

ON SOME P - Q MIXED MODULAR EQUATIONS OF DEGREE 5

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ABSTRACT. In his second notebook, Ramanujan recorded total of 23 P - Q modular equations involving theta-functions $\varphi(q)$, $\psi(q)$ and $f(-q)$. In this paper, modular equations analogous to those recorded by Ramanujan are obtained involving $f(-q)$. As a consequence, several values of quotients of theta-function are evaluated.

Dedicated to Prof. C. Adiga on the occasion of his 62nd Birthday

1. INTRODUCTION

For $|q| < 1$, let $(a; q)_\infty$ denote the infinite product $\prod_{n=0}^{\infty} (1 - aq^n)$, where a, q are complex numbers and $f(a, b)$ be the Ramanujan theta-function:

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1,$$

The following definitions of theta-functions φ , ψ and f follows as special cases of $f(a, b)$:

$$(1.1) \quad \varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2},$$

$$(1.2) \quad \psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2},$$

$$(1.3) \quad f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}.$$

The ordinary or Gaussian hypergeometric function is defined by

$${}_2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \quad 0 \leq |z| < 1,$$

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where a, b, c are complex numbers, $c \neq 0, -1, -2, \dots$, and

$$(a)_0 = 1, \quad (a)_n = a(a+1) \cdots (a+n-1) \quad \text{for any positive integer } n.$$

Let $K(k)$ be the complete elliptic integral of the first kind of modulus k . Recall that

$$(1.4) \quad K(k) := \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}} = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2} k^{2n} = \frac{\pi}{2} \varphi^2(q), \quad (0 < k < 1),$$

and set $K' = K(k')$, where $k' = \sqrt{1-k^2}$ is the so called complementary modulus of k . It is classical to set $q(k) = e^{-\pi K(k')/K(k)}$ so that q is one-to-one increases from 0 to 1.

In the same manner introduce $L_1 = K(\ell_1)$, $L'_1 = K(\ell'_1)$ and suppose that the following equality

$$(1.5) \quad n_1 \frac{K'}{K} = \frac{L'_1}{L_1}$$

holds for some positive integer n_1 . Then a modular equation of degree n_1 is a relation between the moduli k and ℓ_1 which is induced by (1.5). Following Ramanujan, set $\alpha = k^2$ and $\beta = \ell_1^2$. We say that β is of degree n_1 over α . The multiplier m , corresponding to the degree n_1 , is defined by

$$(1.6) \quad m = \frac{K}{L_1} = \frac{\varphi^2(q)}{\varphi^2(q^{n_1})},$$

for $q = e^{-\pi K(k')/K(k)}$.

Let $K, K', L_1, L'_1, L_2, L'_2, L_3$ and L'_3 denote complete elliptic integrals of the first kind corresponding, in pairs, to the moduli $\sqrt{\alpha}, \sqrt{\beta}, \sqrt{\gamma}$ and $\sqrt{\delta}$, and their complementary moduli, respectively. Let n_1, n_2 and n_3 be positive integers such that $n_3 = n_1 n_2$. Suppose that the equalities

$$(1.7) \quad n_1 \frac{K'}{K} = \frac{L'_1}{L_1}, \quad n_2 \frac{K'}{K} = \frac{L'_2}{L_2} \quad \text{and} \quad n_3 \frac{K'}{K} = \frac{L'_3}{L_3},$$

hold. Then a “mixed” modular equation is a relation between the moduli $\sqrt{\alpha}, \sqrt{\beta}, \sqrt{\gamma}$ and $\sqrt{\delta}$ that is induced by (1.7). We say that β, γ and δ are of degrees n_1, n_2 and n_3 , respectively over α . The multipliers $m = K/L_1$ and $m' = L_2/L_3$ are algebraic relation involving α, β, γ and δ .

At scattered places of his second notebook [13], Ramanujan recorded a total of nine $P-Q$ “mixed” modular relations of degrees 1, 3, 5 and 15. These relations were proved by B. C. Berndt and L.-C. Zhang [5], [6] and the same has been reproduced in the book by Berndt [4, pp. 214-235]. In [7], S. Bhargava, C. Adiga and M. S. Mahadeva Naika have established several new $P-Q$ “mixed” modular relations with four moduli. For more information one can see [11] and [12]. Motivated by all these works, we establish some new modular equations of “mixed” degrees and as an application,

we establish some new general formulas for the explicit evaluations of a remarkable product of theta function.

In Section 2, we collect some identities which are useful in proofs of our main results. In Section 3, we establish several new modular equations of degree 5. In Section 4, we establish several new $P-Q$ “mixed” modular equations akin to those recorded by Ramanujan in his notebooks.

Mahadeva Naika, M. C. Maheshkumar and Bairy [9], have defined a new remarkable product of theta-functions $b_{s,t}$:

$$(1.8) \quad b_{s,t} = \frac{te^{\frac{-(t-1)\pi}{4}\sqrt{\frac{s}{t}}}\psi^2\left(-e^{-\pi\sqrt{st}}\right)\varphi^2\left(-e^{-2\pi\sqrt{st}}\right)}{\psi^2\left(-e^{-\pi\sqrt{\frac{s}{t}}}\right)\varphi^2\left(-e^{-2\pi\sqrt{\frac{s}{t}}}\right)},$$

where s, t are real numbers such that $s > 0$ and $t \geq 1$. They have established some new general formulas for the explicit evaluations of $b_{s,t}$ and computed some particular values of $b_{s,t}$. In Section 5, we establish some new modular relations connecting a remarkable product of theta-functions $b_{s,5}$ with $b_{r^2s,5}$ for $r = 2, 4$ and 6 and explicit values of $b_{s,5}$ are deduced.

