

New Modular Relations for the Ratio's of Ramanujan Function $\chi(-q)$ of degree 9

Dedicated to Prof. Chandrashekhar Adiga on his 62nd birthday

B. N. Dharmendra* and P. S. Guruprasad**

*Postgraduate Department of Mathematics

Maharani's Science College for Women

J.L.B. Road, Mysore-570 005, India

**Department of Mathematics

Government First Grade College

Chamarajanagar-571 313, India

bndharma@gmail.com, guruprasad18881@gmail.com

Abstract

In this paper, we establish some new modular equations of Ramanujan's function $\chi(-q)$ of degree 9. Further, we obtain some explicit evaluations of class invariant g_n from them.

Mathematics Subject Classification (2000): Primary 33D10

Keywords: Theta functions.

1 Introduction

In Chapter 16 of his second notebook [1, 7], Ramanujan develops the theory of theta-function and is defined by

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, |ab| < 1, \quad (1.1)$$

$$= (-a; ab)_{\infty} (-b; ab)_{\infty} (ab; ab)_{\infty},$$

where $(a; q)_0 = 1$ and $(a; q)_{\infty} = (1 - a)(1 - aq)(1 - aq^2) \dots$

Following Ramanujan, we define

$$\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q; -q)_{\infty}}{(q; -q)_{\infty}}, \quad (1.2)$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}}, \quad (1.3)$$

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q; q)_{\infty} \quad (1.4)$$

and

$$\chi(q) := (-q; q^2)_\infty. \quad (1.5)$$

Now we define a modular equation in brief. The ordinary hypergeometric series ${}_2F_1(a, b; c; x)$ is defined by

$${}_2F_1(a, b; c; x) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n,$$

where $(a)_0 = 1$, $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$ for any positive integer n , and $|x| < 1$.

Let

$$z := z(x) := {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) \quad (1.6)$$

and

$$q := q(x) := \exp\left(-\pi \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)}\right), \quad (1.7)$$

where $0 < x < 1$.

Let r denote a fixed natural number and assume that the following relation holds:

$$r \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)}. \quad (1.8)$$

Then a modular equation of degree r in the classical theory is a relation between α and β induced by (1.8). We often say that β is of degree r over α and $m := \frac{z(\alpha)}{z(\beta)}$ is called the multiplier. We also use the notations $z_1 := z(\alpha)$ and $z_r := z(\beta)$ to indicate that β has degree r over α .

The function $\chi(q)$ [1, Entry 12(v),(vi), p.56] is intimately connected to Ramanujan's class invariants G_n and g_n , which are defined by

$$G_n = 2^{-1/4} q^{-1/24} \chi(q), \quad g_n = 2^{-1/4} q^{-1/24} \chi(-q), \quad (1.9)$$

where $q = e^{-\pi\sqrt{n}}$, n is a positive rational number,

$$\chi(q) = 2^{1/6} \{ \alpha(1-\alpha)q^{-1} \}^{-1/24} \quad (1.10)$$

and

$$\chi(-q) = 2^{1/6} (1-\alpha)^{1/12} \alpha^{-1/24} q^{-1/24}. \quad (1.11)$$

In section 3, we establish some new modular relations between ratio's of parameter for Ramanujan's function $\chi(-q)$ and explicit evaluations from them.

2 Preliminary results

Lemma 2.1. [3] If $P = \frac{\psi(q)}{q\psi(q^9)}$ and $Q = \frac{\psi(q^2)}{q^2\psi(q^{18})}$, then

$$\frac{P}{Q} + \frac{Q}{P} + 2 = \frac{3}{P} + P. \quad (2.1)$$

Lemma 2.2. [3] If $P = \frac{\psi(q)}{q\psi(q^9)}$ and $Q = \frac{\psi(q^3)}{q^3\psi(q^{27})}$, then

$$\left(3 - P - \frac{3}{P}\right) \left(3 - Q - \frac{3}{Q}\right) = \left(\frac{Q}{P}\right)^2. \quad (2.2)$$

