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SOME MODULAR RELATIONS BETWEEN RAMANUJAN'S FUNCTION $\nu(q)$ AND $\nu(q^n)$

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ABSTRACT. In his 'lost' notebook, S. Ramanujan introduced the parameter $\kappa(q) := R(q)R^2(q^2)$ and $\nu(q) := R^2(q^{\frac{1}{2}})R(q)/R(q^2)$ related to the Rogers-Ramanujan continued fraction $R(q)$. In this paper, we establish some new modular relations connecting $\kappa(q)$ with $\nu(q^n)$ and $\nu(q)$ with $\nu(q^n)$ for $n = 6, 8 \ and \ 10$.

1. INTRODUCTION

The Rogers-Ramanujan continued fraction is defined by

$$R(q) := \frac{q^{1/5}}{1} \, \frac{}{+} \, \frac{q}{1} \, \frac{}{+} \, \frac{q^2}{1} \, \frac{}{+} \, \frac{q^3}{1} \, \frac{}{+} \cdots, \quad |q| < 1, \tag{1.1}$$

was first studied by L. J Rogers [18]. Later, this continued fraction was rediscovered by S. Ramanujan and recorded may interesting results involving $R(q)$. For more details on $R(q)$ one can see, [3], [4], [8], [19], [21], and [22].

In his 'lost' notebook Ramanujan [17], introduced the parameters $\mu(q)$ and $\kappa(q)$ which are related to Rogers-Ramanujan continued fraction. Ramanujan stated several interesting identities involving the parameters $\mu(q)$ and $\kappa(q)$. These results were studied in detail by S. -Y. Kang [11]. Kang also introduced a new parameter $\nu(q)$ which is analogous to $\mu(q)$ and $\kappa(q)$ and established some identities. Recently, C. Gugg [10] established certain identities of Ramanujan using the parameter $\kappa(q)$. S. Cooper [9], also systematically studied several results involving the parameter $\kappa(q)$. M. S. Mahadeva Naika, B. N. Dharmendra and S. Chandankumar [14], [15] established several results involving the parameter $\mu(q)$, $\kappa(q)$ and $\nu(q^n)$.

Recently, Andrews et al. [5] involving combinatorial partition identities associated with the following general family

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$$R(s,t,l,u,v,w) := \sum_{n=0}^{\infty} q^{s\begin{pmatrix} n \\ 2 \end{pmatrix}+tn} r(l,u,v,w;n), \tag{1.2}$$

where

$$r(l,u,v,w;n) := \sum_{j=0}^{[n/u]} (-1)^j \frac{q^{uv\begin{pmatrix} j \\ 2 \end{pmatrix}+(w-ul)j}}{(q;q)_{n-uj}(q^{uv};q^{uv})_j}. \tag{1.3}$$

In particular, we recall the following combinatorial partition identities [5, p.106, Theorem 3]

$$R(2,1,1,1,2,2) := (-q;q^2)_\infty, \tag{1.4}$$

$$R(2,2,1,1,2,2) := (-q^2;q^2)_\infty, \tag{1.5}$$

and

$$R(m,m,1,1,1,2) := \frac{(q^{2m};q^{2m})_\infty}{(q^m;q^{2m})_\infty}. \tag{1.6}$$

Recently, H. M. Srivastava and M. P. Chaudhary, they stated and proved some theorems associated with the family R(s,t,l,u,v,w) defined by (1.2), which depict inter-relationships between q-product identities, continued-fraction identities and combinatorial partition identities.

Putting $m = 1$ in the (1.6), we get

$$R(1,1,1,1,1,2) := \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}. \tag{1.7}$$

which is one of the ramanujan's special case of theta function definition $\psi(q)$.

Recently, C. Adiga at al. [2], they established several modular relations for the Rogers-Ramanujan type functions of order eleven which are analogous to Ramanujan's forty identities for Rogers-Ramanujan functions and also gave interesting partition-theoretic interpretation of some of the modular relations which are derived.

In Chapter 16, of his second notebook [16] Ramanujan defined his theta-function as

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1, \\ = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{1.8}$$

Three special cases of $f(a,b)$ are as follows:

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{1.9}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.10}$$

$$f(-q) := \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2} = (q;q)_\infty, \tag{1.11}$$

where

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \qquad |q| < 1.$$

The ordinary hypergeometric series ${}_2F_1(a,b;c;x)$ is defined by

$${}_2F_1(a,b;c;x) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n,$$

where $(a)_0 = 1, (a)_n = a(a+1)(a+2)\cdots(a+n-1)$ for any positive integer n, and $|\ x\ | < 1$.

Let

$$z := z(x) := {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) \tag{1.12}$$

and

$$q := q(x) := \exp\left(-\pi \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)}\right), \tag{1.13}$$

where $0 < x < 1$.

