Proceedings of the Jangjeon Mathematical Society
23 (2020), No. 2, pp. 283 - 291

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2020.23.2.283

INDUCED SIGNED GRAPHS OF SOME CLASSES OF GRAPHS

ACHU ANIYAN AND SUDEV NADUVATH

Abstract. A signed graph is a graph with positive or negative signs assigned to edges. An induced signed graph is a signed graph constructed from a given graph according to some pre-defined protocols. An induced signed graph of a graph G is a signed graph in which each edge uv receives a sign $(-1)^{|\phi(v)-\phi(u)|}$, where $\phi : V(G) \to \mathbb{Z}$. In this paper, we discuss two types of induced signed graphs and determine the structural properties of these signed graphs such as balancing, clustering, regularity and co-regularity.

2010 Mathematics Subject Classification. 05C22, 05C07.

Keywords and phrases. Signed graphs, induced signed graphs, degree-induced signed graph, eccentricity induced signed graph.

1. Introduction

For terms and definitions in graph theory, we refer to [4, 11] and for the terminology of signed graphs, see [5, 9]. Unless mentioned otherwise, all graphs we consider in this paper are simple, finite, connected and undirected.

A *graph* G is an ordered triple $G = (V, E, \psi)$, where $V = \{v_1, v_2, ..., v_n\}$ is a finite non-empty set of objects called *vertices* of G , E is a collection of two-element subsets of V called its *edges* and ψ is a mapping defined by $\psi : E \to V \times V$ that associates each edge with two vertices (need not necessarily be distinct). The *order* of graph is $|V| = p$ and the *size* of the graph is $|E| = q$. The *degree* of a vertex v_i in G is defined as the number of edges incident on that vertex.

A *signed graph*, denoted by $S(G, \sigma)$, is a graph $G(V, E)$ together with a function $\sigma : E(G) \to \{+, -\}$ (or equivalently, $\sigma : E(G) \to \{+1, -1\}$) that assigns a sign, either $+$ or $-$, to each ordinary edge in graph G (see [5]). The function σ is called the *signature* or *sign function* of S, which is defined on all edges except half edges (the edges having only one end vertex). The signs of free loops are treated to be positive always. The unsigned graph G of a signed graph S, is called the *underlying graph* of S. The notion of signed graphs has been introduced in [5] to solve some socio-psychological problems.

If all the edges of a signed graph S are positive, then the signed graph is called *all-positive* signed graph. If the underlying graph is denoted as G, then the corresponding all-positive signed graph is denoted by $+G$. An all positive complete graph is denoted by $+K_n$ (*cf.* [6]). Similarly, $-G$ denotes an all-negative signed graph and $-K_n$ denotes an all-negative complete graph.

A signed graph is said to be *homogeneous* if it is either all-positive or all-negative and is said to be *heterogeneous* otherwise. The sign of a sub-graph

S' of a signed graph S is defined to be the product of the signs of its edges. If all the cycles in a signed graph are positive, then the graph is said to be *balanced* (*cf.* [4]).

Switching (*cf.* [8]) a vertex in a signed graph means negating the signs of all the edges incident to that vertex. Switching a set of vertices means negating all the edges that have one end in that set and one end in the complementary set. By switching of a signed graph, we mean negating the signs of all the edges in the graph.

Two signed graphs are said to be *isomorphic* if there exist a one to one correspondence between the point sets which preserves adjacency and signs (*cf.* [4]). The number of positive edges incident on a vertex v of a signed graph in S is called the *positive degree* of v and is denoted by $d_s^+(v)$ and the number of negative edges incident on v is called *negative degree* of v and is denoted by $d_s^-(v)$. It is clear that $d_G(v) = d_v^+(v) + d_v^-(v)$.

