) in (3.7), we get

$$B(16n+1) \equiv B(8n+1). \quad (3.9)$$

By induction on  $\alpha$ , we obtain (2.10).

Employing (2.2) and (2.5) in (3.7) and then collecting the coefficients  $q^{2n+1}$  from both sides, we get

$$\sum_{n=0}^{\infty} B(16n+9) q^n \equiv 8 \frac{f_4^5}{f_1^3 f_5} - 2 \frac{f_2^5 f_{20}}{f_4 f_{10} f_1^4} + 8f_2^2 f_1^3 f_5. \quad (3.10)$$

Using (2.2), (2.6) and (2.7) in (3.10), we get

$$\sum_{n=0}^{\infty} B(32n+9) q^n \equiv 8f_2^3 + 8 \frac{f_2^7}{f_1^3 f_5} - 2 \frac{f_2 f_{10}}{f_1 f_5} \quad (3.11)$$

and

$$\sum_{n=0}^{\infty} B(32n+25) q^n \equiv 8f_{10} f_1^3 f_5 + 8f_2^5 f_1^3 f_5 + 8f_8 f_{10}. \quad (3.12)$$

Utilizing (2.7), the equation (3.12) reduces to

$$\sum_{n=0}^{\infty} B(64n+25) q^n \equiv 8f_1^9 \quad (3.13)$$

and

$$\sum_{n=0}^{\infty} B(64n + 57) q^n \equiv 8f_1 f_{20} + 8f_2^3 f_5^3. \quad (3.14)$$

The equation (3.14) is  $\alpha = 0$  case of (2.11). Suppose that the congruence (2.11) is true for  $\alpha \geq 0$ .

Ramanujan recorded the following identity in his notebooks without proof:

$$f_1 = f_{25}(R(q^5)^{-1} - q - q^2 R(q^5)), \quad (3.15)$$

$$\text{where } R(q) = \frac{f(-q, -q^4)}{f(-q^2, -q^3)}.$$

For a proof of (3.15), one can see [6], [14].

Using (3.15) in (2.11) and then collecting the coefficients of  $q^{5n+1}$ , we get

$$\sum_{n=0}^{\infty} B(64 \cdot 5^{2\alpha+1} n + 24 \cdot 5^{2\alpha+1} + 1) q^n \equiv 8f_4 f_5 + 8q f_1^3 f_{10}^3, \quad (3.16)$$

which proves (2.12). Again, using (3.15) in (3.16) and then collecting the coefficients of  $q^{5n+4}$  from the resultant equation, we obtain

$$\sum_{n=0}^{\infty} B(64 \cdot 5^{2\alpha+2} n + 56 \cdot 5^{2\alpha+2} + 1) q^n \equiv 8f_1 f_{20} + 8f_2^3 f_5^3, \quad (3.17)$$

which implies that the congruence (2.11) is true for  $\alpha + 1$ . By mathematical induction, the congruence (2.11) is true for all integers  $\alpha \geq 0$ .

Collecting the coefficients of  $q^{5n+i}$  for  $i = 3, 4$  from (2.11) along with (3.15), we obtain (2.13).

The congruence (3.13) is  $\alpha = \beta = \gamma = 0$  case of (2.14). Suppose that the congruence (2.14) holds for  $\alpha \geq 0$  with  $\beta = \gamma = 0$ . From (2.3), the congruence (2.14) with  $\beta = \gamma = 0$  becomes

$$\sum_{n=0}^{\infty} B(64 \cdot 3^{4\alpha} n + 24 \cdot 3^{4\alpha} + 1) q^n \equiv 8f_1^9. \quad (3.18)$$