Let r denote a fixed natural number and assume that the following relation holds:

$$r\frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)}. \tag{1.14}$$

Then a modular equation of degree r in the classical theory is a relation between α and β induced by (1.14). We often say that β is of degree r over α and $m := \dfrac{z(\alpha)}{z(\beta)}$ is called the multiplier. We also use the notations $z_1 := z(\alpha)$ and $z_r := z(\beta)$ to indicate that β has degree r over α.

In [6] and [23], the authors have defined two parameters $l_{k,n}$ and $l'_{k,n}$ as follows:

$$l_{k,n} := \frac{\psi(-e^{-\pi\sqrt{n/k}})}{k^{1/4}e^{-\frac{(k-1)\pi}{8}\sqrt{n/k}}\psi(-e^{-\pi\sqrt{nk}})}, \tag{1.15}$$

and

$$l'_{k,n} := \frac{\psi(e^{-\pi\sqrt{n/k}})}{k^{1/4}e^{-\frac{(k-1)\pi}{8}\sqrt{n/k}}\psi(e^{-\pi\sqrt{nk}})}. \tag{1.16}$$

They have established several properties and some explicit evaluations of $l_{k,n}$ and $l'_{k,n}$ for different positive rational values of n and k. Recently, M. S. Mahadeva Naika, S. Chandankumar, K. Sushan Bairy [13, 12] have established several new modular equations and also established general formulas for explicit evaluations of the ratios of Ramanujan's theta function ψ. In [15], Mahadeva Naika, Dharmendra and Chandankumar have established several new modular equations of degree 5 and established general formulas for the explicit evaluations of $l_{5,n}$.

In this paper, in Section 2 we record some preliminary results which are useful for our subsequent sections. In Section 3, we establish several new modular relations between $\kappa(q)$ and $\nu(q^n)$ for $n = 4, 6, 8, 10$. In section 4, we establish some modular relations between $\nu(q)$ and $\nu(q^n)$ for $n = 6, 8, 10$ by using section 3. Employing this modular relations we can establish some new modular relations between $P := \dfrac{\psi(-q)}{q^{1/2}\psi(-q^5)}$ and $Q := \dfrac{\psi(-q^n)}{q^{n/2}\psi(-q^{5n})}$ for $n = 6, 8, 10$ and using the above $P - Q$ modular relations of degree 5, we can establish some new general formulas for the explicit evaluations of $l_{5,n}$.

2. Preliminary results

In this section, we collect some identities which are useful in establishing our main results.

Lemma 2.1. [15, p.16] *If* $k := \kappa(q)$ *and* $u := \nu(q)$, *then*

$$(k^2 - 1)u^2 + 2(1 + 2k - k^2)u + k^2 = 1. \tag{2.1}$$

Lemma 2.2. [15, p.17] *If* $k := \kappa(q)$ *and* $u := \nu(q^2)$, *then*

$$(u - 1)k^2 + (u^2 + 1)k + u^2 = u. \tag{2.2}$$

Lemma 2.3. [4, Entry 1.8.1, p. 33] [11] *We have*

$$\frac{\psi(q)}{q^{\frac{1}{2}}\psi(q^5)} = \frac{1+\nu(q)}{1-\nu(q)}. \tag{2.3}$$

Lemma 2.4. [15, p. 15] *If* $u := \nu(q)$ *and* $v := \nu(q^2)$, *then*

$$v^2(u^4+1) + (1-2v-2v^2-2v^3+v^4)(u^3+u) + (-2v^4+10v^2-2)u^2 = 0. \tag{2.4}$$

Lemma 2.5. [15, p. 15] *If* $u := \nu(q)$ *and* $v := \nu(q^3)$, *then*

$$(3u+3u^2-3u^3+u^4)v^3 - 3v^2(u+u^3) + (3u^3-3u+3u^2+1)v = u^3+v^4u. \tag{2.5}$$

Lemma 2.6. [15, p. 15] *If* $u := \nu(q)$ *and* $v := \nu(q^4)$, *then*

$$\begin{aligned}&(v^8+6v^2+6v^6-4v^7-4v-14v^4)(u^7+u) + (32[v^7-v^5+v-v^3]-6v^8\\ &-6-48v^6-48v^2+136v^4)(u^6+u^2) + (15v^8-108v^7-108v+202v^6+15\\ &+80v^5-434v^4+202v^2+80v^3)(u^3+u^5) + (160v^7-20+160v-20v^8\\ &-160v^3-160v^5-320v^6-320v^2+750v^4)u^4+v^4+u^8v^4+u^7+u=0.\end{aligned} \tag{2.6}$$

