Proceedings of the Jangjeon Mathematical Society
23 (2020), No. 2, pp. 271 – 282

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2020.23.2.271

INVESTIGATION ON SPLICE GRAPHS BY EXPLOTING CERTAIN TOPOLOGICAL INDICES

V. LOKESHA, M. MANJUNATH, AND K. ZEBA YASMEEN

Abstract. Let G_1 and G_2 be simple connected graphs with disjoint vertex sets $V(G_1)$ and $V(G_2)$ respectively. For given vertices $a_1 \in V(G_1)$ and $a_2 \in V(G_2)$ a splice of G_1 and G_2 by vertices a_1 and a_2 defined by identifying the vertices a_1 and a_2 in the union of G_1 and G_2. In this article the explicit interpretation of ISI, EM_1, ABC and SK_1 index in terms of the graph size and maximum or minimum vertex degrees of special splice graphs are obtained.

2010 Mathematics Subject Classification: 05C90, 05C76, 05C07;

Keywords and phrases: Topological indices; Graph operations; Splice graph.

Dedicated to Professor Chandrashekar Adiga on his 62^{nd} Birthday.

1. Introduction

A topological index is a numerical quantity associated with the chemical constitution of a chemical compound aiming the correlation of chemical structure with many of its physico-chemical properties, chemical reactivity or biological activities. Topological indices are designed on the ground of transformation of a molecular graph into a number which characterize the topology of that graph [14].

A graph $G(V, E)$ with vertex set V and edge set E is said to be connected, if there exist a connection between any pair of vertices in G. The degree $d_G(u)$ of a vertex u is the number of edges that are incident to it. Δ_G and δ_G represents the maximum and minimum degrees respectively, the notations n_G and e_G denote the number of vertices and edges of G respectively. $d_G(S(u))$ is the degree of selected vertex and $Mr(G \bullet H)$ is the merged vertex in $G \bullet H$.

Estrada et al., [5] proposed a topological index named the atom-bond connectivity index. It is defined as

$$ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_G(u) + d_G(v) - 2}{d_G(u) d_G(v)}}.$$

In [13], Miličević, Nikolić and Trinjastić (2004) reformulated the Zagreb indices as

$$EM_1(G) = \sum_{uv \in E[G]} d_G(e)^2.$$

where $d(e)$ denotes the degree of the edge e in G, which is defined by $d_G(e) = d_G(u) + d_G(v) - 2$ with $e = uv$.

Vukičević and Gašperov (2010) [18, 19] introduced bond-additive topological index namely, inverse sum indeg index. It as a significant predictor of total surface area of octane isomers and is defined as

$$ISI(G) = \sum_{uv \in E(G)} \left[\frac{d_G(u)d_G(v)}{d_G(u) + d_G(v)} \right].$$

Recently, Shegehalli and Kanabur [15] introduced SK_1 index as follows:

$$SK_1(G) = \sum_{uv \in E(G)} \frac{d_G(u) + d_G(v)}{2}.$$

Chemically interesting graphs are obtained by means of different graph operations which can be thought as graph extensions on some general or particular graphs [1, 3, 6, 7, 17]. The reason for studying these operations is to understand how the graph operation can relate the values of the corresponding topological indices of the given graphs. The values of the topological indices of the larger graph obtained as a result of these operations or sometimes to the help us to comment on chemical properties of the resulting graph. Actually this idea motivated from [8].

Splice Graph: [2, 4, 16]Let G and H be two simple connected graphs with disjoint vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$, respectively. Let $b_1 \in V(G)$ and $y_1 \in V(H)$. Then the *splice graph* $G \bullet H$ of G and H by vertices b_1 and y_1, respectively, is defined by identifying the vertices b_1 and y_1 in the union of G and H. It is known that, for splice graphs, the total number of vertices is $n_G + n_H - 1$ while the total number of edges is $e_G + e_H$ (FIGURE 1).