Using (2.3) in (3.18) and then collecting the coefficients of  $q^{3n}$ , we get

$$\begin{aligned} & \sum_{n=0}^{\infty} B(64 \cdot 3^{4\alpha+1} n + 24 \cdot 3^{4\alpha} + 1) q^n \\ & \equiv 8f_1^3 + 8q f_3^9 \equiv 8f_3 + 8q f_3^9 + 8q f_3^3, \end{aligned} \quad (3.19)$$

which implies

$$\begin{aligned} & \sum_{n=0}^{\infty} B(64 \cdot 3^{4\alpha+2} n + 8 \cdot 3^{4\alpha+3} + 1) q^n \\ & \equiv 8f_1^9 + 8f_3^3 \equiv 8q f_6 f_9^3 + 8q^2 f_3 f_9^6 + 8q^3 f_9^9, \end{aligned} \quad (3.20)$$

which yields

$$\sum_{n=0}^{\infty} B(64 \cdot 3^{4\alpha+3} n + 8 \cdot 3^{4\alpha+3} + 1) q^n \equiv 8q f_3^9, \quad (3.21)$$

which reduces to

$$\sum_{n=0}^{\infty} B(64 \cdot 3^{4\alpha+4}n + 24 \cdot 3^{4\alpha+4} + 1) q^n \equiv 8f_1^9, \quad (3.22)$$

which implies that the congruence (2.14) is true for  $\alpha + 1$ . By mathematical induction, the congruence (2.14) is true for all integers  $\alpha \geq 0$  with  $\beta = \gamma = 0$ . Suppose that the congruence (2.14) is true for  $\alpha, \beta \geq 0$  with  $\gamma = 0$ . From the equation (2.14) with  $\gamma = 0$  and employing (3.15), we get

$$\sum_{n=0}^{\infty} B(64 \cdot 3^{4\alpha} \cdot 5^{2\beta+1}n + 56 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} + 1) q^n \equiv 8qf_5^9, \quad (3.23)$$

which implies

$$\sum_{n=0}^{\infty} B(64 \cdot 3^{4\alpha} \cdot 5^{2\beta+2}n + 24 \cdot 3^{4\alpha} \cdot 5^{2\beta+2} + 1) q^n \equiv 8f_1^9, \quad (3.24)$$

which implies that the congruence (2.14) with  $\gamma = 0$  is true for  $\beta + 1$ . So, by induction, the congruence (2.14) with  $\gamma = 0$  is true for all integers  $\alpha, \beta \geq 0$ . Suppose that the congruence (2.14) is true for  $\alpha, \beta, \gamma \geq 0$  and utilizing (2.9) in (2.14), we arrive at

$$\sum_{n=0}^{\infty} B(64 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+1}n + 8 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+1} + 1) q^n \equiv 8q^2f_7^9, \quad (3.25)$$

which proves (2.15). Collecting the coefficients of  $q^{7n+2}$  from (3.25), we obtain

$$\sum_{n=0}^{\infty} B(64 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2}n + 24 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2} + 1) q^n \equiv 8f_1^9, \quad (3.26)$$

which implies that the congruence (2.14) is true for  $\gamma + 1$ . By induction, the congruence (2.14) is true for all integers  $\alpha, \beta, \gamma \geq 0$ .

From the equation (2.15), we obtain (2.16).

Using (3.15) in (2.14), we obtain (2.17).

From the congruence (2.17), we obtain (2.18).

Using (2.3) in (2.14) and then collecting the coefficients of  $q^{3n+1}$  and  $q^{3n+2}$  from the resultant equation, we obtain (2.19) and (2.20) respectively.

Utilizing (3.15) in (2.19) and (2.20), we obtain (2.21) and (2.22) respectively.

From the congruences (2.21) and (2.22), we obtain (2.23) and (2.24) respectively.