Lemma 2.3. [3] If $P = \frac{\psi(q)\psi(q^5)}{q^6\psi(q^9)\psi(q^{45})}$ and $Q = \frac{\psi(q)\psi(q^{45})}{q^{-4}\psi(q^9)\psi(q^5)}$, then

$$\begin{aligned} Q^3 + \frac{1}{Q^3} &= 15 \left(Q^2 + \frac{1}{Q^2} \right) + 45 \left(Q + \frac{1}{Q} \right) + \left(P^2 + \frac{81}{P^2} \right) + 10 \left(P + \frac{9}{P} \right) \\ &\times \left[2 + Q + \frac{1}{Q} \right] + 5 \left(\sqrt{P^3} + \frac{3^3}{\sqrt{P^3}} \right) \left(\sqrt{Q} + \frac{1}{\sqrt{Q}} \right) + 15 \left(\sqrt{P} + \frac{3}{\sqrt{P}} \right) \\ &\times \left[\left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}} \right) + 2 \left(\sqrt{Q} + \frac{1}{\sqrt{Q}} \right) \right] + 40. \end{aligned} \quad (2.3)$$

Lemma 2.4. [3] If $P = \frac{\psi(q)\psi(q^7)}{q^8\psi(q^9)\psi(q^{63})}$ and $Q = \frac{\psi(q)\psi(q^{63})}{q^{-6}\psi(q^9)\psi(q^7)}$, then

$$\begin{aligned} Q^4 + \frac{1}{Q^4} &= 35 \left(Q^3 + \frac{1}{Q^3} \right) + 413 \left(Q^2 + \frac{1}{Q^2} \right) + 1379 \left(Q + \frac{1}{Q} \right) + 1694 \\ &+ \left(P^3 + \frac{9^3}{P^3} \right) + 7 \left(P^2 + \frac{9^2}{P^2} \right) \left[7 + 3 \left(Q + \frac{1}{Q} \right) \right] + 21 \left(P + \frac{9}{P} \right) \\ &\times \left[21 + 14 \left(Q + \frac{1}{Q} \right) + 3 \left(Q^2 + \frac{1}{Q^2} \right) \right] + 7 \left(\sqrt{P^5} + \frac{3^5}{\sqrt{P^5}} \right) \left(\sqrt{Q} + \frac{1}{\sqrt{Q}} \right) \\ &+ 63 \left[\sqrt{P} + \frac{3}{\sqrt{P}} \right] \left[7 \left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}} \right) + 14 \left(\sqrt{Q} + \frac{1}{\sqrt{Q}} \right) + \left(\sqrt{Q^5} + \frac{1}{\sqrt{Q^5}} \right) \right] \\ &+ 21 \left(\sqrt{P^3} + \frac{3^3}{\sqrt{P^3}} \right) \left[2 \left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}} \right) + 7 \left(\sqrt{Q} + \frac{1}{\sqrt{Q}} \right) \right]. \end{aligned} \quad (2.4)$$

Lemma 2.5. [8, p.56],[6]

$$\frac{f^3(-q^2)}{f^3(-q^{18})} = \frac{\psi^2(q)}{\psi^2(q^9)} \left\{ \frac{\psi(q) - 3q\psi(q^9)}{\psi(q) - q\psi(q^9)} \right\}. \quad (2.5)$$

Lemma 2.6. [1, Ch. 16, Entry 24(iii), p.39]

$$\chi(q) = \frac{f(-q^2)}{\psi(-q)}. \quad (2.6)$$

Lemma 2.7. [5]

$$4 \left\{ (g_{2n}g_{8n})^4 + \frac{1}{(g_{2n}g_{8n})^4} \right\} = \left(\frac{g_{8n}}{g_{2n}} \right)^{12}. \quad (2.7)$$