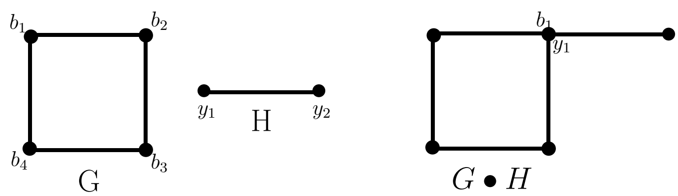

FIGURE 1. Splice of G and H by the vertices b_1 and y_1

The forth coming section includes four subsections viz., subdivision-vertex splice graph, subdivision-edge splice graph, subdivision-vertex neighbourhood splice graph and subdivision-edge neighbourhood splice graph respectively. Also, we recalled the related definitions and reckoned the bounds for the ISI, EM_1, ABC and SK_1 indices.

2. MAIN RESULTS

2.1. **Subdivision-vertex splice graph.** [8] Let G and H be two vertex disjoint graphs and let $b_1 \in V(G)$ and $y_1 \in V(H)$. The subdivision vertex splice G and H is denoted by $G \bullet_v H$ and obtained from $S(G)$ and one copy of H which is identifying the vertices b_1 and y_1 in the union of $S(G)$ and H (FIGURE 2).

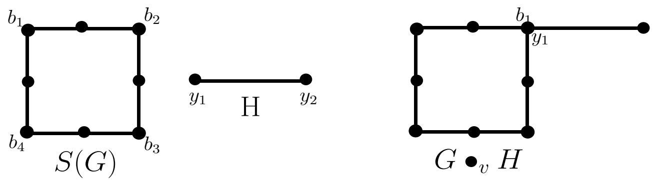

FIGURE 2. S-vertex splice

Theorem 2.1. *Let G and H are two simple connected graphs. Then the bounds for the inverse sum indeg index of $G \bullet_v H$ are given by*

$$ISI[G \bullet_v H] \leq \frac{2\Delta_G[2m_1 - \Delta_G]}{\Delta_G + 2} + \frac{\Delta_H[m_2 - \Delta_H]}{2} + \frac{2\Delta_G(\Delta_G + \Delta_H)}{\Delta_G + \Delta_H + 2} + \frac{\Delta_H^2(\Delta_G + \Delta_H)}{\Delta_G + 2\Delta_H}.$$

$$ISI[G \bullet_v H] \geq \frac{2\delta_G[2m_1 - \delta_G]}{\delta_G + 2} + \frac{\delta_H[m_2 - \delta_H]}{2} + \frac{2\delta_G(\delta_G + \delta_H)}{\delta_G + \delta_H + 2} + \frac{\delta_H^2(\delta_G + \delta_H)}{\delta_G + 2\Delta_H}.$$

Proof. Consider,

$$\begin{aligned}
ISI[G \bullet_v H] = & \sum_{\substack{uv \in E[G \bullet_v H] \\ u \in V[G], v \in I[G]}} \left[\frac{d_G(u).2}{d_G(u) + 2}\right] + \sum_{\substack{uv \in E[G \bullet_v H] \\ u,v \in V[H]}} \left[\frac{d_H(u).d_H(v)}{d_H(u) + d_H(v)}\right] \\
& + \sum_{\substack{uv \in E[G \bullet_v H] \\ u \in M[G \bullet_v H], v \in I[G]}} \left[\frac{(d_G(u) + d_H(v)).2}{d_G(u) + d_H(v) + 2}\right] + \sum_{\substack{uv \in E[G \bullet_v H] \\ u \in M[G \bullet_v H], v \in V[H]}} \left[\frac{(d_G(u) + d_H(v)).d_H(w)}{d_G(u) + d_H(v) + d_H(w)}\right] \\
& = [2m_1 - d_G(S(u))]\left[\frac{2d_G(u)}{d_G(u) + 2}\right] + [m_2 - d_H(S(u))]\left[\frac{d_H(u).d_H(v)}{d_H(u) + d_H(v)}\right] \\
& + d_G(S(u))\left[\frac{2(d_G(u) + d_H(v))}{d_G(u) + d_H(v) + 2}\right] + d_H(S(u))\left[\frac{(d_G(u) + d_H(v)).d_H(w)}{d_G(u) + d_H(v) + d_H(w)}\right] \\
& \leq [2m_1 - \Delta_G]\left[\frac{2\Delta_G}{\Delta_G + 2}\right] + [m_2 - \Delta_H]\left[\frac{\Delta_H^2}{2\Delta_H}\right] + \Delta_G\left[\frac{2(\Delta_G + \Delta_H)}{\Delta_G + \Delta_H + 2}\right] \\
& + \Delta_H\left[\frac{(\Delta_G + \Delta_H).\Delta_H}{\Delta_G + \Delta_H + \Delta_H}\right]
\end{aligned}$$