Employing (2.2), (2.6) and (2.8) in (3.8), we obtain

$$\sum_{n=0}^{\infty} B(16n + 5) q^n \equiv 8 \frac{f_2^7}{f_1^3 f_5} + 8qf_{10}^3 - 2 \frac{f_2 f_{10}}{f_1 f_5} \quad (3.27)$$

and

$$\sum_{n=0}^{\infty} B(16n + 13) q^n \equiv 8f_8 f_{10} + 8f_2^5 f_1^3 f_5 + 8f_{10} f_1^3 f_5. \quad (3.28)$$

Using (2.7) in (3.28), we get

$$\sum_{n=0}^{\infty} B(32n + 13) q^n \equiv 8f_1^9 \quad (3.29)$$

and

$$\sum_{n=0}^{\infty} B(32n+29)q^n \equiv 8f_1f_{20} + 8f_2^3f_5^3. \quad (3.30)$$

The equation (3.30) is  $\alpha = 0$  case of (2.25). The rest of the proofs of the identities (2.25)-(2.27) are similar to the proofs of the identities (2.11)-(2.13). So, we omit the details.

The congruence (3.29) is  $\alpha = \beta = \gamma = 0$  case of (2.28). The rest of the proofs of the identities (2.28)-(2.38) are similar to the proofs of the identities (2.14)-(2.24). So, we omit the details.

Employing (2.2) and (2.8) in (3.6), we get

$$\sum_{n=0}^{\infty} B(8n+3)q^n \equiv 2\frac{f_2f_{10}}{f_1f_5} + 8qf_2f_{10}^2f_1^3f_5 \quad (3.31)$$

and

$$\sum_{n=0}^{\infty} B(8n+7)q^n \equiv 8f_2^5f_1^3f_5 + 8qf_2f_{10}^4. \quad (3.32)$$

Using (2.7) in (3.32), we obtain

$$\sum_{n=0}^{\infty} B(16n+7)q^n \equiv 8f_1^9 \quad (3.33)$$

and

$$\sum_{n=0}^{\infty} B(16n+15)q^n \equiv 8f_1f_{20} + 8f_2^3f_5^3. \quad (3.34)$$

The equation (3.34) is  $\alpha = 0$  case of (2.39). The rest of the proofs of the identities (2.39)-(2.41) are similar to the proofs of the identities (2.11)-(2.13). So, we omit the details.

The congruence (3.33) is  $\alpha = \beta = \gamma = 0$  case of (2.42). The rest of the proofs of the identities (2.42)-(2.52) are similar to the identities (2.14)-(2.24). So, we omit the details.

#### 4. PROOF OF THE THEOREM (2.2)

From (3.31), we have, modulo 4,

$$\begin{aligned} \sum_{n=0}^{\infty} B(8n+3)q^n &\equiv 2\frac{f_1f_5^3}{f_{10}} \\ &\equiv 2f_2^3 + 2qf_{10}^3, \end{aligned} \quad (4.1)$$

which implies

$$\sum_{n=0}^{\infty} B(16n+3)q^n \equiv 2f_1^3 \quad (4.2)$$

and

$$\sum_{n=0}^{\infty} B(16n+11)q^n \equiv 2f_5^3. \quad (4.3)$$

The equation (4.2) is  $\alpha = \beta = \gamma = 0$  case of (2.53). Suppose that the congruence (2.53) is true for  $\alpha \geq 0$  with  $\beta = \gamma = 0$ . From (2.53) with  $\beta = \gamma = 0$ , we have

$$\sum_{n=0}^{\infty} B(16 \cdot 3^{2\alpha} n + 2 \cdot 3^{2\alpha} + 1) q^n \equiv 2f_1^3. \quad (4.4)$$

Utilizing (2.3), the equation (4.4) reduces to

$$\sum_{n=0}^{\infty} B(16 \cdot 3^{2\alpha+1} n + 2 \cdot 3^{2\alpha+2} + 1) q^n \equiv 2f_3^3, \quad (4.5)$$

which yields

$$\sum_{n=0}^{\infty} B(16 \cdot 3^{2\alpha+2} n + 2 \cdot 3^{2\alpha+2} + 1) q^n \equiv 2f_1^3, \quad (4.6)$$

which implies that the congruence (2.53) is true for  $\alpha+1$  with  $\beta = \gamma = 0$ . By mathematical induction, the congruence (2.53) is true for all  $\alpha \geq 0$ . Suppose that the congruence (2.53) holds for  $\alpha, \beta \geq 0$  with  $\gamma = 0$ . Employing (3.15) in (2.53) with  $\gamma = 0$ , we get