2. PRELIMINARY RESULTS

In this section, we list some of the relevant identities which are useful in the proofs of our results.

Lemma 2.1. [3, Ch. 17, Entry 12 (i) and (iii), p. 124] *For $0 < x < 1$, let*

$$(2.1) \quad f(e^{-y}) = \sqrt{z}2^{-1/6}\{x(1-x)e^y\}^{1/24},$$

$$(2.2) \quad f(e^{-2y}) = \sqrt{z}2^{-1/3}\{x(1-x)e^y\}^{1/12},$$

where $z := {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)$ and $y := \pi \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)}$.

Lemma 2.2. [3, Ch. 16, Entry 24 (ii) and (iv), p. 39] *We have*

$$(2.3) \quad f^3(-q) = \varphi^2(-q)\psi(q),$$

$$(2.4) \quad f^3(-q^2) = \varphi(-q)\psi^2(q).$$

Lemma 2.3. [3, Ch. 19, Entry 13 (xii) and (vii), pp. 281-282]

If β is of degree 5 over α , then

$$(2.5) \quad \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} = m,$$

$$(2.6) \quad \left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4} = \frac{5}{m},$$

where m is the multiplier for degree 5.

Lemma 2.4. [14, p. 55] If $X := \frac{f(-q)}{q^{1/6}f(-q^5)}$ and $Y := \frac{f(-q^2)}{q^{1/3}f(-q^{10})}$, then

$$(2.7) \quad XY + \frac{5}{XY} = \left(\frac{Y}{X}\right)^3 + \left(\frac{X}{Y}\right)^3.$$

Lemma 2.5. [14, p. 55] If $X := \frac{f(-q)}{q^{1/6}f(-q^5)}$ and $Y := \frac{f(-q^3)}{q^{1/2}f(-q^{15})}$, then

$$(2.8) \quad (XY)^3 + \left(\frac{5}{XY}\right)^3 + \left[\left(\frac{X}{Y}\right)^6 - \left(\frac{Y}{X}\right)^6\right] + 9 \left[\left(\frac{X}{Y}\right)^3 + \left(\frac{Y}{X}\right)^3\right] = 0.$$

Lemma 2.6. [14, p. 55] If $X := \frac{f(-q)}{q^{1/6}f(-q^5)}$ and $Y := \frac{f(-q^4)}{q^{2/3}f(-q^{20})}$, then

$$(2.9) \quad (XY)^3 + \left(\frac{5}{XY}\right)^3 = \left(\frac{X}{Y}\right)^5 + \left(\frac{Y}{X}\right)^5 - 8 \left\{ \left(\frac{X}{Y}\right)^3 + \left(\frac{Y}{X}\right)^3 \right\} \\ + 4 \left(\frac{X}{Y} + \frac{Y}{X} \right) + 4 / \left(\frac{X}{Y} + \frac{Y}{X} \right).$$

Lemma 2.7. [8] If $P := \frac{\varphi(-q)}{\varphi(-q^5)}$ and $Q := \frac{\varphi(-q^4)}{\varphi(-q^{20})}$, then

$$(2.10) \quad \begin{aligned} & \frac{P^4}{Q^4} + \frac{Q^4}{P^4} + 24 \left[\frac{P^2}{Q^2} + \frac{Q^2}{P^2} \right] + 8 \left[P^2 Q^2 + \frac{5^2}{P^2 Q^2} \right] + 3 \left[Q^4 + \frac{5^2}{Q^4} \right] + 120 \\ & = 20 \left[P^2 + \frac{5}{P^2} \right] + 32 \left[Q^2 + \frac{5}{Q^2} \right] + \left[P^2 Q^4 + \frac{5^4}{P^2 Q^4} \right] + 3 \left[\frac{5P^2}{Q^4} + \frac{Q^4}{P^2} \right]. \end{aligned}$$

Lemma 2.8. [1, Theorem 5.3] If $U := \frac{\varphi^2(q)}{\varphi^2(q^5)}$ and $V := \frac{\psi^2(-q)}{q\psi^2(-q^5)}$, then

$$(2.11) \quad U + UV = 5 + V.$$

Lemma 2.9. [2, Theorem 2.17] If $U := \frac{\varphi(-q)}{\varphi(-q^5)}$ and $V := \frac{\varphi(-q^2)}{\varphi(-q^{10})}$, then

$$(2.12) \quad \frac{U^2}{V^2} + \frac{V^2}{U^2} + 4 = V^2 + \frac{5}{V^2}.$$

3. $P-Q$ MODULAR EQUATIONS OF DEGREE 5

In this section, we establish some new modular equations of degree 5.

Theorem 3.1. If $M := \frac{\varphi(-q)}{\varphi(-q^5)}$ and $N := \frac{\varphi(-q^6)}{\varphi(-q^{30})}$, then

