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$L_d(1)$ IS $\bigcirc(logloglogd)$ FOR ALMOST ALL SQUARE FREE d

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ABSTRACT. Let d be a square free integer.Using Hardy-Ramanujan's value of normal order of $\omega(d)$ we show that $L_d(1) = \bigcirc(logloglogd)$ except on a negligible set. We note that the proof verifies Robin's inequality $\sigma(n) < e^{\gamma}nloglogn$ (equivalent form of Riemann Hypothesis) for such numbers.

Let d be a square free integer with $K = Q(\sqrt{d})$ the corresponding quadratic field. The Legendre symbol $(\frac{d}{n})$ helps us to define the L-series $L_d(s) = \sum_{n=1}^{\infty} \frac{(\frac{d}{n})}{n^s}$. This series converges for Re $s > 0$ and defines an analytic function [3], [7]. The value $L_d(1)$ is of great importance because $L_d(1) = hk$, h =Class number of K and k =structure constant of K. The estimation of $L_d(1)$ (or its exact value) is of interest [3]. Our main result is

Proposition 1. *$L_d(1) = \bigcirc(logloglogd)$ except for d in a negligible set (i.e., a set of natural density zero)*

To prove this we recall an earlier result of ours ([5]). It is the analogue of density computation for square free integers[2]: density= $\frac{1}{\zeta(2)} = \frac{6}{\pi^2}$

Lemma 1. *The natural density of square free ideals of θ_K (the ring of integers of K) is given by*

$$lim_{s\to 1^+} \frac{(s-1)\zeta_K(s)}{\zeta_K(2s)\prod\limits_{p|d}(1+\frac{1}{p})} = \frac{L_d(1)}{\zeta(2)L_d(2)\prod\limits_{p|d}(1+\frac{1}{p})} \leq 1$$

Hence $L_d(1) \leq \zeta(2)L_d(2)\prod\limits_{p|d}(1+\frac{1}{p}) \leq \zeta^2(2)\prod\limits_{p|d}(1+\frac{1}{p})$

(Here ζ_K is the Dedekind Zeta function and ζ the Riemann Zeta function [7])

In view of Lemma 1 our task is to estimate the finite product $\prod\limits_{p|d}(1+\frac{1}{p})$ as $\bigcirc(logloglogd)$ for almost all d.

Lemma 2. $\prod\limits_{p\leq x}(1+\frac{1}{p}) = c_1 logx + \bigcirc(1)$

Proof. We imitate the proof ([1], Thm 13.13) for $\prod\limits_{p\leq x}(1-\frac{1}{p}) = \frac{k}{logx} + \bigcirc(1)$

Let $P(x) = \prod_{p\leq x}(1+\frac{1}{p})$
So

$$logP(x) = \sum_{1\leq x} log(1+\frac{1}{p})$$

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But we have

$$log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - ..... + (-1)^{n+1}\frac{t^n}{n} + ...$$

for $(0 \leq t < 1)$. Taking $t = \frac{1}{p}$ we have, for each prime p,

$$log(1+\frac{1}{p}) = \frac{1}{p} - \frac{1}{2p^2} + \frac{1}{3p^3} - ..... + \frac{(-1)^{n+1}}{np^n} + ...$$

Setting

$$b_p = \frac{1}{p} - log(1+\frac{1}{p})$$

we have

$$\begin{aligned} b_p = \frac{1}{2p^2} - \frac{1}{3p^3} + ... + \frac{(-1)^n}{np^n} + ... &\leq \frac{1}{2p^2} + \frac{1}{3p^3} + ... + \frac{1}{np^n} + ... \\ &< \frac{1}{2p^2}(1 + \frac{1}{p} + \frac{1}{p^2} + ... + \frac{1}{p^n} + ...) \\ &= \frac{1}{2}\frac{1}{p^2}\frac{1}{(1-\frac{1}{p})} = \frac{1}{2p(p-1)} \end{aligned}$$