The *net-degree* of a vertex v of a signed graph S, denoted by $d_S^{\pm}(v)$, is defined as $d_S^{\pm}(v) = d_S^+(v) - d_S^-(v)$ (*cf.* [6]). The signed graph S is said to be *net-regular* if every vertex has same net degree. A signed graph $S = (G, \sigma)$ is said to be *co-regular* if the underlying graph G is r-regular for some integer r and S is net-regular with net-degree k for some integer k (*cf.* [6]). Here, the co-regularity pair can be defined to be ordered pair (r, k).

Theorem 1.1. *Harary's Balance Theorem*The following statements about a signed graph are equivalent.

(i) $\sum$ is balanced.
(ii) $\sum$ has no half edges and the vertex set V can be partitioned into two sets V_1 and V_2 such that $V = V_1 \cup V_2$ and $E^- = E(V_1, V_2)$.
(iii) $\sum$ has no half edges and any of the paths with same endpoints have the same sign.

The idea of an induced signed graph was introduced in [1], where the authors discussed the notion of the set labeling of a signed digraph. In a set-labeled signed digraph, each edge gets the sign based on the cardinality of the symmetric difference of the set-labels of its end vertices. Analogous to this, we introduce the notion of an induced signed graph as follows:

Definition 1.2. An *induced signed graph* of an ordinary undirected graph G is a signed graph obtained from G by assigning a sign to all its edges according to the signature function defined by $\sigma(uv) = (-1)^{|\phi(u)-\phi(v)|}$, where ϕ is a mapping defined by $\phi : V(G) \to \mathbb{Z}$, the set of weights (or labels), associated with each vertex in S.

In view of Definition 1.2, many special induced signed graphs can be created and studied further for many interesting results and characterisations. In this paper, we discuss some signed graphs induced from given undirected unsigned graphs in terms some parameters associated with its vertices.

2. Degree-induced Signed Graphs

In this section, we introduce and study a signed graph degree-induced signed graph, which is induced from a graph in terms of the vertex degrees of the graph. The notion of degree-induced signed graphs is defined as given below:

Definition 2.1. A *degree induced signed graph* or *d-induced signed graph* or of a graph G, denoted by $S_d(G)$ (or simply S_d), is a signed graph obtained from a graph G, by assigning signs to its edges according to the signature function $\sigma_d(uv) = (-1)^{|d(u)-d(v)|}$, where $d(u)$ is the degree of the vertex u or the number of edges incident on u.

The degree induced signed graph of a given graph G is depicted in Figure 1.

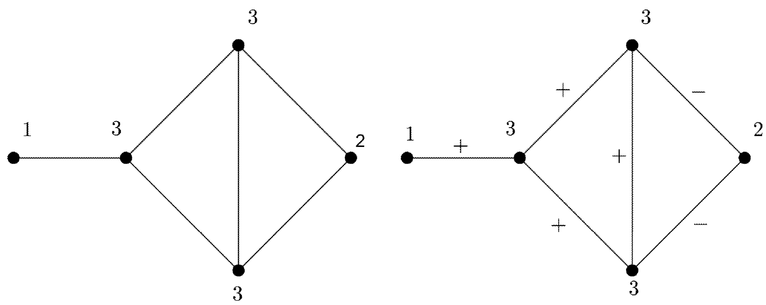

FIGURE 1. A graph G and corresponding d-induced signed graph

Definition 2.2. Two vertices of a d-induced signed graph S_d are said to have *same parity* if they are simultaneously of odd degree or simultaneously of even degree. Similarly, two vertices are said to be of *opposite parity* if one of them is of even degree and other is of odd degree.

A d-induced signed graph is called *positive regular* if every vertex of the graph has same positive degree and *negative regular* if every vertex of graph has same negative degree.

Invoking the above definitions, we now have some immediate observations as given below.