$$(3.1) \quad \begin{aligned} & \left[\frac{M^8}{N^8} + \frac{N^8}{M^8} \right] + 96 \left[\frac{M^6}{N^6} + \frac{N^6}{M^6} \right] + 1146 \left[\frac{M^4}{N^4} + \frac{N^4}{M^4} \right] + 2868 \left[\frac{M^2}{N^2} + \frac{N^2}{M^2} \right] \\ & + \left[N^8 + \frac{5^4}{N^8} \right] - 16 \left[N^6 + \frac{5^3}{N^6} \right] + 188 \left[N^4 + \frac{5^2}{N^4} \right] - 1696 \left[N^2 + \frac{5}{N^2} \right] \\ & - 54 \left[M^6 + \frac{5^3}{M^6} \right] + 498 \left[M^4 + \frac{5^2}{M^4} \right] - 2106 \left[M^2 + \frac{5}{M^2} \right] - 4 \left[\frac{N^8}{M^6} + \frac{5M^6}{N^8} \right] \\ & + 6 \left[\frac{N^8}{M^4} + \frac{5^2M^4}{N^8} \right] - 4 \left[\frac{N^8}{M^2} + \frac{5^3M^2}{N^8} \right] - 144 \left[\frac{N^6}{M^4} + \frac{5M^4}{N^6} \right] + 64 \left[\frac{N^6}{M^2} + \frac{5^2M^2}{N^6} \right] \\ & - 479 \left[\frac{N^4}{M^2} + \frac{5M^2}{N^4} \right] - 165 \left[\frac{M^6}{N^4} + \frac{5N^4}{M^6} \right] + 124 \left[\frac{M^6}{N^2} + \frac{5^2N^2}{M^6} \right] - 936 \left[\frac{M^4}{N^2} + \frac{5N^2}{M^4} \right] \\ & + 10 \left[M^4N^4 + \frac{5^4}{M^4N^4} \right] + 516 \left[M^2N^2 + \frac{5^2}{M^2N^2} \right] - 39 \left[M^2N^4 + \frac{5^3}{M^2N^4} \right] \\ & - 120 \left[M^4N^2 + \frac{5^3}{M^4N^2} \right] + 12 \left[M^6N^2 + \frac{5^4}{M^6N^2} \right] - \left[M^6N^4 + \frac{5^5}{M^6N^4} \right] + 6748 = 0. \end{aligned}$$

Proof. Using the equation (2.8) after changing q to q^2 , we get

$$(3.2) \quad X^9Y^9 + 125X^3Y^3 + X^{12} - Y^{12} + 9X^9Y^3 + 9Y^9X^3 = 0.$$

where

$$X := \frac{f(-q^2)}{q^{1/3}f(-q^{10})} \quad \text{and} \quad Y := \frac{f(-q^6)}{qf(-q^{30})}.$$

Cubing the equation (3.2) and using the equations (2.3) and (2.4), we deduce

$$(3.3) \quad TM^3R^3N^6T_2 + 125MRN^2T + M^4R^4 - N^8T_2^2 + 9M^3R^3N^2T + 9TN^6T_2MR = 0.$$

where

$$R := \frac{\psi^2(q)}{q\psi^2(q^5)}, \quad T := \frac{\psi(q^6)}{q^3\psi(q^{30})} \quad \text{and} \quad T_2 := \frac{\psi^2(q^6)}{q^6\psi^2(q^{30})}.$$

Using the equation (2.11) after changing q to $-q$, we have

$$(3.4) \quad R := \frac{M^2 - 5}{M^2 - 1} \text{ and } T_2 := \frac{N^2 - 5}{N^2 - 1}.$$

Collecting the terms containing T on one side of the equation (3.3) and using the equation (3.4), we get

$$(3.5) \quad A(M, N)B(M, N) = 0,$$

where

$$\begin{aligned} A(M, N) := & (625 - 500M^2 - 20N^6 - 500N^2 + 150M^4 + N^8 + 150N^4 + M^8 - 20M^6 \\ & + 16M^6N^6 + 400M^2N^2 + 120M^4N^2 + 24M^6N^4 + 24M^4N^6 - 300M^4N^4 - 16M^2N^6 \\ & - 16M^6N^2 - 4M^8N^2 + N^8M^8 - 4N^8M^6 + 6N^8M^4 - 4N^8M^2 - 4M^8N^6 \\ & + 120M^2N^4 + 6M^8N^4) \end{aligned}$$

and

$$\begin{aligned} B(M, N) := & (625M^8 + N^{16} - 825N^{12}M^2 + 150M^{12} - 500M^{10} + M^{16} + 12900M^6N^6 \\ & - 4875M^6N^4 - 15000M^4N^6 + 6250M^4N^4 + 7500M^2N^6 - 2000M^8N^2 + 1146M^{12}N^4 \\ & - 720M^{12}N^2 - 2395M^{10}N^4 + 1600M^{10}N^2 + 188N^{12}M^8 - 479N^{12}M^6 + 1146N^{12}M^4 \\ & - 1696N^{10}M^8 + 2868N^{10}M^6 - 4680N^{10}M^4 + 3100N^{10}M^2 + 6748N^8M^8 - 10530N^8M^6 \\ & + 12450N^8M^4 - 6750N^8M^2 + 10M^{12}N^{12} - 120M^{12}N^{10} - 39M^{10}N^{12} + 516M^{10}N^{10} \\ & - 2106M^{10}N^8 + 2868M^{10}N^6 - 8480M^8N^6 + 498M^{12}N^8 - 936M^{12}N^6 - 165M^{14}N^4 \\ & + 96M^{14}N^2 - M^{14}N^{12} + 12M^{14}N^{10} - 54M^{14}N^8 - 20M^{14} + 124M^{14}N^6 + 96N^{14}M^2 \\ & - 16N^{14}M^8 + 64N^{14}M^6 - 144N^{14}M^4 - 4N^{16}M^2 + N^{16}M^8 - 4N^{16}M^6 + 6N^{16}M^4 \\ & - 3125M^2N^4 + 4700M^8N^4). \end{aligned}$$

Expanding in powers of q , the first and second factor of the equation (3.5), one gets respectively,

$$A(M, N) = (256 - 1536q^8 - 512q^9 + 1152q^{10} + 3840q^{11} + 4736q^{12} + \dots)$$

and

$$B(M, N) = q^8 (8448 + 33792q + 33792q^2 - 54528q^3 - 194208q^4 - 268608q^5 + \dots).$$

As $q \rightarrow 0$, the factor $B(M, N)$ of the equation (3.5) vanishes whereas the other factor $A(M, N)$ do not vanish. Hence, we arrive at the equation (3.1) for $q \in (0, 1)$. By analytic continuation the equation (3.1) is true for $|q| < 1$. \square

Remark 1. The modular relation connecting

$$\frac{\psi(q)}{q^{1/2}\psi(q^5)} \text{ and } \frac{\psi(q^6)}{q^3\psi(q^{30})},$$

can be obtained by eliminating M and N from the equation (3.3).

