Proceedings of the Jangjeon Mathematical Society
23 (2020), No. 3, pp. 297 - 304

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2020.23.3.297

THE GROWTH RATE OF RANDOM BALANCING SEQUENCE

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Abstract. In the present paper, a random sequence is defined by the binary recurrence $Z_{n+1} = A\alpha Z_n - qZ_{n-1}$, where α is a random variable which assumes the values $+1$ and -1 with probability $1/2$ each where A is a positive integer and q is a non-zero integer. Furthermore by taking $A = 6$ and $q = 1$, the random balancing case has been defined and the remaining cases for A and q have been further tackled. Apart from that an elementary proof regarding the bounds of the expected value for the absolute value of the n-th term in the random balancing sequence has been provided. Moreover, the bounds for the variance of the absolute value of the n-th term has also been obtained. Furthermore, the growth rate of the random sequence has been graphically depicted.

Keywords: Random sequences; Random variable; Balancing sequence; Growth rate.
2010 Subject classification [A.M.S.]: 11B37,11B39,11D09.

1. Introduction

The binary recurrence of the type $x_{n+1} = Ax_n - qx_{n-1}$ has been studied by several researchers (e.g., [4]). In this paper, we tackle the random case of such sequences. In other words, we consider the three term random binary sequence of the form

$$Z_{n+1} = A\alpha Z_n - qZ_{n-1}, \tag{1.1}$$

where A is a positive integer and q is a non-zero integer and

$$\alpha = \begin{cases} 1, \ with\ p = \frac{1}{2}; \\ -1, \ with\ p = \frac{1}{2} \end{cases}$$

where p stands for probability and initial values $Z_0 = 0$, $Z_1 = 1$. Furthermore, in matrix form, the proposed random sequence can be given by

$$\begin{bmatrix} Z_{n-1} \\ Z_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -q & \pm A \end{bmatrix} \begin{bmatrix} Z_{n-2} \\ Z_{n-1} \end{bmatrix},$$

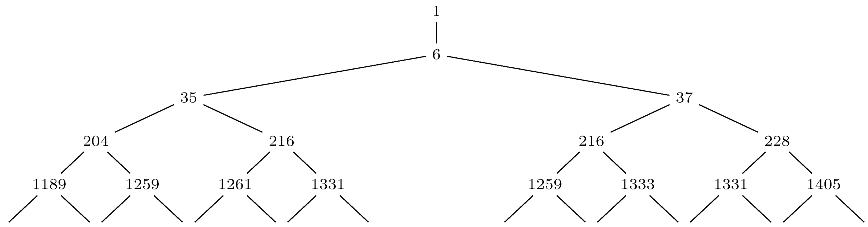
where one of the two matrices

$$G_1 = \begin{bmatrix} 0 & 1 \\ -q & A \end{bmatrix},\ G_2 = \begin{bmatrix} 0 & 1 \\ -q & -A \end{bmatrix}$$

is picked independently with probability $1/2$ at each step. Furthermore, the balancing number B_n is defined by the binary recurrence relation $B_{n+1} = 6B_n - B_{n-1}$([1], [7]) with initial values $B_0 = 0$ and $B_1 = 1$, and which is expressed in the form of matrices in [8]. So, by taking $A = 6$ and $q = 1$, we define the random balancing sequence Z_n as

$$Z_{n+1} = 6\alpha Z_n - Z_{n-1} = \begin{cases} 6Z_n - Z_{n-1}, \ with\ p = \frac{1}{2}; \\ -6Z_n - Z_{n-1}, \ with\ p = \frac{1}{2}, \end{cases} \tag{1.2}$$

with initial values $Z_0 = 0$, $Z_1 = 1$ and α is a random variable as in (1.1). The sequence (1.2) can be graphically shown as



In 1999, Viswanath [9] proposed the concept of random Fibonacci sequence and proved that the sequence grows exponentially at the rate of a named constant which he called Vishwanath constant. His proof involved certain complex floating point computer calculations. Moreover, Embree and Trefethen [2] provided numerical evidence for some general random sequence for the dividing line between sequences which grow and those which decay. In [3], Furstenberg studied regarding the noncommuting random products. Furthermore, in [5], Makover et al. gave a basic proof regarding the growth rate of random Fibonacci sequences.