Proposition 2.3. *The d-induced signed graph of a complete bipartite graph $K_{m,n}$ is an all-positive graph if m and n are of same parity and is an all-negative graphs if m and n are of opposite parity.*

Proof. If m and n are of same parity, then all edges in $K_{m,n}$ are positive and otherwise, all edges are negative. Hence the result. □

Proposition 2.4. *The d-induced signed graphs of Eulerian graphs are always positive regular.*

Proof. The proof is immediate from the fact that the vertices of an Eulerian graph are always of even degree. □

Proposition 2.5. *The d-induced signed graphs of all regular graphs are positive regular.*

In view of Proposition 2.5, we note that the converse of 2.4 need not be true always since odd regular graphs are always positive regular.

The following theorem establishes one of the most significant characteristics of the d-induced signed graph of a graph.

Theorem 2.6. *The d-induced signed graph S_d of a graph G is always balanced.*

Proof. Consider a d-induced signed graph S_d of a graph G. Suppose that the graph contains at least one cycle. Let C be an arbitrary cycle in G (and hence in S_d). Note that all vertices in C have even degree in C.

If all vertices in the cycle C are of the same parity in G (that is, all vertices in C are simultaneously even or simultaneously odd), then all edges in C are positive edges and hence C is a positive cycle. Then the proof is complete. Hence, assume that not all vertices of G are of the same parity in G. Here, the following cases are to be addressed.

Case 1: Suppose that all vertices of the cycle C except one, say v, of the graph G is of same parity. Then two edges incident on v are negative edges and all other edges are positive. Therefore, cycle remains positive. In a similar way another two edges becomes negative when another vertex, not adjacent to v, is of different parity from its neighbours in C.

Case 2: If a vertex u, which is adjacent to v is also of odd parity, then the edge uv becomes positive and the other edges incident on each u and v (the two edges on the cycle having either u or v, but not both, as the end points) become negative. In this case also, number of negative edges is even.

In view of the fact that all other possible situations can be reduced to the above two cases, the result follows. □

Note: Addition or removal of pendant edges or edges adjacent to the cycle will not affect the balancing condition of a d-induced signed graph.

Theorem 2.7. *The switched graph of a d-induced signed graph S_d is balanced if and only if it is bipartite.*

Proof. Let S_d be the induced signed graph of a given graph G and let S'_d be the switched signed graph of S_d. Note that G is the underlying graph of both S_d and S'_d.

First, assume that S'_d is balanced. Hence, every cycle in S'_d is positive (has even number of negative edges).

By Theorem 2.6, the signed graph S_d is also balanced and hence every cycle in S_d also has even number of negative edges. Thus, every cycles in S_d and S'_d has even number of negative edges and even number of positive edges. That is, every cycle in S_d, S'_d (and in G) has even length. Hence G is bipartite.

Conversely, assume that G is bipartite. So every cycle in G (and hence every cycle in S_d and S'_d) is even. Since S_d is balanced (by Theorem 2.6) every cycle in S_d has even number of negative edges and even number of positive edges.

Every cycle in S'_d also has even number of positive and negative edges proving that S'_d is balanced. □

Note: Tree is a graph with zero cycle (that is even number of cycle) and hence bipartite. Hence, it is obvious that tree is balanced.

A subgraph H of a signed graph S is *sign-invariant* if its sign remains unchanged after switching.

Then, in view of the Theorem 2.6, the following result is immediate.

Corollary 2.8. *If a cycle C in a d-induced signed graph is sign invariant if and only if the d-induced signed graph S_d and switched signed graph S'_d are balanced.*

Recall the notion of the net-degree of a signed graph defined in [6] as follows:

Definition 2.9. The *net-degree* of a vertex v in the d-induced signed graph S_d, of a graph G, denoted by $d^{\pm}_{S_d}(v)$, is defined as $d^{\pm}_{S_d}(v) = d^{+}_{S_d}(v) - d^{-}_{S_d}(v)$. The d-induced signed graph S_d is said to be *net-regular* if every vertex of it has same net degree.