4. $P-Q$ “MIXED” MODULAR EQUATIONS

In this section, we establish several new $P-Q$ “mixed” modular equations with four moduli. Throughout this section, we set

$$(4.1) \quad A := \frac{f(-q)f(-q^2)}{q^{1/2}f(-q^5)f(-q^{10})}, \quad B_n := \frac{f(-q^n)f(-q^{2n})}{q^{n/2}f(-q^{5n})f(-q^{10n})} \text{ and } C_n := \frac{q^{n/6}f(-q^n)f(-q^{10n})}{f(-q^{2n})f(-q^{5n})}.$$

Theorem 4.1. For $|q| < 1$,

$$(4.2) \quad \frac{f^2(-q)f^2(-q^2)}{qf^2(-q^5)f^2(-q^{10})} = \frac{U(U-5)}{U-1}, \quad U > 1,$$

$$(4.3) \quad \frac{f^2(-q)f^2(-q^2)}{qf^2(-q^5)f^2(-q^{10})} = \frac{V(V-5)}{V-1}, \quad V > 1,$$

where $U := \frac{\varphi^2(-q)}{\varphi^2(-q^5)}$ and $V := \frac{\psi^2(q)}{q\psi^2(q^5)}$.

Proof of (4.2). The equations (2.5) and (2.6) can be rewritten as

$$(4.4) \quad m \left\{ \frac{\alpha(1-\alpha)}{\beta(1-\beta)} \right\}^{1/4} + 1 = \frac{5}{m} + \left\{ \frac{\alpha(1-\alpha)}{\beta(1-\beta)} \right\}^{1/4}.$$

Employing the equation (2.1) after changing q to $-q$ and equation (2.2) in the equation (4.4), we arrive at the equation (4.2). \square

Proof of (4.3). Using the equation (2.11) in the equation (4.2), we arrive at the equation (4.3). \square

Theorem 4.2. For $|q| < 1$,

$$(4.5) \quad \frac{qf^6(-q)f^6(-q^{10})}{f^6(-q^2)f^6(-q^5)} = \frac{U(U-1)}{U-5},$$

$$(4.6) \quad \frac{qf^6(-q)f^6(-q^{10})}{f^6(-q^2)f^6(-q^5)} = \frac{(V-5)}{V(V-1)},$$

where $U := \frac{\varphi^2(-q)}{\varphi^2(-q^5)}$ and $V := \frac{\psi^2(q)}{q\psi^2(q^5)}$.

Proof. The proof of equations (4.5) and (4.6) are similar to the proof of (4.2) and (4.3). Hence, we omit the details. \square

Theorem 4.3. If $P := AB_2$ and $Q := \frac{A}{B_2}$, then

$$(4.7) \quad \left(Q^4 + \frac{1}{Q^4}\right) - 3\left(Q^2 + \frac{1}{Q^2}\right) - \left(P + \frac{5^2}{P}\right)\left(Q + \frac{1}{Q}\right) - 12 = 0.$$

Proof. Taking the cube of both sides of the equation (2.7), we deduce

$$(4.8) \quad X^3Y^3 + \frac{125}{X^3Y^3} + 15\left(\frac{X^3}{Y^3} + \frac{Y^3}{X^3}\right) = \frac{X^9}{Y^9} + \frac{Y^9}{X^9},$$

where $X := \frac{f(-q)}{q^{1/6}f(-q^5)}$ and $Y := \frac{f(-q^2)}{q^{1/3}f(-q^{10})}$. Using (2.3), (2.4) and (4.1), we deduce

$$(4.9) \quad A^4V_1^4B_2^4V_2^4 + 125A^2V_1^2B_2^2V_2^2 + 12A^4V_1^4B_2^2V_2^2 + 12B_2^4V_2^4A^2V_1^2 = A^6V_1^6 + B_2^6V_2^6,$$

where $V_1 := \frac{\psi(q)}{q^{1/2}\psi(q^5)}$ and $V_2 := \frac{\psi(q^2)}{q\psi(q^{10})}$.

Using the equation (4.3) in the equation (4.9), we deduce

$$(4.10) \quad \begin{aligned} & 12B_2^6A^2vu + A^6B_2^6uv + 5A^6B_2^4uv + 5A^4B_2^6uv + 12A^6B_2^2uv + 625A^2B_2^2v \\ & + 625A^2B_2^2u - 18A^8u - 18B_2^8v + 185A^4B_2^4v + 185A^4B_2^4u + 300A^4B_2^2u \\ & + 60A^4B_2^2uv + 300B_2^4A^2v + 60B_2^4A^2vu + A^8B_2^6v - 250A^6 - 250B_2^6 \\ & + 1350A^4B_2^4 + 2125A^4B_2^2 + A^6B_2^8u + 37A^4B_2^6v + 40A^6B_2^4v + 8A^6B_2^6v \\ & + 40A^4B_2^6u + 37A^6B_2^4u + 5A^4B_2^8u + 8A^6B_2^6u + 60A^6B_2^2u + 60B_2^6A^2v \\ & + 425A^4B_2^2v - 50A^6u - 50B_2^6v + 12A^2B_2^8u + 5A^8B_2^4v + 425A^2B_2^4u \\ & + 12A^8B_2^2v + 125A^2B_2^2uv + 64A^6B_2^6 + 96A^6B_2^2v + 96B_2^6A^2u + 25A^4B_2^4uv \\ & - 2B_2^{14}v - 2A^{14}u + 2125A^2B_2^4 + 3125A^2B_2^2 + A^8B_2^8 + 8A^8B_2^6 + 37A^8B_2^4 \\ & + 8A^6B_2^8 + 296A^6B_2^4 + 37A^4B_2^8 + 296A^4B_2^6 + 60A^8B_2^2 + 480A^6B_2^2 \\ & + 60A^2B_2^8 + 480A^2B_2^6 - 120B_2^8 - 120A^8 - 24B_2^{14} - 24A^{14} - 2A^{12} - 2B_2^{12} = 0. \end{aligned}$$