Panda and Rout [6] studied a class of binary recurrences of the form $x_{n+1} = Bx_n - x_{n-1}$ for $B = 6$ with initial terms $x_0 = 0, x_1 = 1$. The sequence $\{x_n\}_{n=1}^{\infty}$ coincides with the natural numbers when $B = 2$ and represents the class of even indexed Fibonacci numbers when $B = 3$. In this research article, we define the random balancing sequence. we evaluate the bounds of the expected value for the absolute value of the n-th term in the proposed random sequence via an elementary technique. Furthermore, we also obtain the bounds on the variance for x_n. Apart from this, we obtain the approximate value of A below which the proposed random sequences decay and also provide a graph to validate the arguments and theorems.

2. Bounds of expected value of n^{th} label of the proposed sequence

In this section we consider the sequence

$$Z_{n+1} = \begin{cases} AZ_n - qZ_{n-1}, \ \textit{with } p = \frac{1}{2}; \\ -AZ_n - qZ_{n-1}, \ \textit{with } p = \frac{1}{2}, \end{cases} \tag{2.1}$$

where A is a positive integer. Here the set of possible random sequences has been considered as a binary tree structure where each leaf Z_i have two branches leading to the child leaves $|AZ_i - qZ_{i-1}|$ and $|-AZ_i - qZ_{i-1}|$. Such a tree lists the 2^n such sequences of length n. Furthermore, let the expected value for the absolute value of the n-th term in the random sequence (2.1) is taken as $E(|Z_n|)$.

In general, the binary tree has been constructed beginning with an element a at label $n = 0$ which give rise to two children b_1 and b_2 at label $n = 1$ which subsequently have four grand-children in the subsequent labels following the sequence as in (2.1) and so on. The basic set in Fig 1 displays the binary random recurrence in (2.1). The detailed setup of the binary tree has been depicted in the following figure.

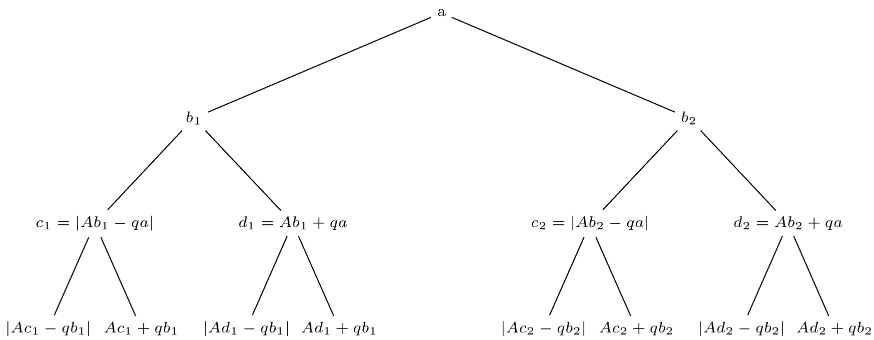


Fig. 1. *Binary tree for the seq.* (2.1)

2.1. **Expectation bounds for the random sequence in general.** By referring the binary tree structure in Fig 1, the sum of the row in label 3 of the binary tree has been utilized to find the upper and lower bounds of the expected value $E(|Z_n|)^{1/n}$ of the absolute value of the n-th term in the proposed random sequence in Eq. (2.1).

Here we assume that q is a positive integer and since the calculations for the negative assumptions of q is similar, it is omitted. Furthermore, let the sum of the row values in the label 3 be denoted by μ. Then,

$$\begin{aligned}\mu &= |Ac_1 - qb_1| + Ac_1 + qb_1 + |Ad_1 - qb_1| + Ad_1 + qb_1 + |Ac_2 - qb_2| \\ &\quad + Ac_2 + qb_2 + |Ad_2 - qb_2| + Ad_2 + qb_2,\end{aligned}$$

which can be simplified as

$$\begin{aligned}\mu &= 2q(b_1 + b_2) + A(c_1 + c_2 + d_1 + d_2) + |Ac_1 - qb_1| + |Ad_1 - qb_1| \\ &\quad + |Ac_2 - qb_2| + |Ad_2 - qb_2|.\end{aligned}$$

Now assuming that $Ab_1 \geq qa$ and $Ab_2 \geq qa$, we obtain

$$\begin{aligned}\mu =&2q(b_1 + b_2) + A(c_1 + c_2 + d_1 + d_2) + |A^2b_1 - Aqa - qb_1| + |A^2b_1 + Aqa - qb_1| \\ &+ |A^2b_2 - Aqa - qb_2| + |A^2b_2 + Aqa - qb_2|.\end{aligned}$$

which can be rewritten as

$$\begin{aligned}\mu &= 2qb_1 + A(c_1 + d_1) + |A^2b_1 - Aqa - qb_1| + |A^2b_1 + Aqa - qb_1| + 2qb_2 + A(c_2 + d_2) \\ &\quad + |A^2b_2 - Aqa - qb_2| + |A^2b_2 + Aqa - qb_2|.\end{aligned}$$