The d-induced signed graph S_d of an r-regular graph is always net regular with net regularity r. The d-induced signed graph S_d of a regular graph (regularity r) is always co-regular with co-regularity pair (r, r).

For cycles, net degree, net regularity and co-regularity is $+2$. Complete graph K_n has net degree, net regularity and co-regularity is $n-1$.

Definition 2.10. [10] A signed graph S is said to be 2-*clusterable* if its vertex set can be partitioned into two subsets such that the lines between points of same sets has positive sign and the lines between points of different sets has negative sign.

Note that a d-induced signed graph of a graph G is 2-clusterable implies that it is balanced. But its converse need not be true as all positive signed graphs are balanced but not 2-clusterable.

Theorem 2.11. *The d-induced signed graph of a complete bipartite graph $K_{m,n}$ is 2-clusterable if and only if m and n are of opposite parity.*

The vertex set of d-induced signed graph of a complete bipartite graph can be partitioned into two subsets such that vertices within the same partition are joined by positive edges and negative edges have end vertices across the partition.

The following result immediate from the facts that all vertices of an Eulerian graph are of even parity.

Theorem 2.12. *The non-homogeneous d-induced signed graph is 2-clusterable if graph is non-Eulerian.*

A necessary and sufficient condition for the d-induced signed graph of a wheel graph to be 2-clusterable is discussed in the following result.

Proposition 2.13. *The d-induced signed graph of a wheel graph $W_{1,n}$ is 2-clusterable if and only if n is even, with rim vertices in one cluster and central vertex in other cluster.*

Proof. Suppose that d-induced signed graph of $W_{1,n}$ is 2-clusterable. Then, there should be at least one negative edge in it. Since all rim vertices are of degree 3, the above condition is possible only when the degree of the central vertex is even. That is, n is even.

Conversely, assume that n is even. Then, the degree of central vertex is even. For the wheel graph $W_{1,n}$, the rim vertices has degree 3, which is odd. Hence, the central vertex comes in one partition and rim vertices in other partition. Therefore, it is 2-clusterable. □

Two signed graphs are said to be *isomorphic*, if there is an ismorphism between their underlying graphs that preserves the signs of edges (see [10]).

Since signed graph isomorphism preserves adjacency and signs, the following result is immediate.

Proposition 2.14. *The d-induced signed graphs of isomorphic graphs are also isomorphic.*

Proof. The proof is immediate from the fact that isomorphic graphs preserve the order, size and the adjacency. □

3. Eccentricity-Induced Signed Graphs

In this section, we discuss another type of induced signed graph of a graph, called the eccentricity-induced signed graph which is defined as follows:

Definition 3.1. An *eccentricity-induced signed graph* or *ϵ-induced signed graph* of an ordinary graph G, denoted by $S_\epsilon(G)$ or simply S_ϵ, is a signed graph obtained from G by assigning signs to its edges according to the signature function $\sigma_\epsilon(uv) = (-1)^{|\epsilon(u)-\epsilon(v)|}$, where $\epsilon(u)$ is the eccentricity of the vertex u.

Figure 2 illustrates the eccentricity induced signed graph of a graph.

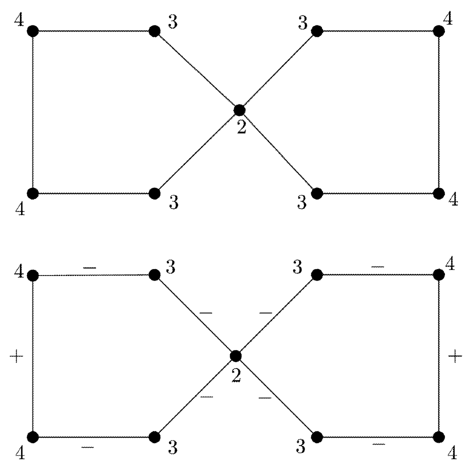

Figure 2. A graph G and ϵ-induced signed graph of G

In this context, we say that two vertices are of *same parity* if their eccentricities are simultaneously even or simultaneously odd, otherwise they are said to be of *opposite parity.*

A *self-centric graph G* is a graph whose centre is the graph G itself. In view of Definition 3.1 and the above mentioned concepts the following results are immediate.