$$\sum_{n=0}^{\infty} B(16 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} + 1) q^n \equiv 2f_5^3, \quad (4.7)$$

which implies

$$\sum_{n=0}^{\infty} B(16 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} + 1) q^n \equiv 2f_1^3, \quad (4.8)$$

which implies that the congruence (2.53) is true for  $\beta+1$  with  $\gamma = 0$ . By mathematical induction, the congruence (2.53) is true for all non-negative integers  $\alpha, \beta$  with  $\gamma = 0$ . Suppose that the congruence (2.53) holds for  $\alpha, \beta, \gamma \geq 0$ . Employing (2.9) in (2.53), we get

$$\sum_{n=0}^{\infty} B(16 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+1} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2} + 1) q^n \equiv 2f_7^3, \quad (4.9)$$

which proves (2.54). The congruence (4.9) reduces to

$$\sum_{n=0}^{\infty} B(16 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2} + 1) q^n \equiv 2f_1^3, \quad (4.10)$$

which implies that the congruence (2.53) is true for  $\gamma+1$ . By mathematical induction, the congruence (2.53) is true for all integers  $\alpha, \beta, \gamma \geq 0$ .

Using (2.3) in (2.53) and then collecting the coefficients of  $q^{3n}, q^{3n+1}$  and  $q^{3n+2}$ , we obtain (2.55), (2.56) and (2.57) respectively.

Collecting the coefficients of  $q^{3n+1}$  and  $q^{3n+2}$  from (2.56), we get (2.58).

Employing (3.15) in (2.53) and then collecting the coefficients of  $q^{5n+3}$ , we obtain (2.59).

Collecting the coefficients of  $q^{5n+2}$  and  $q^{5n+4}$  from (2.53) along with (3.15), we obtain (2.60).

Collecting the coefficients of  $q^{5n+i}$  for  $i = 1, 2, 3, 4$  from (2.59), we arrive at (2.61).

From (4.3), we deduce

$$\sum_{n=0}^{\infty} B(80n + 11) q^n \equiv 2f_1^3. \quad (4.11)$$

The congruence (4.11) is  $\alpha = \beta = \gamma = 0$  case of (2.62). The rest of the proofs of the identities (2.62)-(2.70) are similar to the proofs of the identities (2.53)-(2.61). So, we omit the details.

From (3.27), we arrived at

$$\sum_{n=0}^{\infty} B(16n+5) q^n \equiv 2 \frac{f_1 f_5^3}{f_{10}}. \quad (4.12)$$

Employing (2.8) in (4.12), we get

$$\sum_{n=0}^{\infty} B(32n+5) q^n \equiv 2 f_1^3 \quad (4.13)$$

and

$$\sum_{n=0}^{\infty} B(32n+21) q^n \equiv 2 f_5^3. \quad (4.14)$$

The rest of the proofs of the identities (2.71)-(2.88) are similar to the proofs of the identities (2.53)-(2.61). So, we omit the details.

From the equation (3.11), we have

$$\begin{aligned} \sum_{n=0}^{\infty} B(32n+9) q^n &\equiv 2 \frac{f_1 f_{10}}{f_5} \\ &\equiv 2 f_2^3 + 2 q f_{10}^3, \end{aligned} \quad (4.15)$$

which yields

$$\sum_{n=0}^{\infty} B(64n+9) q^n \equiv 2 f_1^3 \quad (4.16)$$

and

$$\sum_{n=0}^{\infty} B(64n+41) q^n \equiv 2 f_5^3. \quad (4.17)$$

The rest of the proofs of the identities (2.89)-(2.106) are similar to the proofs of the identities (2.53)-(2.61). So, we omit the details.

#### ACKNOWLEDGMENT

The author are thankful to the referee for his/her comments which improves the quality of our paper.

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