Furthermore, assuming

$$\mu_1 = 2qb_1 + A(c_1 + d_1) + |A^2b_1 - Aqa - qb_1| + |A^2b_1 + Aqa - qb_1|$$

and

$$\mu_2 = 2qb_2 + A(c_2 + d_2) + |A^2b_2 - Aqa - qb_2| + |A^2b_2 + Aqa - qb_2|,$$

we have

$$\mu = \mu_1 + \mu_2. \tag{2.2}$$

So for μ_1, we need six conditions.
Case 1. $Ab_1 < qa,\ Aqa + A^2b_1 \geq qb_1$. Then $\mu_1 = A(c_1+d_1) + 2qb_1 + 2A^2b_1$, provided $A^2 < \frac{q}{2}$.
Case 2. $Ab_1 < qa,\ A^2b_1 > Aqa + qb_1$. Then $\mu_1 = A(c_1+d_1) + 2qb_1 + 2A^2b_1$.
Case 3. $Ab_1 < qa,\ Aqa + A^2b_1 < qb_1$. Then $\mu_1 = A(c_1+d_1) + 2qb_1 + 2A^2b_1$.
Case 4. $Ab_1 \geq qa,\ Aqa + A^2b_1 \geq qb_1$. Then $\mu_1 = A(c_1+d_1) + 2qb_1 + 2A^2b_1$, provided $A^2 > \frac{q}{2}$
Case 5. $Ab_1 \geq qa,\ Aqa + A^2b_1 < qb_1$. Then $\mu_1 = A(c_1+d_1) + 4qb_1 - 2Aa$, provided $A < q$.
Case 6. $Ab_1 \geq qa,\ Aqa > A^2b_1 + qb_1$. Then $\mu_1 = A(c_1+d_1) + 2qb_1 + 2Aa$, provided $A > q$.
Remark: Since μ_2 is just a replication of μ_1 i.e. symmetrical where b_1 is replaced by b_2, hence the similar conditions for μ_2 are omitted.

Hence substituting the values of μ_1 and μ_2 in (2.2), we obtain μ for the six cases as

$$A(c_1 + c_2 + d_1 + d_2) + (2A^2 + 2q)(b_1 + b_2),$$

$$A(c_1 + c_2 + d_1 + d_2) + (2A^2 + 2q)(b_1 + b_2),$$

$$A(c_1 + c_2 + d_1 + d_2) + (2A^2 + 2q)(b_1 + b_2),$$

$$A(c_1 + c_2 + d_1 + d_2) + (2A^2 + 2)(b_1 + b_2),$$

$$A(c_1 + c_2 + d_1 + d_2) + 2Aa + 2q(b_1 + b_2),$$

and

$$A(c_1 + c_2 + d_1 + d_2) - 2Aa + 2q(b_1 + b_2).$$

Now, if we denote the sum of row values in label n by $S[n]$, then we get the six recurrence relations corresponding to each of the above sums as

$$S[n] = AS[n-1] + (2A^2 + 2)S[n-2], ...(\textbf{Case 1},\ \textbf{Case 2},\ \textbf{Case 3},\ \textbf{Case 4}),$$

$$S[n] = AS[n-1] + 2qS[n-2] + 2AS[n-3], ...(\textbf{Case 5})$$

and

$$S[n] = AS[n-1] + 2qS[n-2] - 2AS[n-3], ...(\textbf{Case 6}).$$

These recurrence relations can be solved in terms of cubic and quadratic equations respectively.

2.2. **Bounds for the Expectation of random balancing sequence.** Let us assume $A = 6$ and $q = 1$ in (1.1) in which the random sequence in Eq. (1.1) takes the form

$$Z_{n+1} = 6\alpha Z_n - Z_{n-1}, \tag{2.3}$$

which we call as the random balancing sequence where α takes the value 1 with probability $\frac{1}{2}$ and value -1 with probability $\frac{1}{2}$. In this case, starting from initial value a and its two branches b_1 and b_2 which proceeds in the subsequent labels as in the case of Fig 1 but following (2.3). So, the binary tree takes the following form.