Lemma 2.7. [15, p. 15] *If* $u := \nu(q)$ *and* $v := \nu(q^5)$, *then*

$$\begin{aligned}&(u^5+10u^3+6-5u^2-5u^4-10u)v^4 - (6u^5-35u+25u^3-25u^4+20u^2)v^3\\ &+(11u^5+25u^2-35u^4-25u+20u^3)v^2 - (6u^5-5u-10u^4-5u^3+10u^2)v\\ &+v+u^5-v^5-11v^3+6v^2=0.\end{aligned} \tag{2.7}$$

3. Modular Relations Between $\kappa(q)$ and $\nu(q^n)$

In this section, we establish several new modular relations connecting $\kappa(q)$ with $\nu(q^n)$

Theorem 3.1. *If* $u := \kappa(q)$ *and* $v := \nu(q^4)$, *then*

$$\begin{aligned}&(-1+v)u^4 + (-1-v^3-v+3v^2)u^3 + (-v^3-v)u^2\\ &+(v^3-3v^2+v+v^4)u - v^4+v^3 = 0.\end{aligned} \tag{3.1}$$

Proof. Replace q to q^2 in equation (2.4) and using the equation (2.2), we find that

$$\begin{aligned} &16(1 - v - u - 3v^2u^3 + 3v^2u + vu^3 + vu^2 - vu + v^3u^3 + v^3u^2 \\ &- v^3u + v^4u^4 + v^4u^3 - v^3u^4)(vu - v^4 + v^3 - u^4 - u^3 + 3v^2u^3 \\ &- 3v^2u + vu^4 - vu^3 - vu^2 - v^3u^3 - v^3u^2 + v^3u + v^4u) = 0. \end{aligned} \tag{3.2}$$

By examining the behavior of the above factors near $q = 0$, we can find a neighborhood about the origin, where the second factor is zero; whereas other factors are not zero in this neighborhood. By the Identity Theorem second factor vanishes identically. This completes the proof. □

Theorem 3.2. *If* $u := \kappa(q)$ *and* $v := \nu(q^6)$, *then*

$$\begin{aligned} &(v^3 - 4v^4 + 6v^5 + v^7 - 4v^6)u^8 + (-17v^6 + 21v^2 + 5v^7 - 7v + 27v^4 \\ &- 33v^3 + 1 + 3v^5)u^7 + (-6v^6 - 2 - 33v^5 - 42v^2 + 29v^3 + 34v^4 + 3v^7 \\ &+ 17v)u^6 + (23v^5 + v^8 - 117v^4 - 3v - 15v^7 + 49v^6 + 3v^2 + 59v^3)u^5 \\ &+ (-3v^5 + 2 + 44v^6 - 21v + 2v^8 - 60v^4 + 44v^2 - 3v^3 - 21v^7)u^4 + \\ &(-3v^6 - 23v^3 - 1 + 117v^4 + 3v^7 - 59v^5 + 15v - 49v^2)u^3 + (29v^5 + 3v \\ &- 2v^8 + 34v^4 + 17v^7 - 42v^6 - 33v^3 - 6v^2)u^2 + (17v^2 + 33v^5 - 21v^6 \\ &- 3v^3 - 5v - v^8 - 27v^4 + 7v^7)u + v + v^5 - 4v^2 - 4v^4 + 6v^3 = 0. \end{aligned} \tag{3.3}$$

Proof. Replace q to q^2 in equation (2.5) and using the equation (2.2), we arrive at the equation (3.3). □

Theorem 3.3. *If* $u := \kappa(q)$ *and* $v := \nu(q^8)$, *then*

$$\begin{aligned} &(v^4 - v^5)u^8 + (-1 + 6v^3 + 7v^4 - 10v^2 - 7v^5 + 5v)u^7 + (8v^3 + 3v \\ &- v^7 + 1 - 16v^2 - 12v^5 + 3v^6 + 14v^4)u^6 + (-10v + 15v^6 + 25v^2 \\ &- 32v^4 - 5v^7 - 3v^3 + 10v^5)u^5 + (-2v - 30v^4 + 5v^3 + 13v^2 - 2v^7 \\ &+ 5v^5 + 13v^6)u^4 + (32v^4 - 15v^2 + 3v^5 - 10v^3 + 5v + 10v^7 - 25v^6)u^3 \\ &+ (-v + 3v^2 - 16v^6 + 3v^7 + 14v^4 + 8v^5 + v^8 - 12v^3)u^2 + (v^8 + 7v^3 \\ &- 7v^4 - 6v^5 + 10v^6 - 5v^7)u + v^4 - v^3 = 0. \end{aligned} \tag{3.4}$$