$$ISI[G \bullet_v H] \leq \frac{2\Delta_G[2m_1 - \Delta_G]}{\Delta_G + 2} + \frac{\Delta_H[m_2 - \Delta_H]}{2} + \frac{2\Delta_G(\Delta_G + \Delta_H)}{\Delta_G + \Delta_H + 2} + \frac{\Delta_H^2(\Delta_G + \Delta_H)}{\Delta_G + 2\Delta_H}.$$

One can analogously compute the following,

$$ISI[G \bullet_v H] \geq \frac{2\delta_G[2m_1 - \delta_G]}{\delta_G + 2} + \frac{\delta_H[m_2 - \delta_H]}{2} + \frac{2\delta_G(\delta_G + \delta_H)}{\delta_G + \delta_H + 2} + \frac{\delta_H^2(\delta_G + \delta_H)}{\delta_G + 2\Delta_H}.$$

□

Theorem 2.2. *Let G and H are two simple connected graphs. Then the bounds for the EM_1 index of $G \bullet_v H$ are given by*

$$EM_1[G \bullet_v H] \leq \Delta_G^2[2m_1 - \Delta_G] + 4[m_2 - \Delta_H][\Delta_H - 1]^2 + \Delta_G[\Delta_G + \Delta_H]^2 + \Delta_H[\Delta_G + 2\Delta_H - 2]^2.$$

$$EM_1[G \bullet_v H] \geq \delta_G^2[2m_1 - \delta_G] + 4[m_2 - \delta_H][\delta_H - 1]^2 + \delta_G[\delta_G + \delta_H]^2 + \delta_H[\delta_G + 2\delta_H - 2]^2.$$

Proof. Consider,

$$EM_1[G \bullet_v H] = \sum_{\substack{uv\in E[G\bullet_v H]\\ u\in V[G], v\in I[G]}} [d_G(u) + 2 - 2]^2 + \sum_{\substack{uv\in E[G\bullet_v H]\\ u,v\in V[H]}} [d_H(u) + d_H(v) - 2]^2$$
$$+ \sum_{\substack{uv\in E[G\bullet_v H]\\ u\in Mr[G\bullet_v H], v\in I[G]}} [d_G(u) + d_H(v)) + 2 - 2]^2 + \sum_{\substack{uv\in E[G\bullet_v H]\\ u\in Mr[G\bullet_v H], v\in V[H]}} [d_G(u) + d_H(v) + d_H(w) - 2]^2$$
$$= [2m_1 - d_G(S(u))][d_G(u)]^2 + [m_2 - d_H(S(u))][d_H(u) + d_H(v) - 2]^2$$
$$+ d_G(S(u))[d_G(u) + d_H(v)]^2 + d_H(S(u))[d_G(u) + d_H(v) + d_H(w) - 2]^2$$
$$\leq \Delta_G^2[2m_1 - \Delta_G] + [m_2 - \Delta_H][\Delta_H + \Delta_H - 2]^2 + \Delta_G[\Delta_G + \Delta_H]^2$$
$$+ \Delta_H[\Delta_G + \Delta_H + \Delta_H - 2]^2$$
$$EM_1[G \bullet_v H] \leq \Delta_G^2[2m_1 - \Delta_G] + 4[m_2 - \Delta_H][\Delta_H - 1]^2 + \Delta_G[\Delta_G + \Delta_H]^2 + \Delta_H[\Delta_G + 2\Delta_H - 2]^2.$$