where $u := \pm\sqrt{A^4 + 6A^2 + 25}$ and $v := \pm\sqrt{B_2^4 + 6B_2^2 + 25}$.

Eliminating u and v from the equation (4.10), we find and squaring both sides, we deduce

$$\begin{aligned}
 & (A^8 - A^6 B_2^4 - 3A^6 B_2^2 - 12A^4 B_2^4 - 25A^4 B_2^2 - A^4 B_2^6 - 3A^2 B_2^6 - 25A^2 B_2^4 \\
 & + B_2^8) (A^{16} + B_2^{16} - 187500A^6 B_2^4 - 187500A^4 B_2^6 - 26900A^6 B_2^8 \\
 & - 390625A^4 B_2^4 - 93125A^6 B_2^6 - 57500A^4 B_2^8 - 15625A^2 B_2^8 - 9600B_2^{10} A^4 \\
 & - 5000B_2^{10} A^2 - 875B_2^{12} A^2 - 7676A^8 B_2^8 - 57500A^8 B_2^4 - 15625A^8 B_2^2 \\
 (4.11) \quad & - 26900A^8 B_2^6 - 1076A^8 B_2^{10} - 92A^8 B_2^{12} - A^8 B_2^{14} - 35A^{14} B_2^4 - 384A^{12} B_2^6 \\
 & - A^{14} B_2^8 - A^{12} B_2^{12} - 52A^{14} B_2^2 - 1024A^{12} B_2^4 - 875A^{12} B_2^2 - 3987A^{10} B_2^6 \\
 & - 9600A^{10} B_2^4 - 8A^{14} B_2^6 - 5000A^{10} B_2^2 - 92A^{12} B_2^8 - 1076A^{10} B_2^8 \\
 & - 149A^{10} B_2^{10} - 12A^{10} B_2^{12} - 384A^6 B_2^{12} - 3987A^6 B_2^{10} - 8A^6 B_2^{14} \\
 & - 35A^4 B_2^{14} - 12A^{12} B_2^{10} - 1024B_2^{12} A^4 - 52A^2 B_2^{14}) = 0.
 \end{aligned}$$

Expanding in powers of q , the first and second factors of (4.11), one gets respectively

$$-4q^{11} (4 + 8q - 27q^2 - 68q^3 + 40q^4 + 278q^5 + 62q^6 - 723q^7 + \dots)$$

and

$$(-2 + 2q^2 - 10q^3 + 30q^4 + 552q^5 - 2016q^6 + 1038q^7 + 15620q^8 + \dots).$$

As q tends to 0 the first factor of (4.11) vanishes whereas the second factor does not vanish. Hence we arrive at (4.7) for $q \in (0, 1)$. By analytic continuation (4.7) is true for $|q| < 1$. \square

Theorem 4.4. If $P = AB_4$ and $Q = \frac{A}{B_4}$, then

$$\begin{aligned}
 & \left(Q^8 + \frac{1}{Q^8} \right) - 52 \left(Q^6 + \frac{1}{Q^6} \right) - 1024 \left(Q^4 + \frac{1}{Q^4} \right) - 3987 \left(Q^2 + \frac{1}{Q^2} \right) \\
 & - \left(P + \frac{5^2}{P} \right) \left[1076 \left(Q + \frac{1}{Q} \right) + 384 \left(Q^3 + \frac{1}{Q^3} \right) + 35 \left(Q^5 + \frac{1}{Q^5} \right) \right] \\
 (4.12) \quad & - \left(P^2 + \frac{5^4}{P^2} \right) \left[92 \left(Q^2 + \frac{1}{Q^2} \right) + 8 \left(Q^4 + \frac{1}{Q^4} \right) + 149 \right] - \left(P^3 + \frac{5^6}{P^3} \right) \\
 & \times \left[12 \left(Q + \frac{1}{Q} \right) + \left(Q^3 + \frac{1}{Q^3} \right) \right] - \left(P^4 + \frac{5^8}{P^4} \right) - 7676 = 0.
 \end{aligned}$$