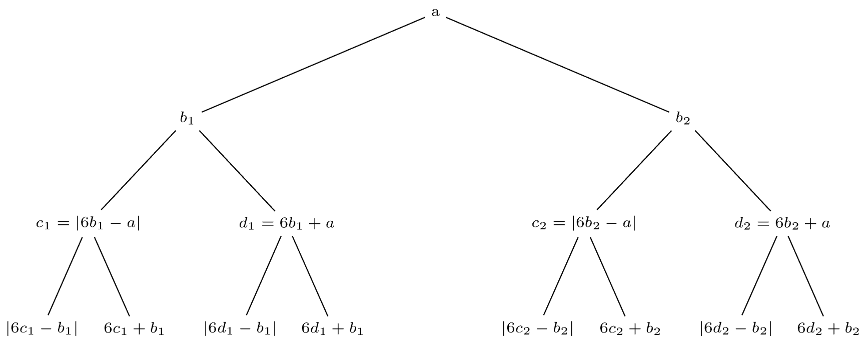


Fig. 2. Binary tree for the seq. (2.3)

Similar to the previous case, the sum of the 3-rd row is denoted by μ. Furthermore, the sum of third row of the binary tree in Fig. 2 is utilized to find the upper and lower bounds of $E(|Z_n|)^{1/n}$. So, we have

$$\begin{aligned}\mu &= |6c_1 - b_1| + 6c_1 + b_1 + |6d_1 - b_1| + 6d_1 + b_1 + |6c_2 - b_2| + 6c_2 + b_2 \\ &\quad + |6d_2 - b_2| + 6d_2 + b_2\end{aligned}$$

which can be simplified as

$$\mu = 72b_1 + 6(c_1 + c_2 + d_1 + d_2) + 37b_2 + 6a + |6|6b_2 - a| - b_2|. \tag{2.4}$$

Now assuming that $6b_2 \geq a$, we get

$$|6|6b_2 - a| - b_2| = |35b_2 - 6a| = 35b_2 - 6a.$$

Now the bounds of $35b_2 - 6a$ can be written as

$$35b_2 - 7a < 35b_2 - 6a < 35b_2 - 5a,$$

which provides a bound for (2.4) as

$$72b_1+6(c_1+c_2+d_1+d_2)+37b_2+6a+35b_2-7a < \mu < 72b_1+6(c_1+c_2+d_1+d_2)+37b_2+6a+35b_2-5a,$$

which on further simplification gives

$$72(b_1 + b_2) + 6(c_1 + c_2 + d_1 + d_2) - a < \mu < 72(b_1 + b_2) + 6(c_1 + c_2 + d_1 + d_2) + a.$$

Now, if we denote the sum of the n-th row by $S[n]$, then the lower bound of μ can be written in the form of a recurrence relation as

$$S[n] = 6S[n-1] + 72S[n-2] - S[n-3],$$

which reduces to the cubic equation $x^3 - 6x^2 - 72x + 1 = 0$, which renders the value approximately as $x = 11.99537$. Similarly the upper bound of μ, can be written as a recurrence relation

$$S[n] = 6S[n-1] + 72S[n-2] + S[n-3],$$

which reduces to the cubic equation $x^3 - 6x^2 - 72x - 1 = 0$, which renders the value approximately as $x = 12.00463$.

2.3. **Generalization of the bounds of Expectation.**

Cases	Recurrence relations	Equations
$n=3$	$S[n] = 6S[n-1] + 72S[n-2] \pm S[n-3]$	$x^3 - 6x^2 - 72x \pm 1 = 0,$
$n=4$	$S[n] = 6S[n-1] + 2S[n-2] + 840S[n-3] \pm S[n-4]$	$x^4 - 6x^3 - 2x^2 - 840x \pm 1 = 0$
$n=5$	$S[n] = 6S[n-1] + 2S[n-2] + 10080S[n-4] \pm S[n-5]$	$x^5 - 6x^4 - 2x^3 - 10080x \pm 1 = 0$
$n=6$	$S[n] = 6S[n-1] + 2S[n-2] + 120960S[n-5] \pm S[n-6]$	$x^6 - 6x^5 - 2x^4 - 120960x \pm 1 = 0$