Proof. Replace q to q^2 in equation (2.6) and using the equation (2.2), we find that

$$\begin{aligned}
&256(-10vu^3+25v^2u^3-10v^2u+30v^4u^4-5v^3u^4-32v^4u^3+2vu^4-u+7v^4u\\
&-3vu^2+5vu-3v^3u^3-8v^3u^2+6v^3u+12u^6v^3-3v^6u^2-10u^5v^3+10v^5u^3\\
&+15v^6u^3+12v^5u^2-u^2-v^4+16v^2u^2-13v^2u^4-6v^5u^7+10v^6u^7-25v^6u^5\\
&-5v^5u^4+32v^4u^5-8v^5u^6+3v^5u^5-5v^7u^7-3v^7u^6+10v^7u^5-13v^6u^4\\
&-7u^7v^4-14u^6v^4+2v^7u^4-u^8v^4+16v^6u^6-3u^6v^2-15u^5v^2+v^5+u^6v\\
&+5u^5v+7u^7v^3+u^8v^3-7v^5u-5v^7u^3+v^7u^2-14v^4u^2-v^8u^6+v^8u^7)\\
&(5vu^3-15v^2u^3-30v^4u^4+5v^3u^4+32v^4u^3-2vu^4-7v^4u-vu^2-10v^3u^3\\
&-12v^3u^2+7v^3u+8u^6v^3-16v^6u^2-3u^5v^3+3v^5u^3-25v^6u^3+8v^5u^2+v^4\\
&+3v^2u^2+13v^2u^4-7v^5u^7+15v^6u^5+5v^5u^4-v^5u^8-32v^4u^5-12v^5u^6\\
&+10v^5u^5-v^7u^6-5v^7u^5+13v^6u^4+7u^7v^4+14u^6v^4-2v^7u^4+u^8v^4\\
&-v^3+3v^6u^6-16u^6v^2+25u^5v^2+u^6-u^7+3u^6v-10u^5v+5u^7v-10u^7v^2\\
&+6u^7v^3-6v^5u+10v^6u+10v^7u^3+3v^7u^2-5v^7u+v^8u^2+v^8u+14v^4u^2)=0.
\end{aligned} \tag{3.5}$$

By examining the behavior of the above factors near $q=0$, we can find a neighborhood about the origin, where the second factor is zero; whereas other factors are not zero in this neighborhood. By the Identity Theorem second factor vanishes identically. This completes the proof. □

Theorem 3.4. *If* $u:=\kappa(q)$ *and* $v:=\nu(q^{10})$, *then*

$$\begin{aligned}
&(5v^2+3v+v^9+14v^5+13v^6-3v^8-5v^7-14v^4-13v^3-1)u^{10}+(10v^9-35v^8\\
&-15v-15v^7+10v^2+15v^3+110v^6+5v^5-90v^4+5)u^9+(-510v^5+35v^9-200v^2\\
&+45v-145v^8+285v^4+205v^3+225v^7+70v^6-10)u^8+(-220v^8+45v^9-1070v^3\\
&+1170v^4+10+570v^2+60v^5-105v-1530v^6+1070v^7)u^7+(160v-5-20v^9+65v^8\\
&+650v^7+2790v^3-5820v^4-3440v^6+6420v^5-800v^2)u^6+(619v^8+4388v^6\\
&-127v-127v^9-2496v^7+1+4388v^4-2496v^3+v^{10}-4738v^5+619v^2)u^5+(5v^{10}
\end{aligned}$$

$$-65v^2+800v^8+20v-6420v^5-650v^3-160v^9-2790v^7+5820v^6+3440v^4)u^4+$$
$$(-220v^2+570v^8+45v+1170v^6-105v^9-1070v^7+1070v^3+60v^5-1530v^4$$
$$+10v^{10})u^3+(-45v^9+10v^{10}-70v^4+145v^2-225v^3+200v^8-35v-205v^7$$
$$+510v^5-285v^6)u^2+(-15v^9+110v^4-15v^3+15v^7+10v-90v^6+5v^5-35v^2$$
$$+5v^{10}+10v^8)u+13v^7+3v^2-13v^4-5v^8+v^{10}+14v^6-3v^9-v-14v^5+5v^3=0. \tag{3.6}$$

Proof. Replace q to q^2 in equation (2.7) and using the equation (2.2), we arrive at the equation (3.6). □

4. Modular Relations Between $\nu(q)$ and $\nu(q^n)$

In this section, we establish several new modular relations connecting $\nu(q)$ with $\nu(q^n)$