One can analogously compute the following,

$$EM_1[G \bullet_v H] \geq \delta_G^2[2m_1 - \delta_G] + 4[m_2 - \delta_H][\delta_H - 1]^2 + \delta_G[\delta_G + \delta_H]^2 + \delta_H[\delta_G + 2\delta_H - 2]^2.$$

□

Theorem 2.3. *Let G and H are two simple connected graphs. Then the bounds for the Atom-bond connectivite index and SK_1 index of $G \bullet_v H$ are given by*

$$ABC[G \bullet_v H] \leq \frac{[2m_1 - \Delta_G]}{\sqrt{2}} + [m_2 - \Delta_H]\sqrt{\frac{2(\Delta_H - 1)}{\Delta_H^2}} + \frac{\Delta_G}{\sqrt{2}} + \Delta_H\sqrt{\frac{\Delta_G + 2\Delta_H - 2}{(\Delta_G + \Delta_H).\Delta_H}}$$

$$ABC[G \bullet_v H] \geq \frac{[2m_1 - \delta_G]}{\sqrt{2}} + [m_2 - \delta_H]\sqrt{\frac{2(\delta_H - 1)}{\delta_H^2}} + \frac{\delta_G}{\sqrt{2}} + \delta_H\sqrt{\frac{\delta_G + 2\delta_H - 2}{(\delta_G + \delta_H).\delta_H}}$$

$$SK_1[G \bullet_v H] \leq [2m_1 - \Delta_G]\left[\frac{\Delta_G + 2}{2}\right] + [m_2 - \Delta_H]\Delta_H + \Delta_G\left[\frac{\Delta_G + \Delta_H + 2}{2}\right] + \Delta_H\left[\frac{\Delta_G + 2\Delta_H}{2}\right].$$
$$SK_1[G \bullet_v H] \geq [2m_1 - \delta_G]\left[\frac{\delta_G + 2}{2}\right] + [m_2 - \delta_H]\delta_H + \delta_G\left[\frac{\delta_G + \delta_H + 2}{2}\right] + \delta_H\left[\frac{\delta_G + 2\delta_H}{2}\right].$$

Proof. The proof technique is identical to the proof of Theorem 2.2. □

2.2. **Subdivision-edge splice graph:** [8] Let $p_2 \in I(G)$ be the inserted vertex of $S(G)$ and $y_1 \in V(H)$. Then the S-edge splice of G and H is denoted by $G \bullet_e H$ that is obtained from $S(G)$ and one copy of H identifying the vertices p_2 and y_1 in the union of $S(G)$ and H (FIGURE 3).

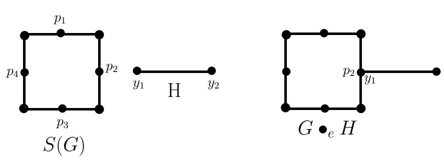

FIGURE 3. S-edge splice graph

Theorem 2.4. *Let G and H are two simple connected graphs. Then the bounds for the inverse sum indeg index of $G \bullet_e H$ are given by*

$$ISI[G \bullet_e H] \leq \frac{4\Delta_G[m_1-1]}{\Delta_G+2} + \frac{\Delta_H[m_2-\Delta_H]}{2} + 2\left[\frac{\Delta_G(\Delta_H+2)}{\Delta_G+\Delta_H+2}\right] + \left[\frac{\Delta_H^2(\Delta_H+2)}{2(\Delta_H+1)}\right].$$
$$ISI[G \bullet_e H] \geq \frac{4\delta_G[m_1-1]}{\delta_G+2} + \frac{\delta_H[m_2-\delta_H]}{2} + 2\left[\frac{\delta_G(\delta_H+2)}{\delta_G+\delta_H+2}\right] + \left[\frac{\delta_H^2(\delta_H+2)}{2(\delta_H+1)}\right].$$