Proposition 3.2. *The ϵ-induced signed graph of a complete bipartite graph $K_{m,n}$ is positive homogeneous.*

Proposition 3.3. *The ϵ-induced signed graphs of self-centric graphs are all positive homogeneous signed graphs.*

Every complete graph is a self-centric graph and hence in view of Proposition 3.3, the following theorem is immediate.

Proposition 3.4. *The ϵ-induced signed graphs of complete graphs are non-clusterable.*

Proposition 3.5. *The ϵ-induced signed graphs of wheel graphs $W_{1,n}$, $n \geq 4$ are 2-clusterable with rim vertex in one cluster and central vertex in other cluster.*

Proposition 3.6. *The ϵ-induced signed graphs of isomorphic graphs are also isomorphic.*

Proof. The proof follows from the fact that corresponding vertices in the isomorphic graphs have the same eccentricity. □

The following theorem establishes one of the most interesting characteristic of the d-induced signed graph of a graph.

Theorem 3.7. *The ϵ-induced signed graph S_ϵ of a graph G is always balanced.*

Proof. The result is an immediate consequence of the fact that any two adjacent vertices in a cycle with opposite parity will make two edges (incident on one of them) negative. The proof is exactly as that of Theorem 2.6. □

The following table gives the eccentricity sequence of some fundamental classes of graphs such as complete graph K_p, complete bipartite graphs $K_{m,n}$, cycle C_n of order n and the ϵ-induced signed graphs for the same are given in Figure 3, Figure 5 and Figure 4 respectively.

Graphs	**Conditions**	**Eccentricity Sequences**
K_n	$n \geq 2$	$1, 1, 1, ..., 1$
$K_{1,n}$	$n \geq 2$	$1, 2, 2, ..., 2$
$K_{m,n}$	$2 \leq m \leq n$	$2, 2, 2, ..., 2$
C_n	n is even	$\frac{n}{2}, \frac{n}{2}, \frac{n}{2}, ..., \frac{n}{2}$
	n is odd	$\frac{n-1}{2}, \frac{n-1}{2}, \frac{n-1}{2}, ..., \frac{n-1}{2}$