3 Modular Relation between $q^{1/3} \frac{\chi(-q)}{\chi(-q^9)}$ and $q^{n/3} \frac{\chi(-q^n)}{\chi(-q^{9n})}$

Theorem 3.1. If $P := q^{1/3} \frac{\chi(-q)}{\chi(-q^9)}$ and $Q := q^{2/3} \frac{\chi(-q^2)}{\chi(-q^{18})}$ then

$$\frac{P^2}{Q} + \frac{P}{Q^2} + \frac{Q}{P^2} + \frac{Q^2}{P} = PQ + \frac{1}{PQ}. \quad (3.1)$$

Proof. Replace q by $-q$ and also replace q by $-q^9$ in the lemma (2.6) to obtain

$$\chi(-q) = \frac{f(-q^2)}{\psi(q)}, \quad (3.2)$$

$$\chi(-q^9) = \frac{f(-q^{18})}{\psi(q^9)}. \quad (3.3)$$

Divide the equation (3.2) by (3.3), we get

$$\frac{\chi(-q)}{\chi(-q^9)} = \frac{\psi(q^9)}{\psi(q)} \frac{f(-q^2)}{f(-q^{18})}. \quad (3.4)$$

Raising the power on both sides by three and also multiplying q on both sides of the above equation, we get

$$q \frac{\chi^3(-q)}{\chi^3(-q^9)} = \frac{\psi^3(q^9)}{\psi^3(q)} \left\{ q \frac{f^3(-q^2)}{f^3(-q^{18})} \right\}. \quad (3.5)$$

Using the equations (2.5) and (3.5), we obtain

$$ba^2 - ba - a + 3 = 0, \quad (3.6)$$

where $a := \frac{\psi(q)}{q\psi(q^9)}$, $b := q \frac{\chi^3(-q)}{\chi^3(-q^9)}$.

Solving (3.6) for a , we get

$$a = \frac{b + 1 + \sqrt{b^2 - 10b + 1}}{2b}. \quad (3.7)$$

Using (3.7) and (2.1), we obtain

$$(PQ^4 + Q^3 - PQ - Q^3P^3 + P^3 + P^4Q)(P^2Q^8 + P^4Q^7 - PQ^7 + P^6Q^6 - 2Q^6P^3 + Q^6 + 2P^5Q^5 - 2P^2Q^5 + P^7Q^4 - 3P^4Q^4 + PQ^4 - 2Q^3P^6 + 2Q^3P^3 + P^8Q^2 - 2P^5Q^2 + P^2Q^2 - P^7Q + P^4Q + P^6) = 0, \quad (3.8)$$

by examining the behavior of the above factors near $q = 0$, we can find a neighborhood about the origin, where the first factor is zero; whereas other factor is not zero in this neighborhood. By the Identity Theorem first factor vanishes identically. This completes the proof. \square

New modular relations for the ratio's of Ramanujan function $\chi(-q)$ of degree 9

Remark 1. Using (1.9), P and Q in the above theorem can also be written as
 $P := \frac{g_n}{g_{81n}}$ and $Q := \frac{g_{4n}}{g_{324n}}$.

Theorem 3.2. If $P := q^{1/3} \frac{\chi(-q)}{\chi(-q^9)}$ and $Q := q \frac{\chi(-q^3)}{\chi(-q^{27})}$ then