Proof. Consider,

$$\begin{aligned}
ISI[G \bullet_e H] &= \sum_{\substack{uv\in E[G\bullet_e H]\\ u\in V[G],v\in I[G]}} \left[\frac{d_G(u).2}{d_G(u)+2}\right] + \sum_{\substack{uv\in E[G\bullet_e H]\\ u,v\in V[H]}} \left[\frac{d_H(u).d_H(v)}{d_H(u)+d_H(v)}\right] \\
&+ \sum_{\substack{uv\in E[G\bullet_e H]\\ u\in Mr[G\bullet_e H],v\in V[G]}} \left[\frac{(d_H(v)+2).d_G(u)}{(d_H(v)+2)+d_G(u)}\right] + \sum_{\substack{uv\in E[G\bullet_e H]\\ u\in Mr[G\bullet_e H],v\in V[H]}} \left[\frac{(d_H(u)+2).d_H(v)}{(d_H(v)+2)+d_H(v)}\right] \\
&= [2m_1-2]\left[\frac{d_G(u).2}{d_G(u)+2}\right] + [m_2-d_H(S(u))]\left[\frac{d_H(u).d_H(v)}{d_H(u)+d_H(v)}\right] + 2\left[\frac{(d_H(v)+2).d_G(u)}{(d_H(v)+2)+d_G(u)}\right] \\
&+ d_H(S(u))\left[\frac{(d_H(u)+2).d_H(v)}{(d_H(v)+2)+d_H(v)}\right] \\
&\leq [2m_1-2]\left[\frac{2\Delta_G}{\Delta_G+2}\right] + [m_2-\Delta_H]\left[\frac{\Delta_H.\Delta_H}{\Delta_H+\Delta_H}\right] + 2\left[\frac{(\Delta_H+2).\Delta_G}{\Delta_H+\Delta_G+2}\right] + \Delta_H\left[\frac{\Delta_H(\Delta_H+2)}{\Delta_H+\Delta_H+2}\right] \\
&\leq 2[m_1-1]\left[\frac{2\Delta_G}{\Delta_G+2}\right] + [m_2-\Delta_H]\left[\frac{\Delta_H^2}{2\Delta_H}\right] + 2\left[\frac{(\Delta_H+2).\Delta_G}{\Delta_H+\Delta_G+2}\right] + \Delta_H\left[\frac{\Delta_H(\Delta_H+2)}{2\Delta_H+2}\right] \\
ISI[G \bullet_e H] &\leq \frac{4\Delta_G[m_1-1]}{\Delta_G+2} + \frac{\Delta_H[m_2-\Delta_H]}{2} + 2\left[\frac{\Delta_G(\Delta_H+2)}{\Delta_G+\Delta_H+2}\right] + \left[\frac{\Delta_H^2(\Delta_H+2)}{2(\Delta_H+1)}\right].
\end{aligned}$$

One can analogously compute the following,

$$ISI[G \bullet_e H] \geq \frac{4\delta_G[m_1-1]}{\delta_G+2} + \frac{\delta_H[m_2-\delta_H]}{2} + 2\left[\frac{\delta_G(\delta_H+2)}{\delta_G+\delta_H+2}\right] + \left[\frac{\delta_H^2(\delta_H+2)}{2(\delta_H+1)}\right].$$