Cases	Lower bound of $E(\lvert x_n\rvert)^{1/n}$	Upper bound of $E(\lvert x_n\rvert)^{1/n}$
$n=3$	$\frac{x}{2} = 5.99768\cdots$	$\frac{x}{2} = 6.00231\cdots$
$n=4$	$\frac{x}{2} = 5.9998\cdots$	$\frac{x}{2} = 6.00014\cdots$
$n=5$	$\frac{x}{2} = 5.9999\cdots$	$\frac{x}{2} = 6.0000\cdots$
$n=6$	$\frac{x}{2} = 5.99999\cdots$	$\frac{x}{2} = 6.00000\cdots$

Table 1. Table for the bounds of Expectation

For $n = 7, 8, 9, \ldots$, applying the similar procedure, we get the bounds for expectation of the respective sequences. Now the growth rate of $S[n]$ will be again given by the real roots divided by 2 and proceeding in a similar way, we get the theorem as

Theorem 2.1. *For any positive integer n,*

$$5.99\cdots < (E(|Z_n|))^{1/n} < 6.000\cdots.$$

In the succeeding section, we find the bounds of the variance and state it as a theorem as

Theorem 2.2. *For any positive integer n,*

$$\lim_{n\to\infty} (\mu_2(|Z_n|))^{1/n} = 36 + 10\sqrt{13},$$

where $\mu_2(|Z_n|)$ is the variance of the sequence (Z_n) in (2.3).

Proof. We begin the proof by denoting the sum of squares of a given row by $V[n]$ and considering the same Fig 2. By the definition of variance say m_2, we have $m_2 = m_2' - (m_1')^2$, where $m_2' = E(|Z_n|^2)$ and $m_1' = E(|Z_n|)$. Now for the case of m_2', we take the sum of the squares of the values in the row at label 3 as

$$|6b_1 - a|^2 + (6b_1 + a)^2 + |6b_2 - a|^2 + (6b_2 + a)^2 = 4a^2 + 72(b_1^2 + b_2^2), \tag{2.5}$$

which can be clearly seen as the sum of four times the square of first row plus seventy two times the sum of squares of second row. So (2.5) can be expressed in the form of a recurrence relation as

$$V[n] = 72V[n-1] + 4V[n-2],$$

from which we get the quadratic equation

$$y^2 - 72y - 4 = 0,$$

which provides the solution in the form of a growth factor for m_2' as $y = 36 + 10\sqrt{13}$. Now for m_1', as already calculated, in the previous case and we found that as $lim_{n\to\infty}(\mu_1')^2 = 0$. Hence we obtain the desired result. □

Note: From the conditions succeeding (2.2) and from the subsequent results, it has been very much clear that both the successors of a will satisfy one set of conditions, for which the square of growth factor A^2 must be less than $q/2$, which implies that $A < \frac{\sqrt{q}}{\sqrt{2}}$. So, if we let $q = 1$, then $A < \frac{1}{\sqrt{2}}$.

3. Additional remark

For $A = 7, 8, 9, ...$ and $q = 2, 3, 4, ...$ and so on, we follow the similar procedure to obtain the results and hence are omitted here.

4. Graph

In this section the graphical representation have been depicted regarding possible growth and decay of the random balancing sequence. The third and sixth cases succeeding (2.2) have been graphically depicted taking $q = 1$ in the following figure.

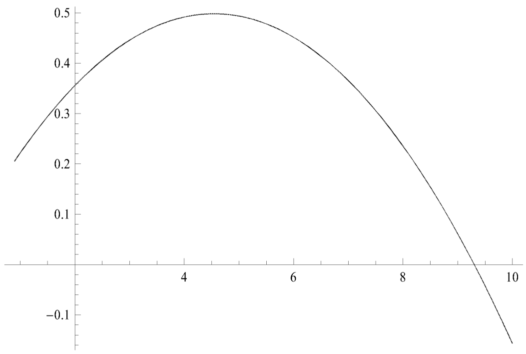


Figure 1. Growth rate for the Random balancing sequence

Conclusion

In this paper, we have studied a random sequence $\{Z_n\}$ defined by the binary recurrence $Z_{n+1} = A\alpha Z_n - qZ_{n-1}$, where $\alpha = \pm 1$, to evaluate the bounds of the expected value for the absolute value of the n-th term. Exact bounds can be calculated in the case of a random balancing sequence as an example where A is replaced by 6 and q is replaced by 1. Following the similar technique, other cases are also obtained.

Acknowledgement

This work is dedicated to Prof. Chandrashekar Adiga on his 62nd Birthday. Furthermore, we are thankful to the learned ananymous reviewers for their suggestions.

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