$$\begin{aligned} & \left\{ \frac{P^9}{Q^9} + \frac{Q^9}{P^9} \right\} - 8 \left\{ \frac{P^6}{Q^6} + \frac{Q^6}{P^6} \right\} - 35 \left\{ \frac{P^3}{Q^3} + \frac{Q^3}{P^3} \right\} - \left\{ \frac{P^9}{Q^6} + \frac{Q^6}{P^9} + \frac{P^6}{Q^9} + \frac{Q^9}{P^6} \right\} \\ & + \left\{ \frac{P^9}{Q^3} + \frac{Q^3}{P^9} + \frac{P^3}{Q^9} + \frac{Q^9}{P^3} \right\} + 17 \left\{ \frac{P^6}{Q^3} + \frac{Q^3}{P^6} + \frac{P^3}{Q^6} + \frac{Q^6}{P^3} \right\} = \left\{ P^6 Q^6 + \frac{1}{P^6 Q^6} \right\} \\ & + 28 \left\{ P^3 Q^3 + \frac{1}{P^3 Q^3} \right\} + 10 \left\{ P^6 Q^3 + \frac{1}{P^6 Q^3} + P^3 Q^6 + \frac{1}{P^3 Q^6} \right\} + 19 \left\{ P^6 + \frac{1}{P^6} \right\} \\ & + 19 \left\{ Q^6 + \frac{1}{Q^6} \right\} - 46 \left\{ P^3 + \frac{1}{P^3} \right\} - 46 \left\{ Q^3 + \frac{1}{Q^3} \right\} + 74. \end{aligned} \tag{3.9}$$

Proof. Employing the equation (3.7) and equation (2.2), we obtain (3.9). \square

Theorem 3.3. If $P := q^{1/3} \frac{\chi(-q)}{\chi(-q^9)}$ and $Q := q^{4/3} \frac{\chi(-q^4)}{\chi(-q^{36})}$ then

$$\begin{aligned} & \left\{ \frac{P^3}{Q^3} + \frac{Q^3}{P^3} \right\} + 7 \left\{ \frac{P^2}{Q^2} + \frac{Q^2}{P^2} \right\} + 14 \left\{ \frac{P}{Q} + \frac{Q}{P} \right\} \\ & + \left\{ \frac{P^4}{Q} + \frac{Q}{P^4} + \frac{P}{Q^4} + \frac{Q^4}{P} \right\} + 16 = \left\{ P^3 Q^3 + \frac{1}{P^3 Q^3} \right\} \\ & - 5 \left\{ P^2 Q + \frac{1}{P^2 Q} + P Q^2 + \frac{1}{P Q^2} \right\} - 4 \left\{ P^3 + \frac{1}{P^3} \right\} - 4 \left\{ Q^3 + \frac{1}{Q^3} \right\}. \end{aligned} \tag{3.10}$$

Proof. The equation (2.1) can be deduced to $b^2 + (-a^2 + 2a - 3)b + a^2 = 0$, where $a = \frac{\psi(q)}{q\psi(q^9)}$, $b = \frac{\psi(q^2)}{q^2\psi(q^{18})}$. Solving the above quadratic equation for b , we obtain

$$b := -a + \frac{3}{2} + \frac{a^2}{2} + \frac{\sqrt{6a^2 - 12a - 4a^3 + 9 + a^4}}{2}. \tag{3.11}$$

Replacing q by q^2 in equation (2.1) and using (3.11), we obtain

$$\begin{aligned} & 27c - 27c^2 - a^4 - 36ca + 18ca^2 - 8ca^3 + 3ca^4 + 36c^2a - 24c^2a^2 \\ & + 12c^2a^3 - 3c^2a^4 + 6c^3a^2 - 8c^3a + c^3a^4 - 4c^3a^3 - c^4 + 9c^3 = 0, \end{aligned} \tag{3.12}$$

where $c = \frac{\psi(q^4)}{q^4\psi(q^{36})}$.