Proof. Using the equation (2.10) in the equation (4.2), we deduce

$$\begin{aligned}
 & 625 - 125u + 125v - A^4uB_4^6v - 5A^4uB_4^4v - 90A^2uB_4^2v - 6A^2uB_4^6v \\
 & - 38A^2uB_4^4v + 40A^2u + 2A^6u - 1215B_4^2v - 63B_4^6v - A^6v - 56A^6B_4^2 \\
 & - 36A^6B_4^4 - 8A^6B_4^6 - A^6B_4^8 + 3A^4v - 592A^4B_4^2 - 360A^4B_4^4 - 80A^4B_4^6 \\
 & - 9A^4B_4^8 - 15A^2v - 3080A^2B_4^2 - 1732A^2B_4^4 - 376A^2B_4^6 - 39A^2B_4^8 \\
 & - 25uv - 1360uB_4^2 - 688uB_4^4 - 144uB_4^6 - 13uB_4^8 - 125A^2 - 6000B_4^2 \\
 (4.13) \quad & + 29A^4 - 3A^6 + 2A^8 - 3216B_4^4 - 688B_4^6 - 63B_4^8 - 9A^4u - 13A^6B_4^2v \\
 & - A^6B_4^6v - 5A^6B_4^4v - 129A^4B_4^2v - 9A^4B_4^6v - 53A^4B_4^4v - A^4uv \\
 & - 56A^4uB_4^2 - 36A^4uB_4^4 - 8A^4uB_4^6 - A^4uB_4^8 - 643A^2B_4^2v - 39A^2B_4^6v \\
 & - 259A^2B_4^4v + 6A^2uv - 424A^2uB_4^2 - 252A^2uB_4^4 - 56A^2uB_4^6 - 6A^2uB_4^8 \\
 & - 269uB_4^2v - 13uB_4^6v - 105uB_4^4v - 13A^4uB_4^2v - 499B_4^4v = 0.
 \end{aligned}$$

where $u := \pm\sqrt{A^4 + 6A^2 + 25}$ and $v := \pm\sqrt{B_4^4 + 6B_4^2 + 25}$.

Eliminating u and v from the equation (4.13), we arrive at (4.12). \square

Theorem 4.5. If $P = AB_6$ and $Q = \frac{A}{B_6}$, then

$$\begin{aligned}
 & \mathbb{Q}^{16} - 363\mathbb{Q}^{14} - 30882\mathbb{Q}^{12} - 698682\mathbb{Q}^{10} - 6183702\mathbb{Q}^8 - 16140317\mathbb{Q}^6 \\
 & + 37225608\mathbb{Q}^4 + 231497788\mathbb{Q}^2 + \mathbb{P}\{60133800\mathbb{Q} + 21753498\mathbb{Q}^3 \\
 & - 1148442\mathbb{Q}^5 - 2210604\mathbb{Q}^7 - 406488\mathbb{Q}^9 - 26740\mathbb{Q}^{11} - 519\mathbb{Q}^{13}\} \\
 & + \mathbb{P}^2\{6287236\mathbb{Q}^2 + 858465\mathbb{Q}^4 - 462222\mathbb{Q}^6 - 150099\mathbb{Q}^8 - 12840\mathbb{Q}^{10} \\
 (4.14) \quad & - 267\mathbb{Q}^{12} + 10229305\} + \mathbb{P}^3\{1132002\mathbb{Q} + 362832\mathbb{Q}^3 - 42462\mathbb{Q}^5 - 37066\mathbb{Q}^7 \\
 & - 4323\mathbb{Q}^9 - 78\mathbb{Q}^{11}\} + \mathbb{P}^4\{74418\mathbb{Q}^2 + 4471\mathbb{Q}^4 - 5955\mathbb{Q}^6 - 1026\mathbb{Q}^8 - 12\mathbb{Q}^{10} \\
 & + 130902\} + \mathbb{P}^5\{9171\mathbb{Q} + 2028\mathbb{Q}^3 - 588\mathbb{Q}^5 - 171\mathbb{Q}^7 - \mathbb{Q}^9\} + \mathbb{P}^6\{300\mathbb{Q}^2 \\
 & - 27\mathbb{Q}^4 - 18\mathbb{Q}^6 + 679\} + \mathbb{P}^7\{24\mathbb{Q} - \mathbb{Q}^5\} + \mathbb{P}^8 + 36965548 = 0,
 \end{aligned}$$

where $\mathbb{P}^n = \left(P^n + \frac{5^{2n}}{P^n}\right)$ and $\mathbb{Q}^n = \left(Q^n + \frac{1}{Q^n}\right)$.

Proof. The proof of the equation (4.14) is similar to the proof of the equation (4.12); Notice that now (3.1) is used in place of (2.10). \square

Theorem 4.6. If $P = C_1 C_2$ and $Q = \frac{C_1}{C_2}$, then

$$(4.15) \quad \left(P + \frac{1}{P} \right) \left(Q^3 + \frac{1}{Q^3} \right) + 2 = \left(P^2 + \frac{1}{P^2} \right).$$

Proof. Using the equation (2.12) in the equation (4.5), we deduce

$$(4.16) \quad \begin{aligned} & 10v - 10u - 2vC_2^6 C_1^6 - 2vuC_2^6 + 6vu + 4uC_1^6 + 2vC_2^6 - 2C_2^{12} C_1^6 - 6 \\ & + 24C_2^6 C_1^6 - 2uC_2^{12} + 24uC_2^6 + 6vC_1^6 - 46C_1^6 - 8C_2^6 + 4C_1^{12} + 2C_2^{12} = 0. \end{aligned}$$

where $u := \pm\sqrt{C_1^{12} - 18C_1^6 + 1}$ and $v := \pm\sqrt{C_2^{12} - 18C_2^6 + 1}$.