So

$$B = \sum_p b_p < \sum_p \frac{1}{p(p-1)} < \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \sum_{n=2}^{\infty}(\frac{1}{n-1} - \frac{1}{n}) = 1$$

Now

$$B - \sum_{p \leq x} b_p = \sum_{p > x} b_p < \sum_{n > x} \frac{1}{n(n-1)} = \bigcirc(\frac{1}{x})$$

So

$$B - (\sum_p (\frac{1}{p} - log(1+\frac{1}{p}))) = \bigcirc(\frac{1}{x})$$

Hence $\sum_{p \leq x} log(1+\frac{1}{p}) = \sum_{p \leq x}\frac{1}{p} - B + \bigcirc(\frac{1}{x}) = loglogx + A - B + \bigcirc(\frac{1}{logx}) + \bigcirc(\frac{1}{x})$
([1], Theorem 4.12)
Taking exponentials

$$P(x)=\prod_{p\le x}(1+\frac{1}{p}) = e^{(A-B)}.logx.e^{\bigcirc(\frac{1}{logx})}e^{\bigcirc(\frac{1}{x})}$$
$$= e^{(A-B)}.logx(1+\bigcirc(\frac{1}{logx}))(1+\bigcirc(\frac{1}{x}))$$
$$= e^{(A-B)}logx\{1+\bigcirc(\frac{1}{xlogx})+\bigcirc(\frac{1}{x})+\bigcirc(\frac{1}{logx})\}$$
$$= e^{(A-B)}logx+\bigcirc(\frac{1}{x})+\bigcirc(\frac{logx}{x})+\bigcirc(1)$$
$$= e^{(A-B)}logx+\bigcirc(1).$$

□

Lemma 3. $\prod_{p|d}(1+\frac{1}{p}) \le \prod_{p\le p_{\omega(d)}}(1+\frac{1}{p}) = c_1 log(p_{\omega(d)})+\bigcirc(1)$
($\omega(n)$ denotes the number of distinct prime divisors of n)

Proof. Use Lemma 2 with $x = p_{\omega(d)}$ together with the fact that all prime divisors of d are between 2 and $p_{\omega(d)}$. □

Lemma 4. $logp_{\omega(d)} = \bigcirc(log\omega(d))$

Proof. $p_n \le 12(nlogn + nlog(\frac{12}{e}))$ by Cheybshev ([1], Th 4.7) so $logp_n = \bigcirc(logn)$.Choose $n=\omega(d)$ □

Lemma 5. *Given $\epsilon>0$, $|\omega(n)-loglogn|<\epsilon loglogn$ for n in a set of density 1. Hence $\omega(n)<(1+\epsilon)loglogn$ for almost all n (the exceptional set is negligible i.e., of density zero)*

Proof. This is the famous result on normal order of $\omega(n)$ of Hardy-Ramanujan: see [2],[6] for several proofs. □

Lemma 6. $logp_{\omega(d)} = \bigcirc(logloglogd)$ *for almost all d.*
Hence $L_d(1) = \bigcirc(logloglogd)$ for almost all square free d.

Proof. $logp_{\omega(d)} = \bigcirc(log\omega(d))$ by Lemma 4. Let $\epsilon>0$.

$$log\omega(d) < log[(1+\epsilon)loglogd]$$
$$= log(1+\epsilon)+logloglogd \; for \; almost \; all \; d \; by \; Lemma \; 5$$

Hence $logp_{\omega(d)} = \bigcirc(logloglogd)$ □

Proof of Proposition1: Combining Lemmas 1-6, we have

$$L_d(1) \le \zeta^2(2)\prod_{p|d}(1+\frac{1}{p})$$
$$\le c\prod_{p\le p_{\omega(d)}}(1+\frac{1}{p})$$
$$= c_1 log(p_{\omega(d)})+\bigcirc(1)$$
$$= \bigcirc(log\omega(d))$$
$$= \bigcirc(logloglogd) \; for \; almost \; all \; square \; free \; d$$

Corollary 1. $\prod\limits_{p|d}(1+\frac{1}{p}) = \frac{\sigma(d)}{d} < e^{\gamma} loglogd$ for almost all d. For such d, therefore, Robin's inequality $\frac{\sigma(n)}{n} < e^{\gamma} loglogn$ is valid : the validity for all $n > 7!$ is equivalent to the Riemann Hypothesis[4]($\sigma(n)$ =sum of divisors of n).Since the density of square free integers d is $\frac{6}{\pi^2}$, Robin's inequality is valid for a large subset of positive integers.