From the equations (3.7) and (3.12), we obtain (3.10). \square

Theorem 3.4. If $P := q^{1/3} \frac{\chi(-q)}{\chi(-q^9)}$ and $Q := q^{5/3} \frac{\chi(-q^5)}{\chi(-q^{45})}$ then

$$\begin{aligned} & \frac{P^3}{Q^3} + \frac{Q^3}{P^3} + 5 \left\{ \frac{P^2}{Q} + \frac{Q}{P^2} + \frac{Q^2}{P} + \frac{P}{Q^2} \right\} + 10 \\ &= \left\{ P^2 Q^2 + \frac{1}{P^2 Q^2} \right\} - 5 \left\{ PQ + \frac{1}{PQ} \right\}. \end{aligned}$$

Proof. Employing the equations (3.7) and (2.3), we obtain

$$\begin{aligned} & (-P^5 Q^5 + Q^6 + 5P^4 Q^4 + 5P^2 Q^5 + 5P^2 Q^2 + 10P^3 Q^3 + P^6 - PQ \\ &+ 5P^4 Q + 5PQ^4 + 5P^5 Q^2)(P^5 Q^{11} - 10P^5 Q^5 - 5P^5 Q^8 - 10P^5 Q^2 \\ &+ 35P^4 Q^4 - 5P^{10} Q - 10P^{10} Q^7 + P^{10} Q^{10} + 20P^{10} Q^4 + 20P^2 Q^8 \\ &- 5P^2 Q^{11} + Q^{12} - 10P^2 Q^5 + P^2 Q^2 - 5P^9 Q^3 + 5P^3 Q^3 - 5P^3 Q^9 \\ &- 20P^6 Q^9 - 20P^9 Q^6 - 20P^3 Q^6 - 20P^6 Q^3 - 5P^{11} Q^2 + 35P^8 Q^8 \\ &+ 5P^9 Q^9 + 26P^6 Q^6 - 5P^4 Q^7 + 20P^4 Q^{10} - 5PQ^{10} + P^{12} + P^{11} Q^5 \\ &- 10P^7 Q^7 - 10P^7 Q^{10} - 5P^7 Q^4 - 5P^8 Q^5 + 20P^8 Q^2 + P^7 Q + PQ^7) = 0. \end{aligned}$$

by examining the behavior of the above factors near $q = 0$, we can find a neighborhood about the origin, where the first factor is zero; whereas other factor is not zero in this neighborhood. By the Identity Theorem first factor vanishes identically. This completes the proof. \square

Theorem 3.5. If $P := q^{1/3} \frac{\chi(-q)}{\chi(-q^9)}$ and $Q := q^{7/3} \frac{\chi(-q^7)}{\chi(-q^{63})}$ then

$$\begin{aligned} & \left\{ \frac{P^4}{Q^4} + \frac{Q^4}{P^4} \right\} - 14 \left\{ \frac{P^2}{Q^2} + \frac{Q^2}{P^2} \right\} - 35 \left\{ \frac{P}{Q} + \frac{Q}{P} \right\} \\ &+ 7 \left\{ P^3 + \frac{1}{P^3} \right\} + 7 \left\{ Q^3 + \frac{1}{Q^3} \right\} = \left\{ P^3 Q^3 + \frac{1}{P^3 Q^3} \right\} \\ &- 21 \left\{ P^2 Q + \frac{1}{P^2 Q} + PQ^2 + \frac{1}{PQ^2} \right\} + 42. \end{aligned}$$