Theorem 4.1. *If* $u := \nu(q)$ *and* $v := \nu(q^6)$, *then*

$$28v^6-56v^7+(5782v^7-14v-17906v^6+5226v^5-14v^{15}+2092v^{12}+2092v^4$$
$$+5226v^{11}+10440v^8+5782v^9-17906v^{10}+354v^{14}-1778v^{13}+354v^2$$
$$-1778v^3)u^5-56v^9+(-420v^2-2608v^{12}+1780v^{13}-7652v^5-9244v^9$$
$$-9244v^7+19604v^6+1780v^3+12v^{15}-2608v^4+19604v^{10}-420v^{14}$$
$$-17280v^8-7652v^{11}+12v)u^7+70v^8+28v^{10}+(5506v^5+1-34282v^9$$
$$+92146v^8+766v^2+6782v^4-12022v^6-3850v^{13}+5506v^{11}-34282v^7$$
$$-12022v^{10}-14v^{15}-14v+766v^{14}+6782v^{12}-3850v^3+v^{16})u^8+(5782v^7$$
$$-14v-17906v^6+5226v^5-14v^{15}+2092v^{12}+2092v^4+5226v^{11}+10440v^8$$
$$+5782v^9-17906v^{10}+354v^{14}-1778v^{13}+354v^2-1778v^3)u^{11}+(18437v^7$$
$$-105v^2+403v^{11}+403v^5-375v^{10}+1009v^3+18437v^9-375v^6-28700v^8$$
$$-9v-1426v^{12}+1009v^{13}-105v^{14}-9v^{15}-1426v^4)u^6+(64v^{11}-176v^6$$
$$-8v^4+192v^7-176v^{10}-144v^8-8v^{12}+64v^5+192v^9)u+(-2450v^9-6v^{15}$$

$$+178v^{11}+66v^2-282v^{13}+6984v^8-6v-1522v^6-2450v^7-1522v^{10}+524v^4$$
$$-282v^3+524v^{12}+66v^{14}+178v^5)u^{13}+(861v^{13}-3185v^5-922v^{12}-1787v^7$$
$$-922v^4+15v-207v^2+15v^{15}-3185v^{11}-207v^{14}+10127v^{10}+10127v^6+861v^3$$
$$-9548v^8-1787v^9)u^{12}+(178v^{11}-2450v^9-6v^{15}+66v^2-282v^{13}+66v^{14}-6v$$
$$+6984v^8-1522v^6-2450v^7-1522v^{10}+524v^4-282v^3+524v^{12}+178v^5)u^3+v^4$$
$$-8v^5+(v-134v^{12}+37v^{11}+235v^{10}+55v^{13}+55v^3-1460v^8-134v^4+235v^6$$
$$+v^{15}-11v^2+37v^5+547v^7+547v^9-11v^{14})u^2+(1780v^{13}-420v^2-2608v^{12}$$
$$-7652v^5-9244v^9-9244v^7+19604v^6+1780v^3+12v^{15}-2608v^4+19604v^{10}$$
$$-420v^{14}-17280v^8-7652v^{11}+12v)u^9+(v-134v^{12}+37v^{11}+235v^{10}+v^{15}$$
$$+55v^{13}+547v^9+55v^3-1460v^8-134v^4+235v^6-11v^2+37v^5+547v^7$$
$$-11v^{14})u^{14}-8v^{11}+v^{12}+(403v^{11}-105v^2+18437v^7+403v^5-375v^{10}$$
$$+1009v^3+18437v^9-375v^6-28700v^8-1426v^{12}+1009v^{13}-9v-105v^{14}$$
$$-1426v^4-9v^{15})u^{10}+(-56v^7+v^4-56v^9-8v^{11}+v^{12}+70v^8+28v^{10}$$
$$+28v^6-8v^5)u^{16}+(861v^{13}-3185v^5-922v^{12}-1787v^7-922v^4+15v-207v^2$$
$$+15v^{15}-3185v^{11}-207v^{14}+10127v^{10}+10127v^6+861v^3-9548v^8-1787v^9)u^4$$
$$+(-176v^6-8v^4+64v^{11}+192v^7-176v^{10}-144v^8-8v^{12}+64v^5+192v^9)u^{15}=0. \tag{4.1}$$

Proof. Using the equations (2.1) and (3.3), we arrive at the equation (4.1). □

Theorem 4.2. *If* $u:=\nu(q)$ *and* $v:=\nu(q^8)$, *then*

$$3003v^7-2002v^6+3003v^9-3432v^8-2002v^{10}+v+(10571436v^{10}+7636v^2$$
$$+7636v^{14}+10571436v^6-6v-3598294v^{11}-20445730v^9+25564822v^8-6v^{15}$$
$$+v^{16}-20445730v^7+813108v^4-114178v^3-114178v^{13}-3598294v^5+1$$
$$+813108v^{12})u^8+(8v^{15}-6169280v^{10}+11635096v^7-492352v^{12}-4160v^2$$
$$-14406528v^8+2158888v^{11}-492352v^4+66872v^3+2158888v^5+66872v^{13}$$
$$+8v-6169280v^6+11635096v^9-4160v^{14})u^7+(-1168v^{14}-4376704v^8$$