Eliminating u and v from the equation (4.16) leads to

$$(4.17) \quad \begin{aligned} & (C_1^6 - C_2^6 C_1^6 + C_2^6 - C_2^2 C_1^2 + C_2^2 C_1^8 + 2C_2^4 C_1^4 + C_2^8 C_1^2) (C_2^4 C_1^{16} \\ & + C_2^8 C_1^{14} - C_2^2 C_1^{14} + C_2^{12} C_1^{12} - 4C_2^6 C_1^{12} + C_1^{12} + 4C_2^{10} C_1^{10} - 4C_2^4 C_1^{10} \\ & + C_2^{14} C_1^8 + C_2^8 C_1^8 + C_2^2 C_1^8 - 4C_2^{12} C_1^6 + 4C_2^6 C_1^6 + C_2^{16} C_1^4 - 4C_2^{10} C_1^4 \\ & + C_2^4 C_1^4 - C_2^{14} C_1^2 + C_2^8 C_1^2 + C_2^{12}) = 0. \end{aligned}$$

Expanding in powers of q , the first and second factors of (4.17), one gets respectively

$$q^{11} (8 - 32q - 8q^2 + 168q^3 - 220q^4 + 196q^5 - 760q^6 + 1748q^7 + \dots)$$

and

$$(3 - 24q + 117q^2 - 456q^3 + 1356q^4 - 3192q^5 + 7242q^6 - 17304q^7 + \dots).$$

As q tends to 0 the first factor of (4.17) vanishes whereas the second factor does not vanish. Hence we arrive at (4.15) for $q \in (0, 1)$. By analytic continuation (4.15) is true for $|q| < 1$. \square

Theorem 4.7. If $P = C_1 C_4$ and $Q = \frac{C_1}{C_4}$, then

$$(4.18) \quad \begin{aligned} & \left(P^3 + \frac{1}{P^3} \right) \left[19 \left(Q + \frac{1}{Q} \right) + 8 \left(Q^3 + \frac{1}{Q^3} \right) + \left(Q^5 + \frac{1}{Q^5} \right) \right] \\ & + \left(Q^6 + \frac{1}{Q^6} \right) + 13 \left(Q^4 + \frac{1}{Q^4} \right) + 52 \left(Q^2 + \frac{1}{Q^2} \right) + 82 = \left(P^6 + \frac{1}{P^6} \right). \end{aligned}$$

Proof. The proof of the equation (4.18) is similar to the proof of (4.15); Notice that now (2.10) is used in place of (2.12). \square

5. REMARKABLE PRODUCT OF THETA-FUNCTIONS

In this section, we establish several new modular identities connecting the remarkable product of theta-functions $b_{s,5}$ with $b_{r^2s,5}$ for $r = 2, 4$, and 6 .

Lemma 5.1. [9] *If s and t are any positive rational, then*

$$(5.1) \quad b_{2s,t}b_{\frac{2}{s},t} = 1.$$

Lemma 5.2. [10] *$0 < b_{s,t} \leq 1$ for all $s \geq 2$ and t positive integer greater than 1.*

Theorem 5.1. *If $X = \sqrt{b_{s,5}b_{4s,5}}$ and $Y = \sqrt{\frac{b_{s,5}}{b_{4s,5}}}$, then*

$$(5.2) \quad \left(Y^4 + \frac{1}{Y^4}\right) - 3\left(Y^2 + \frac{1}{Y^2}\right) - 5\left(X + \frac{1}{X}\right)\left(Y + \frac{1}{Y}\right) - 12 = 0.$$

Proof. Using the equation (1.8) in the equation (4.7) we arrive at the equation (5.2). \square

Corollary 5.1.

$$(5.3) \quad b_{4,5} = \frac{\sqrt{2 + 2\sqrt{5} - 2\sqrt{2 + 2\sqrt{5}}}}{2},$$

$$(5.4) \quad b_{1,5} = \frac{\sqrt{2 + 2\sqrt{5} + 2\sqrt{2 + 2\sqrt{5}}}}{2}.$$

Proof. Putting $s = 1/2$, in (5.2) and using the fact that $b_{1,5}b_{4,5} = 1$, we deduce

$$(5.5) \quad (h^8 - 2h^6 - 2h^4 - 2h^2 + 1)(h^2 + h + 1)(h^2 - h + 1) = 0,$$

where $h := b_{4,5}$.

We observe that the first factor of (5.5) vanishes for specific value of $q := e^{-\pi\sqrt{4/5}}$, whereas the other factors does not vanish. Hence, we have

$$(5.6) \quad t^2 - 2t - 4 = 0,$$

where $t := h^2 + \frac{1}{h^2}$.

On solving the equation (5.6) for h and $t > 0$, we deduce

$$(5.7) \quad h^2 + \frac{1}{h^2} = 1 + \sqrt{5}.$$

On solving the equation (5.7) for h and $0 < h < 1$, we arrive at (5.3) and (5.4). \square

Theorem 5.2. If $X = \sqrt{b_{s,5}b_{16s,5}}$ and $Y = \sqrt{\frac{b_{s,5}}{b_{16s,5}}}$, then

$$(5.8) \quad \begin{aligned} & \left(Y^8 + \frac{1}{Y^8} \right) - 52 \left(Y^6 + \frac{1}{Y^6} \right) - 1024 \left(Y^4 + \frac{1}{Y^4} \right) - 3987 \left(Y^2 + \frac{1}{Y^2} \right) \\ & - 5 \left(X + \frac{1}{X} \right) \left[1076 \left(Y + \frac{1}{Y} \right) + 384 \left(Y^3 + \frac{1}{Y^3} \right) + 35 \left(Y^5 + \frac{1}{Y^5} \right) \right] \\ & - 5^2 \left(X^2 + \frac{1}{X^2} \right) \left[92 \left(Y^2 + \frac{1}{Y^2} \right) + 8 \left(Y^4 + \frac{1}{Y^4} \right) + 149 \right] - 5^3 \left(X^3 + \frac{1}{X^3} \right) \\ & \times \left[12 \left(Y + \frac{1}{Y} \right) + \left(Y^3 + \frac{1}{Y^3} \right) \right] - 5^4 \left(X^4 + \frac{1}{X^4} \right) - 7676 = 0. \end{aligned}$$

Proof. Using the equation (1.8) in the equation (4.12) we arrive at the equation (5.8). \square

Corollary 5.2.

$$(5.9) \quad b_{8,5} = \sqrt{(\sqrt{2} - 1)(\sqrt{5} - 2)},$$

$$(5.10) \quad b_{1/2,5} = \sqrt{(\sqrt{2} + 1)(\sqrt{5} + 2)}.$$

Proof. Putting $s = 1/4$, in (5.8) and using the fact that $b_{1/2,5}b_{8,5} = 1$, we deduce

$$(5.11) \quad \begin{aligned} & (h^8 - 8h^6 - 22h^4 - 8h^2 + 1) (h^4 + 3h^2 + 1) (h^4 - h^3 + h^2 + h + 1) \\ & (h^4 + h^3 + h^2 - h + 1) = 0, \end{aligned}$$

where $h := b_{8,5}$.