Table 1. Eccentricity sequences of some classes of graphs

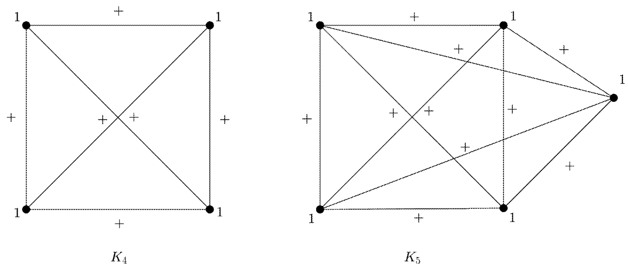

FIGURE 3. ϵ-induced signed graphs of complete graphs K_4 and $K+5$

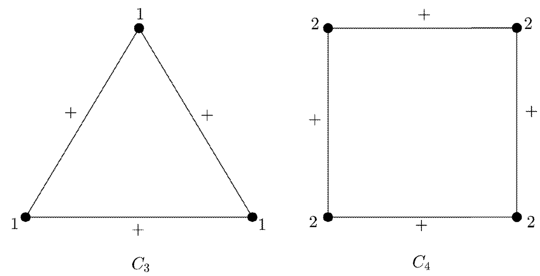

FIGURE 4. ϵ-induced signed graphs of cycles C_3 and C_4

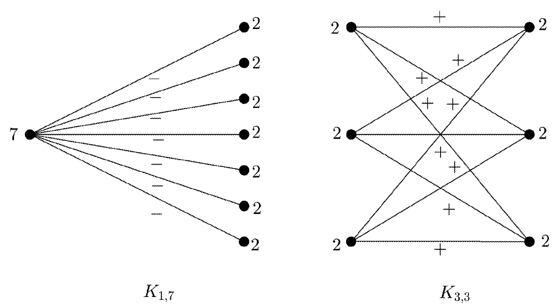

FIGURE 5. ϵ-induced signed graphs of bipartite graphs $K_{1,7}$ and $K_{3,3}$

Theorem 3.8. *The switched signed graph of an ϵ-induced signed graph S_ϵ of a graph G is balanced if and only if it is bipartite.*

Proof. The proof is similar to the proof of Theorem 2.7. □

Hence, the following result is straight forward.

Corollary 3.9. *A cycle C in an ϵ-induced signed graph S is sign invariant if and only if the ϵ-induced signed graph S_ϵ and its switched signed graph S'_ϵ are balanced.*

Theorem 3.10. *The non-homogeneous ϵ-induced signed graph S_ϵ and d-induced signed graph S_d of a graph G are 2-clusterable.*

Proof. Consider an ϵ-induced signed graph S_ϵ and d-induced signed graph S_d of a graph G. From Theorem 2.6 and Theorem 3.7, it is clear that both d- induced and ϵ-induced signed graphs are balanced. Since the graph is non-homogeneous and balanced it contains at least two negative edges. By Theorem 1.1, every balanced graphs is 2-clusterable. Hence both ϵ-induced and d-induced signed graphs are 2-clusterable. □

4. Conclusion

In this paper, we have introduced two types of signed graphs - degree induced and eccentricity induced - induced from given graph classes and studied their characteristics such as balance, clusterability etc. The study can be extended to many other graph classes, graph products and graph powers. The signed graphs corresponding to many other parameters, such as distances, can also be constructed and studied in details. The signed graphs can be defined from ordinary colored graphs, and thus can study a few interesting results. All these points show a wide scope for the research.

References

[1] B. D. Acharya, *Set-valuations of signed digraphs*, J. Combin. Inform. System Sci, 37 (2012), 145-167.
[2] J. Akiyama, D. Avis, V. Chvátal and H. Era, *Balancing signed graphs*, Discrete Appl. Math. 3 (1981), 227-233.
[3] L. W. Beineke and Frank Harary, *Consistency in marked graphs*, J. Math. Psychol. 18 (1978), 260-269.
[4] F. Harary, *Graph theory*, Narosa Publ. New Delhi (2001)
[5] Frank Harary, *On the notion of balanced signed graphs*, Michigan Math. J. 2 (1953), 143-146.
[6] S. Hameed, V. Paul, and K. A. Germina, *On co-regular signed graphs*, Australas. J. Combin. 62 (1984), 8-16.
[7] E. Sampathkumar, *Point signed and line signed graphs*, Nat. Acad. Sci. Letters, 7 (1984), 91-93.
[8] N. K. Sudev, K. P. Chithra, and K. A. Germina, *Switched signed graphs of integer additive set-valued signed graphs*, Discrete Math. Algorithm. Appl. 9 (2017), 1750043.
[9] Thomas Zaslavsky, *Signed graphs*, Discrete Appl. Math. 4 (1982), 47-74.
[10] Thomas Zaslavsky, *Signed graphs and geometry*, J. Combin. Inform. System Sci., 37 (2012), 2-4.
[11] D. B. West, *Introduction to graph theory*, Prentice Hall of India, New Delhi (2001)

Department of Mathematics, CHRIST (Deemed to be University), Bengaluru, India.
Email address: achu.aniyan@res.christuniversity.in

Department of Mathematics, CHRIST (Deemed to be University), Bengaluru, India.
Email address: sudev.nk@christuniversity.in