□

Theorem 2.5. *Let G and H are two simple connected graphs. Then the bounds for the EM_1 index of $G \bullet_e H$ are given by*

$$EM_1[G \bullet_e H] \leq 2\Delta_G^2[m_1-1] + 4[m_2-\Delta_H][\Delta_H-1]^2 + 2[\Delta_G+\Delta_H]^2 + 4\Delta_H^3.$$
and
$$EM_1[G \bullet_e H] \geq 2\delta_G^2[m_1-1] + 4[m_2-\delta_H][\delta_H-1]^2 + 2[\delta_G+\delta_H]^2 + 4\delta_H^3.$$

Proof. Consider,

$$
\begin{aligned}
EM_1[G\bullet_e H] =& \sum_{\substack{uv\in E[G\bullet_e H]\\ u\in V[G], v\in I[G]}} [d_G(u)+2-2]^2 + \sum_{\substack{uv\in E[G\bullet_e H]\\ u,v\in V[H]}} [d_H(u)+d_H(v)-2]^2\\
&+ \sum_{\substack{uv\in E[G\bullet_e H]\\ u\in Mr[G\bullet_e H], v\in V[G]}} [(d_H(v)+2)+d_G(v)-2]^2 + \sum_{\substack{uv\in E[G\bullet_e H]\\ u\in Mr[G\bullet_e H], v\in V[H]}} [(d_H(u)+2)+d_H(v)-2]^2\\
&= [2m_1-2][d_G(u)]^2 + [m_2 - d_H(S(u))][d_H(u)+d_H(v)-2]^2 + 2[d_H(u)+2+d_G(v)-2]^2\\
&+ d_H(S(u))[d_H(u)+2+d_H(v)-2]^2\\
&\leq 2\Delta_G^2[m_1-1] + [m_2-\Delta_H][\Delta_H+\Delta_H-2]^2 + 2[\Delta_G+\Delta_H]^2 + \Delta_H[\Delta_H+\Delta_H]^2\\
&\leq 2\Delta_G^2[m_1-1] + [m_2-\Delta_H][2\Delta_H-2]^2 + 2[\Delta_G+\Delta_H]^2 + \Delta_H[2\Delta_H]^2\\
EM_1[G\bullet_e H] &\leq 2\Delta_G^2[m_1-1] + 4[m_2-\Delta_H][\Delta_H-1]^2 + 2[\Delta_G+\Delta_H]^2 + 4\Delta_H^3.
\end{aligned}
$$

One can analogously compute the following,

$$EM_1[G\bullet_e H] \geq 2\delta_G^2[m_1-1] + 4[m_2-\delta_H][\delta_H-1]^2 + 2[\delta_G+\delta_H]^2 + 4\delta_H^3.$$

□

Theorem 2.6. *Let G and H are two simple connected graphs. Then the bounds for the Atom-bond connectivite index and SK_1 index of $G\bullet_e H$ are given by*

$$ABC[G\bullet_e H] \leq \sqrt{2}[m_1-1] + [m_2-\Delta_H]\frac{\sqrt{2(\Delta_H-1)}}{\Delta_H} + 2\sqrt{\frac{\Delta_H+\Delta_G}{\Delta_G.(\Delta_H+2)}} + \Delta_H\sqrt{\frac{2}{\Delta_H+2}}.$$

$$ABC[G\bullet_e H] \geq \sqrt{2}[m_1-1] + [m_2-\delta_H]\frac{\sqrt{2(\delta_H-1)}}{\delta_H} + 2\sqrt{\frac{\delta_H+\delta_G}{\delta_G.(\delta_H+2)}} + \delta_H\sqrt{\frac{2}{\delta_H+2}}.$$

$$SK_1[G\bullet_e H] \leq [m_1-1][\Delta_G+2] + \Delta_H[m_2-\Delta_H] + [\Delta_G+\Delta_H+2] + \Delta_H[\Delta_H+1].$$

$$SK_1[G\bullet_e H] \geq [m_1-1][\delta_G+2] + \delta_H[m_2-\delta_H] + [\delta_G+\delta_H+2] + \delta_H[\delta_H+1].$$

