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LOCALLY BOUNDED NOT A PRIORI CONTINUOUS HOMOMORPHISMS OF NONCOMPACT SEMISIMPLE LIE GROUPS INTO CONNECTED AMENABLE LIE GROUPS ARE IDENTITY HOMOMORPHISMS

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Abstract. We prove that every not a priori continuous locally bounded homomorphism of a noncompact connected simple Lie group into a connected amenable Lie group is the identity homomorphism.

§ 1. Introduction

Let G be a connected noncompact simple Lie group, let H be a connected amenable Lie group, and let $\pi\colon G \to H$ be a (not a priori continuous) locally bounded homomorphism. The main result of the present note is that $\pi(G) = \{e_H\}$, where e_H stands for the identity element of the group H.

§ 2. Preliminaries

Several theorems, having applications to not necessarily continuous homomorphisms of connected Lie groups and even of some connected locally compact groups, were proved recently (see [1–3]). The corresponding automatic continuity results are especially efficient for perfect connected Lie groups and, in particular, for connected semisimple Lie groups. One of important results obtained using these automatic continuity theorems was given in [4].

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It claims that a connected Lie group admitting a not necessarily continuous embedding in a connected amenable Lie group is automatically amenable. In this note we use this result to prove that, if G is a connected noncompact simple Lie group, H is a connected amenable Lie group, and $\pi\colon G \to H$ is a (not a priori continuous) homomorphism, then $\pi(G) = \{e_H\}$, where e_H stands for the identity element of the group H, i.e., the homomorphism π is trivial.

§ 3. Main theorem

Theorem. *Let G be a connected noncompact simple Lie group, let H be a connected amenable Lie group, and let $\pi\colon G \to H$ be a (not a priori continuous) locally bounded homomorphism. Then $\pi(G) = \{e_H\}$, where e_H stands for the identity element of the group H.*

Proof. Let $\mathfrak{h}$ be the Lie algebra of the group H. and let ad_H be the adjoint representation of H on $\mathfrak{h}$. The kernel of ad_H is the center of H, and hence the image of H in the general linear group $\mathrm{GL}(\mathfrak{h})$ is isomorphic to the quotient group of H by the center Z_H of H. Let us consider the composite $\mathrm{ad}_H \circ \pi$ of the locally bounded homomorphism π (which is not a priori continuous) and the continuous representation ad_H. This composite is obviously locally bounded. Since every locally bounded finite-dimensional representation of a connected semisimple Lie group is automatically continuous [3], it follows that the image of G in the quotient Lie group H/Z_H is an analytic subgroup of H/Z_H. Since H/Z_H is an amenable Lie group (together with H, Proposition 1.13 of [5]), it cannot contain a nontrivial analytic subgroup isomorphic to a noncompact simple Lie group ([4] and Theorem 3.8 of [5]). This means that the composite $\mathrm{ad}_H \circ \pi$ is trivial (the image of this composite is the identity element of H/Z_H), and therefore the image of π belongs to the center Z_H of H.

Every homomorphism of the connected simple Lie group G into a commutative group defines corresponding homomorphisms of the triple systems in G into the same commutative group. Since the commutator subgroup of the triangular group of such a system is the corresponding unipotent group, it follows that this unipotent group is taken to the identity element of the commutative group. The well-known formula expressing the diagonal elements in the triangular group using a fixed element of the triple system and the unipotent elements shows that the image of the diagonal subgroup is a fixed element, and hence the identity element of the image. Thus, the image of every triple system is the identity element of H. Since these triple systems generate the entire group G, it follows that the image of G is the singleton formed by the identity element of H, as was to be proved.

§ 4. Discussion

The result of the above theorem admits a generalization. Let G be a connected Lie group, let H be a connected amenable Lie group, let $\pi\colon G \to H$ be a homomorphism which is not a priori continuous. Let R be the radical of G, let L be a Levi subgroup of G, and let L have no compact connected normal subgroups. Then $\pi(L) = \{e_h\}$ by the above theorem, and therefore $\pi(N) = \{e\}$ for the normal subgroup of G generated by L. In particular, for groups of this kind, we have $\pi([L, R]) = \{e\}$. Indeed, since

$$\pi(l) = e_H \quad \text{for every} \quad l \in L,$$

it follows that

$$\begin{aligned}\pi(l^{-1}r^{-1}lr) &= \pi(l^{-1})\pi(r^{-1})\pi(l)\pi(r) \\ &= \pi(r^{-1})\pi(r) = \pi(e_G) = e_H \quad \text{for every} \quad l \in L \quad \text{and} \quad r \in R,\end{aligned}$$

where e_G stands for the identity element of G. For the same reason, the following assertion holds.

Corollary. *Let G be a perfect connected Lie group with commutative radical and let a Levi subgroup of G have no compact connected normal subgroups. Let H be a connected amenable Lie group and let π be a (not a priori continuous) locally bounded homomorphism of G into H. Then $\pi(G) = e_H$.*

In particular, for the Poincaré group P, every (not a priori continuous) locally bounded homomorphism of P into an amenable group is trivial.

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