Proof. Employing the equations (3.7) and (2.4), we obtain

$$\begin{aligned} & (Q^8 - 14P^6 Q^2 + 21P^3 Q^2 - 35P^3 Q^5 + 7QP^4 + 21P^6 Q^5 - Q^7 P^7 + 7Q^7 P^4 + P^8 \\ &+ 7Q^4 P + 7Q^4 P^7 - 42Q^4 P^4 - 35Q^3 P^5 + 21Q^3 P^2 + 21Q^6 P^5 - 14Q^6 P^2 - QP) \\ & (Q^{15} P^7 + 21P^{12} Q^{13} + P^{16} + 98P^9 Q^7 - 70P^3 Q^{13} + 140P^6 Q^{13} + Q^{16} - 161P^3 Q^7 \\ &+ 238P^{12} Q^4 - 7Q^{15} P^4 - 350P^9 Q^4 + 49Q^2 P^8 - 7QP^{12} - 14Q^2 P^5 - 21Q^2 P^{11} \\ &- 21Q^{11} P^2 + 14Q^2 P^{14} + QP^9 - 224P^6 Q^7 - 161P^9 Q^{13} - 350P^{12} Q^7 + 21P^3 Q^4 \\ &+ P^{15} Q^7 + 427Q^8 P^5 + 427Q^8 P^{11} + 49Q^8 P^2 + 49Q^8 P^{14} - 70Q^3 P^{13} - 342Q^8 P^8 \\ &+ 259P^{12} Q^{10} + 140P^3 Q^{10} - 7P^{15} Q^4 - 7Q^{12} P + Q^{14} P^{14} + 259P^6 Q^4 + 315P^6 Q^{10} \\ &+ 140Q^3 P^{10} - 161Q^3 P^7 + Q^2 P^2 + 21Q^{12} P^{13} + 21Q^3 P^4 + 259Q^{12} P^{10} - 350Q^{12} P^7 \\ &- 21Q^{14} P^5 + 98Q^9 P^7 + 14Q^{14} P^2 - 14Q^{11} P^{14} + 427Q^{11} P^8 - 21Q^5 P^{14} + 427Q^5 P^8 \end{aligned}$$

$$\begin{aligned}
& + 140Q^6P^{13} + Q^9P + 49Q^{14}P^8 - 224Q^6P^7 + 259Q^6P^4 - 259Q^5P^5 - 14Q^5P^2 \\
& - 14Q^{14}P^{11} - 161Q^9P^{13} - 224Q^9P^{10} - 350Q^9P^4 - 357Q^{11}P^5 - 259Q^{11}P^{11} \\
& - 224P^9Q^{10} - 357Q^5P^{11} + 238Q^{12}P^4 + 315Q^6P^{10}) = 0.
\end{aligned}$$

by examining the behavior of the above factors near $q = 0$, we can find a neighborhood about the origin, where the first factor is zero; whereas other factor is not zero in this neighborhood. By the Identity Theorem first factor vanishes identically. This completes the proof. \square

4 Explicit evaluation of g_n

Theorem 4.1.

$$g_{72} = \frac{(\sqrt{3}+1)(\sqrt{3}+\sqrt{2})^{\frac{1}{3}} \left\{ (9\sqrt{6}-22)(\sqrt{3}+\sqrt{2})^{\frac{1}{3}} \sqrt{35\sqrt{2}+28\sqrt{3}+1} \right\}^{\frac{1}{4}}}{\sqrt{2}}, \quad (4.1)$$

$$g_{\frac{8}{9}} = \left\{ (9\sqrt{6}-22)(\sqrt{3}+\sqrt{2})^{\frac{1}{3}} \sqrt{35\sqrt{2}+28\sqrt{3}+1} \right\}^{\frac{1}{4}}. \quad (4.2)$$

Proof. Setting $n = \frac{2}{9}$ in Theorem 3.1 and using the fact that $g_{2n}g_{\frac{2}{n}} = 1$, we obtain

$$b^4a^6 + b^3a^8 - ba^6 - b^3a^2 + a^2 + b = 0, \quad (4.3)$$

where $a = g_{18} = (\sqrt{2} + \sqrt{3})^{1/3}$ and $b = \frac{g_{\frac{8}{9}}}{g_{72}}$. Solve the above equation (4.3) for b to get

$$\frac{g_{\frac{8}{9}}}{g_{72}} = \frac{(\sqrt{3} - \sqrt{2})(\sqrt{3} - 1)(\sqrt{3} + \sqrt{2})^{\frac{2}{3}}}{\sqrt{2}}. \quad (4.4)$$

Employing the equation (2.7) with $n = \frac{1}{9}$, we obtain

$$4 \left\{ \left(\frac{g_{\frac{8}{9}}}{g_{18}} \right)^4 + \left(\frac{g_{18}}{g_{\frac{8}{9}}} \right)^4 \right\} = \left(g_{18}g_{\frac{8}{9}} \right)^{12}. \quad (4.5)$$