$$+ 16032v^3 - 48v^{15} - 98112v^4 - 48v - 1554416v^6 + 458816v^{11} + 3367248v^7$$
$$+ 458816v^5 - 1554416v^{10} + 16032v^{13} - 98112v^{12} + 3367248v^9 - 1168v^2)u^6$$
$$+ (-9207752v^9 + 376320v^{12} - 9207752v^7 + 40v^{15} + 11449344v^8 + 2880v^{14}$$
$$- 50888v^{13} + 4820160v^{10} + 376320v^4 + 40v + 4820160v^6 - 1665432v^{11}$$
$$- 1665432v^5 + 2880v^2 - 50888v^3)u^5 + (6334002v^9 - 3432032v^{10} + 22v$$
$$- 2080v^2 + 6334002v^7 + 1227734v^5 + 1227734v^{11} - 7769792v^8 - 3432032v^6$$
$$+ 36322v^{13} + 36322v^3 - 2080v^{14} + 22v^{15} - 279072v^{12} - 279072v^4)u^4$$
$$+ (111488v^{12} - 2371472v^9 - 480368v^{11} - 2371472v^7 + 1152v^{14} + 2883840v^8$$
$$- 480368v^5 + 111488v^4 + 1152v^2 - 48v^{15} - 48v - 15312v^{13} + 1312640v^6$$
$$- 15312v^3 + 1312640v^{10})u^3 + (28v^{15} + 111596v^{11} - 626048v^8 - 28096v^4$$
$$- 496v^2 - 28096v^{12} + 111596v^5 + 518612v^7 - 293264v^6 + 4644v^{13} + 28v$$
$$+ 4644v^3 + 518612v^9 - 293264v^{10} - 496v^{14})u^2 + (73344v^8 - 8v^{15} + 128v^2$$
$$- 61592v^9 + 36480v^{10} - 952v^3 - 952v^{13} + 4544v^{12} - 15272v^5 - 15272v^{11} - 8v$$
$$+ 36480v^6 - 61592v^7 + 4544v^4 + 128v^{14})u + (3003v^7 - 364v^4 - 2002v^6$$
$$- 14v^{14} + 1001v^5 - 3432v^8 + 1001v^{11} - 2002v^{10} + 3003v^9 + v - 14v^2 - 364v^{12}$$
$$+ 91v^{13} + 91v^3 + v^{15})u^{16} + (73344v^8 - 8v^{15} + 128v^2 - 61592v^9 + 36480v^{10}$$
$$- 952v^3 - 952v^{13} + 4544v^{12} - 15272v^5 - 15272v^{11} - 8v + 36480v^6 - 61592v^7$$
$$+ 4544v^4 + 128v^{14})u^{15} + (111488v^{12} - 2371472v^9 - 480368v^{11} - 2371472v^7$$
$$+ 1152v^{14} + 2883840v^8 - 480368v^5 + 111488v^4 + 1152v^2 - 48v^{15} - 15312v^{13}$$
$$- 48v + 1312640v^6 - 15312v^3 + 1312640v^{10})u^{13} + (6334002v^9 - 3432032v^{10}$$
$$- 2080v^2 + 6334002v^7 + 1227734v^5 + 1227734v^{11} - 7769792v^8 - 3432032v^6$$
$$+ 22v + 36322v^{13} + 36322v^3 - 2080v^{14} + 22v^{15} - 279072v^{12} - 279072v^4)u^{12}$$
$$+ (-9207752v^9 + 376320v^{12} - 9207752v^7 + 40v^{15} + 11449344v^8 + 2880v^{14}$$
$$- 50888v^{13} + 4820160v^{10} + 376320v^4 + 4820160v^6 - 1665432v^{11} + 40v$$
$$- 1665432v^5 + 2880v^2 - 50888v^3)u^{11} + (-1168v^{14} - 4376704v^8 + 16032v^3$$
$$- 48v^{15} - 98112v^4 - 48v - 1554416v^6 + 458816v^{11} + 3367248v^7 + 458816v^5$$
$$- 1554416v^{10} + 16032v^{13} - 98112v^{12} + 3367248v^9 - 1168v^2)u^{10} + (8v^{15}$$

$$
\begin{aligned}
&- 6169280v^{10} + 11635096v^7 - 492352v^{12} - 4160v^2 - 14406528v^8 + 2158888v^{11}\\
&- 492352v^4 + 66872v^3 + 2158888v^5 + 66872v^{13} + 8v - 6169280v^6 + 11635096v^9\\
&- 4160v^{14})u^9 + (28v^{15} + 111596v^{11} - 626048v^8 - 28096v^4 - 496v^2 - 28096v^{12}\\
&+ 111596v^5 + 518612v^7 - 293264v^6 + 4644v^{13} + 28v + 4644v^3 + 518612v^9\\
&- 293264v^{10} - 496v^{14})u^{14} + 1001v^{11} + 91v^{13} - 364v^{12} + v^{15} - 14v^{14} - 14v^2\\
&- 364v^4 + 91v^3 + 1001v^5 = 0.
\end{aligned}
\tag{4.2}
$$