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References

[1] Tom Apostol, *Introduction to Analytic Number Theory*, Springer 1976
[2] G H Hardy, *Ramanujan Twelve lectures; Chelsea 1959*
[3] M. Hindry, *Introduction to Zeta and L-functions from Arithmetic Geometry and some applications*, (Mini curso, XXI Escola de Algebra, Brasilia, julho 2010)
[4] J. Lagarias, *An Elementary problem equivalent to the Riemann Hypothesis*, Amer Math Monthly 109 (2002) pp 534-543
[5] S. Srinivas Rau and B.Uma, *Squarefree ideals in Quadratic fields and the Dedekind Zeta function*, Vikram Math Journal Vol 13 (1993) pp 35-44
[6] G.Tenenbaum, *Introduction to Analytic and Probabilitic Number Theory*, Cmbridge University Press 1995.
[7] TIFR Pamphlet 4, Algebraic Number Theory, 1966
[8] G Sudhaamsh Mohan Reddy, SS Rau, B Uma *Applications of Tauberian Theorems to Dirichlet Series* , (Lap Lambert Academic Publishing, Germany, ISBN:978-613-9-45507-2, 2019)
[9] G Sudhaamsh Mohan Reddy, SS Rau, B Uma *A remark on Hardy-Ramanujan's approximation of divisor functions*, International Journal of Pure and Applied Mathematics 118 (4), 997-999, 2018
[10] G Sudhaamsh Mohan Reddy, SS Rau, B Uma, *Some Dirichlet Series and Means of Their Coefficients*,Southeast Asian Bulletin of Mathematics 40 (4), 585-591, 2016
[11] G Sudhaamsh Mohan Reddy, SS Rau, B Uma, *Some arithmetic functions and their means*,International Journal of Pure and Applied Mathematics 119 (2), 369-374, 2018
[12] G Sudhaamsh Mohan Reddy, SS Rau, B Uma, *A Bertrand Postulate for a subclass of primes*, Boletim da Sociedade Paranaense de Matemtica 31 (2), 109-111, 2013
[13] G Sudhaamsh Mohan Reddy, SS Rau, B Uma, *Converegence of a series leading to an analogue of Ramanujan's assertion on squarefree integers*, Boletim da Sociedade Paranaense de Matemtica 38 (2), 83-87, 2020
[14] G Sudhaamsh Mohan Reddy, SS Rau, B Uma, *A direct proof of Convergence of Euler product for $L_d(1)$*, International Journal of Advanced Science and Technology, Vol. 28, No. 16, (2019), pp. 1308-1311
[15] G Sudhaamsh Mohan Reddy, SS Rau, B Uma, *Kronecker's lemma and a converse with applications*, International Journal of Advanced Science and Technology, Vol. 28, No. 16, (2019), pp. 1312-1314
[16] G Sudhaamsh Mohan Reddy, SS Rau, B Uma, *An equivalent form of the Prime Number Theorem*, (Accepted in Sarajevo Journal of Mathematics ANUBIH)
[17] G Sudhaamsh Mohan Reddy, SS Rau, B Uma, *A note on Dirichlet series connected to $\frac{L_d(1)}{L_d(2)}$-I*, (Accepted in AIP Conference Proceedings)
[18] G Sudhaamsh Mohan Reddy, SS Rau, *Bounds for special values of L_d for a Quadratic Field $Q(\sqrt{d})$*, (Accepted in AIP Conference Proceedings)
[19] G Sudhaamsh Mohan Reddy, SS Rau, *An analogue 0f Landau-Walfisz Theorem*, (Accepted in AIP Conference Proceedings)

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