We observe that the first factor of (5.11) vanishes for specific value of $q := e^{-\pi\sqrt{8/5}}$, whereas the other factors does not vanish. Hence, we have

$$(5.12) \quad t^2 - 8t - 24 = 0,$$

where $t := h^2 + \frac{1}{h^2}$.

On solving the equation (5.12) for h and $t > 0$, we deduce

$$(5.13) \quad h^2 + \frac{1}{h^2} = 4 + 2\sqrt{10}.$$

On solving the equation (5.13) for h and $0 < h < 1$, we arrive at (5.9) and (5.10). \square

Theorem 5.3. If $X = \sqrt{b_{s,5}b_{36s,5}}$ and $Y = \sqrt{\frac{b_{s,5}}{b_{36s,5}}}$, then

$$(5.14) \quad \begin{aligned} & \mathbb{Y}^{16} - 363\mathbb{Y}^{14} - 30882\mathbb{Y}^{12} - 698682\mathbb{Y}^{10} - 6183702\mathbb{Y}^8 - 16140317\mathbb{Y}^6 \\ & + 37225608\mathbb{Y}^4 + 231497788\mathbb{Y}^2 + 5\mathbb{X} \{ 60133800\mathbb{Y} + 21753498\mathbb{Y}^3 \\ & - 1148442\mathbb{Y}^5 - 2210604\mathbb{Y}^7 - 406488\mathbb{Y}^9 - 26740\mathbb{Y}^{11} - 519\mathbb{Y}^{13} \} \\ & + 5^2\mathbb{X}^2 \{ 6287236\mathbb{Y}^2 + 858465\mathbb{Y}^4 - 462222\mathbb{Y}^6 - 150099\mathbb{Y}^8 - 12840\mathbb{Y}^{10} \\ & - 267\mathbb{Y}^{12} + 10229305 \} + 5^3\mathbb{X}^3 \{ 1132002\mathbb{Y} + 362832\mathbb{Y}^3 - 42462\mathbb{Y}^5 \\ & - 37066\mathbb{Y}^7 - 4323\mathbb{Y}^9 - 78\mathbb{Y}^{11} \} + 5^4\mathbb{X}^4 \{ 74418\mathbb{Y}^2 + 4471\mathbb{Y}^4 - 5955\mathbb{Y}^6 \\ & - 1026\mathbb{Y}^8 - 12\mathbb{Y}^{10} + 130902 \} + 5^5\mathbb{X}^5 \{ 9171\mathbb{Y} + 2028\mathbb{Y}^3 - 588\mathbb{Y}^5 - 171\mathbb{Y}^7 \\ & - \mathbb{Y}^9 \} + 5^6\mathbb{X}^6 \{ 300\mathbb{Y}^2 - 27\mathbb{Y}^4 - 18\mathbb{Y}^6 + 679 \} + 5^7\mathbb{X}^7 \{ 24\mathbb{Y} - \mathbb{Y}^5 \} + 5^8\mathbb{X}^8 \\ & + 36965548 = 0, \end{aligned}$$

where $\mathbb{X}^n = \left(X^n + \frac{1}{X^n} \right)$ and $\mathbb{Y}^n = \left(Y^n + \frac{1}{Y^n} \right)$.

Proof. Using the equation (1.8) in the equation (4.14) we arrive at the equation (5.14). \square

Corollary 5.3.

$$(5.15) \quad b_{12,5} = \sqrt{\frac{(2 - \sqrt{3})(7 - 3\sqrt{5})}{2}},$$

$$(5.16) \quad b_{1/3,5} = \sqrt{\frac{(2 + \sqrt{3})(7 + 3\sqrt{5})}{2}}.$$

Proof. Putting $s = 1/6$, in (5.14) and using the fact that $b_{1/3,5}b_{12,5} = 1$, we deduce,

$$(5.17) \quad \begin{aligned} & (h^8 - 28h^6 + 63h^4 - 28h^2 + 1)(h^{12} + 10h^{10} + 15h^8 + 28h^6 + 15h^4 + 10h^2 + 1) \\ & (h^8 - 2h^7 + 4h^6 - h^5 + 7h^4 + h^3 + 4h^2 + 2h + 1)(h^8 + 2h^7 + 4h^6 + h^5 + 7h^4 \\ & - h^3 + 4h^2 - 2h + 1) = 0, \end{aligned}$$

where $h := b_{12,5}$.

We observe that the first factor of (5.17) vanishes for specific value of $q := e^{-\pi\sqrt{12/5}}$, whereas the other factors does not vanish. Hence, we have

$$(5.18) \quad t^2 - 28t + 61 = 0,$$

where $t := h^2 + \frac{1}{h^2}$.

On solving the equation (5.18) for h and $t > 0$, we deduce

$$(5.19) \quad h^2 + \frac{1}{h^2} = 14 + 3\sqrt{15}.$$

On solving the equation (5.19) for h and $0 < h < 1$, we arrive at (5.15) and (5.16). \square

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