Proof. Using the equations (2.1) and (3.4), we arrive at the equation (4.2). □

Theorem 4.3. *If* $u := \nu(q)$ *and* $v := \nu(q^{10})$, *then*

$$
\begin{aligned}
&(117474v^5 + 117474v^{15} + 381640v^3 + 1502740v^8 - 2951050v^{13} - 123160v^{18}\\
&+ 21730v^{19} + 1502740v^{12} - 614020v^4 + 1556200v^{14} - 1680v^{20} + 2341630v^9\\
&- 2951050v^7 - 614020v^{16} + 21730v - 4437408v^{10} - 123160v^2 + 1556200v^6\\
&+ 381640v^{17} + 2341630v^{11} - 1680)u^{15} + (610250v^7 + 20960v^2 - 54776v^{15}\\
&+ 260 - 342910v^{12} + 125570v^4 + 20960v^{18} - 478645v^{11} - 3505v - 478645v^9\\
&- 273350v^{14} + 260v^{20} - 3505v^{19} - 54776v^5 - 69740v^{17} - 342910v^8 + 125570v^{16}\\
&- 69740v^3 + 610250v^{13} + 874428v^{10} - 273350v^6)u^{16} + (13346v^{15} - 2180v^{18} + 340v\\
&+ 29160v^6 + 87440v^9 + 8040v^3 + 69620v^8 - 17020v^4 - 191880v^{10} + 69620v^{12}\\
&- 24 - 2180v^2 - 92270v^7 + 13346v^5 + 8040v^{17} + 340v^{19} + 29160v^{14} + 87440v^{11}\\
&- 24v^{20} - 92270v^{13} - 17020v^{16})u^{17} + (5535v^{13} + 25490v^{10} - 3775v^9 + 1070v^{14}\\
&+ 5535v^7 + 105v^{18} - 450v^{17} - 3775v^{11} + 1300v^{16} - 15v - 15v^{19} + 1070v^6 + v^{20}\\
&+ 105v^2 - 2319v^{15} - 450v^3 - 2319v^5 - 11125v^{12} + 1300v^4 + 1 - 11125v^8)u^{18}\\
&+ (-26v^{15} + 240v^{14} - 1952v^{10} + 240v^6 - 580v^{11} - 580v^9 + 1760v^8 - 930v^{13}\\
&+ 1760v^{12} - 930v^7 - 26v^5)u^{19} + (45v^7 - 120v^{12} - 10v^6 + 210v^9 - 252v^{10}\\
&+ v^{15} - 120v^8 + 45v^{13} + 210v^{11} + v^5 - 10v^{14})u^{20} + (15391800v^6 - 26253450v^7\\
&- 1399720v^2 - 41472800v^{10} - 58142v^{15} - 21672v^{20} - 5668420v^{16} + 3982640v^3
\end{aligned}
$$