Using (4.4), (4.5) and $a = g_{18} = (\sqrt{2} + \sqrt{3})^{1/3}$, we obtain (4.1) and (4.2). \square

Theorem 4.2.

$$g_{54} = \left\{ 2^{\frac{1}{3}}r + \left(3 \times 2^{\frac{1}{6}}r^2 - 2 \times 2^{\frac{2}{3}}r^2 + \sqrt{2} + 1 \right) \sqrt{1 + \sqrt{2}} \right\}^{\frac{1}{3}}, \quad (4.6)$$

$$g_{162} = \frac{\{15000 - 2760k^2 + 1140\sqrt{6}k^2 + 1000k + 6000\sqrt{6}\}^{\frac{1}{3}}}{10}, \quad (4.7)$$

$$\text{where } r = (7\sqrt{2} + 10) \left(\sqrt{1 + \sqrt{2}} \right), \quad k = (13144 + 5366\sqrt{6})^{\frac{1}{3}}.$$

Proof. Setting $n = \frac{2}{27}$ in Theorem 3.2 and using the fact that $g_{2n}g_{\frac{2}{n}} = 1$, we obtain

$$(m^{18}n^{18} - 18m^{15}n^{15} + 27m^{12}n^{12} - 52m^9n^9 + 27m^6n^6 - 18m^3n^3 + 1) \\ (-1 + mn)^2(m^2n^2 + mn + 1)^2 = 0, \quad (4.8)$$

where $m = g_6 = (1 + \sqrt{2})^{\frac{1}{6}}$ and $n = g_{54}$.

By examining the behavior of the above factors near $q = 0$, we can find a neighborhood about the origin, where the first factor is zero; whereas other factor are not zero in this neighborhood. By the Identity Theorem first factor vanishes identically. Solving the first factor for n , we obtain (4.6).

Again setting $n = \frac{2}{9}$ in Theorem 3.2 and using the fact that $g_{2n}g_{\frac{2}{n}} = 1$, we obtain

$$h^9 - 45h^6 - 18h^6\sqrt{6} + 45h^3 + 18h^3\sqrt{6} - 20\sqrt{6} - 49 = 0, \quad (4.9)$$

where $h = g_{162}$. Solve the above equation to obtain (4.7). \square

Acknowledgment The second author is thankful to UGC, New Delhi, for the financial support under the minor research project MRP(S)-1341/11-12/KAMY073/UGC-SWRO dated 17-Jan-13 under which this work has been done.

References

- [1] B. C. Berndt, Ramanujan's Notebooks, Part III, Springer-Verlag, New York (1991).
- [2] B. C. Berndt, Ramanujan's Notebooks, Part IV, Springer-Verlag, New York (1994).
- [3] M. S. Mahadeva Naika, S. Chandankumar and K. Sushan Bairy, Modular equations for the ratios of Ramanujan's theta function ψ and evaluations. New Zealand J. Math., **40(1)** (2010), 33-48.
- [4] N. D. Baruah and N Saikia, Two parameters for Ramanujan's theta-functions and their explicit values. Rocky Mountain J.Math., **36(6)** (2007), 1747-1790.
- [5] Nipen Saikia, Ramanujans modular equations and Weber-Ramanujan class invariants G_n and g_n , Bull. Math. Sci., **2** (2012), 205-223.
- [6] S. -Y. Kang, *Some theorems on the Rogers-Ramanujan continued fraction and associated theta function identities in Ramanujan's lost notebook*, Ramanujan J., **3 (1)** (1999), 91-11.
- [7] S. Ramanujan, Notebooks (2 volumes). Bombay. Tata Institute of Fundamental Research (1957).
- [8] S. Ramanujan, The 'lost' notebook and other unpublished papers. New Delhi. Narosa (1988).