$$
\begin{aligned}
&+22309390v^9+264810v^{19}+12144620v^8+264810v-1399720v^{18}-5668420v^4\\
&-21672+22309390v^{11}+12144620v^{12}+3982640v^{17}+15391800v^{14}-26253450v^{13}\\
&-58142v^5)u^{13}+45v^7-10v^{14}+(7210v^{20}-90255v-8298680v^9-5626620v^{14}\\
&-8298680v^{11}+15962474v^{10}+491335v^2+9950890v^7+491335v^{18}-1447865v^3\\
&-164778v^{15}+2165570v^{16}-5626620v^6-90255v^{19}-4844140v^8-4844140v^{12}\\
&+2165570v^4-164778v^5-1447865v^{17}+9950890v^{13}+7210)u^{14}+(-54897580v^{14}\\
&+86515v^{20}+89630310v^7+2031946v^{15}+5239930v^{18}-1027790v-1027790v^{19}\\
&+86515-54897580v^6+5239930v^2-39426780v^8+141481188v^{10}+2031946v^5\\
&-39426780v^{12}+89630310v^{13}-14309220v^{17}-76950910v^{11}-14309220v^3\\
&-76950910v^9+19292585v^4+19292585v^{16})u^{10}+(34554940v^8+12424270v^3\\
&-4529580v^{18}+47876040v^6+34554940v^{12}-4529580v^2-78534180v^7\\
&-16843860v^{16}+47876040v^{14}+66626380v^9+12424270v^{17}+66626380v^{11}\\
&+884990v^{19}-78534180v^{13}-1639028v^{15}-121427896v^{10}-1639028v^5\\
&-74280v^{20}+884990v-16843860v^4-74280)u^{11}+(764523v^{15}-44191895v^{11}\\
&+764523v^5-31486360v^{14}+2922370v^{18}+46928-8124495v^{17}+2922370v^2\\
&+52411245v^{13}-564210v-44191895v^9-23299010v^{12}+80021580v^{10}\\
&-8124495v^3+46928v^{20}-564210v^{19}-31486360v^6+52411245v^7\\
&-23299010v^8+11207010v^4+11207010v^{16})u^{12}-10v^6+(610250v^7\\
&+20960v^2-54776v^{15}+260-342910v^{12}+125570v^4+20960v^{18}-478645v^{11}\\
&-3505v-478645v^9-273350v^{14}+260v^{20}-3505v^{19}-54776v^5-69740v^{17}\\
&-342910v^8+125570v^{16}-69740v^3+610250v^{13}+874428v^{10}-273350v^6)u^4\\
&+(117474v^5+117474v^{15}+381640v^3+1502740v^8-2951050v^{13}-123160v^{18}\\
&+21730v^{19}+1502740v^{12}-614020v^4+1556200v^{14}-1680v^{20}+2341630v^9\\
&-2951050v^7-614020v^{16}+21730v-4437408v^{10}-123160v^2+1556200v^6\\
&+381640v^{17}+2341630v^{11}-1680)u^5+(7210v^{20}-90255v-8298680v^9\\
&-5626620v^{14}-8298680v^{11}+15962474v^{10}+491335v^2+9950890v^7\\
&+491335v^{18}-1447865v^3-164778v^{15}+2165570v^{16}-5626620v^6\\
&-90255v^{19}-4844140v^8-4844140v^{12}+2165570v^4-164778v^5\\
&-1447865v^{17}+9950890v^{13}+7210)u^6+(15391800v^6-26253450v^7\\
&-1399720v^2-41472800v^{10}-58142v^{15}-21672v^{20}-5668420v^{16}
\end{aligned}
$$

$$
\begin{aligned}
&+ (764523v^{15} - 44191895v^{11} + 764523v^5 - 31486360v^{14} + 2922370v^{18}\\
&+ 46928 - 8124495v^{17} + 2922370v^2 + 52411245v^{13} - 564210v - 44191895v^9\\
&- 23299010v^{12} + 80021580v^{10} - 8124495v^3 + 46928v^{20} - 564210v^{19}\\
&- 31486360v^6 + 52411245v^7 - 23299010v^8 + 11207010v^4 + 11207010v^{16})u^8\\
&+ (34554940v^8 + 12424270v^3 - 4529580v^{18} + 47876040v^6 + 34554940v^{12}\\
&- 4529580v^2 - 78534180v^7 - 16843860v^{16} + 47876040v^{14} + 66626380v^9\\
&+ 12424270v^{17} + 66626380v^{11} + 884990v^{19} - 78534180v^{13} - 1639028v^{15}\\
&- 121427896v^{10} - 1639028v^5 - 74280v^{20} + 884990v - 16843860v^4 - 74280)u^9\\
&+ (-26v^{15} + 240v^{14} - 1952v^{10} + 240v^6 - 580v^{11} - 580v^9 + 1760v^8 - 930v^{13}\\
&+ 1760v^{12} - 930v^7 - 26v^5)u + (5535v^{13} + 25490v^{10} - 3775v^9 + 1070v^{14}\\
&+ 5535v^7 + 105v^{18} - 450v^{17} - 3775v^{11} + 1300v^{16} - 15v - 15v^{19} + 105v^2\\
&+ 1070v^6 - 2319v^{15} + v^{20} - 450v^3 - 2319v^5 - 11125v^{12} + 1300v^4 + 1\\
&- 11125v^8)u^2 + v^5 + v^{15} + (13346v^{15} - 2180v^{18} + 29160v^6 + 87440v^9 + 8040v^3\\
&+ 340v + 69620v^8 - 17020v^4 - 191880v^{10} + 69620v^{12} - 24v^{20} - 24 - 2180v^2\\
&- 92270v^7 + 13346v^5 + 8040v^{17} + 340v^{19} + 29160v^{14} + 87440v^{11} - 92270v^{13}\\
&- 17020v^{16})u^3 + 210v^9 - 120v^8 - 252v^{10} + 210v^{11} = 0.
\end{aligned}
\tag{4.3}
$$

Proof. Using the equations (2.1) and (3.6), we arrive at the equation (4.3). □

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