Proof. The proof technique is identical to the proof of Theorem 2.5. □

2.3. **Subdivision-vertex neighbourhood splice Graph:** Let $b_1 \in V(G)$ and $y_1 \in V(H)$. The S-vertex neighbourhood splice of G and H is denoted by $G\bullet_{nv}H$ and obtained from $S(G)$ and $d(b_1)$ copies of H and identifying the neighbourhood vertices of b_1. For $y_1 \in V(H)$, the union of the corresponding neighbourhood separated vertices $b_1 \in V(G)$ of $S(G)$ (FIGURE 4).

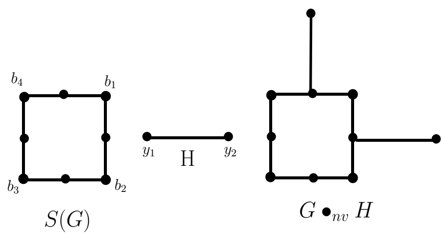

FIGURE 4. S- vertex neighbourhood splice

Theorem 2.7. *Let G and H are two simple connected graphs. Then the bounds for the Inverse sum indeg index of $G \bullet_{nv} H$ are given by*

$$ISI[G \bullet_{nv} H] \leq 2[m_1 - \Delta_G]\left[\frac{2\Delta_G}{\Delta_G + 2}\right] + \frac{\Delta_G\Delta_H}{2}[m_2 - \Delta_H] + 2\Delta_G\left[\frac{\Delta_G(2+\Delta_H)}{\Delta_G + \Delta_H + 2}\right] + \Delta_G\Delta_H\left[\frac{\Delta_H(2+\Delta_H)}{2(1+\Delta_H)}\right].$$

and

$$ISI[G \bullet_{nv} H] \geq 2[m_1 - \delta_G]\left[\frac{2\delta_G}{\delta_G + 2}\right] + \frac{\delta_G\delta_H}{2}[m_2 - \delta_H] + 2\delta_G\left[\frac{\delta_G(2+\delta_H)}{\delta_G + \delta_H + 2}\right] + \delta_G\delta_H\left[\frac{\delta_H(2+\delta_H)}{2(1+\delta_H)}\right].$$

Proof. Consider,

$$\begin{aligned} ISI[G \bullet_{nv} H] &= \sum_{\substack{uv\in E[G\bullet_{nv}H]\\ u\in V[G], v\in I[G]}} \left[\frac{d_G(u).2}{d_G(u)+2}\right] + \sum_{\substack{uv\in E[G\bullet_{nv}H]\\ u,v\in V[H]}} \left[\frac{d_H(u).d_H(v)}{d_H(u)+d_H(v)}\right] \\ &+ \sum_{\substack{uv\in E[G\bullet_{nv}H]\\ u\in Mr[G\bullet_{nv}H], v\in V[G]}} \left[\frac{d_G(u).(2+d_H(v))}{d_G(u)(2+d_H(v))}\right] + \sum_{\substack{uv\in E[G\bullet_{nv}H]\\ u\in Mr[G\bullet_{nv}H], v\in V[H]}} \left[\frac{(2+d_H(u)).d_H(v)}{(2+d_H(u))+d_H(v)}\right] \\ &= 2[m_1 - d_G(S(u))]\left[\frac{2d_G(u)}{d_G(u)+2}\right] + d_G(S(u))[m_2 - d_H(S(u))]\left[\frac{d_H(u).d_H(v)}{d_H(u)+d_H(v)}\right] \\ &+ 2d_G(S(u))\left[\frac{d_G(u)(2+d_H(v))}{d_G(u)+d_H(v)+2}\right] + d_G(S(u))d_H(S(u))\left[\frac{d_H(v)(2+d_H(u))}{2+d_H(u)+d_H(v)}\right] \\ &= 2[m_1 - d_G(S(u))]\left[\frac{2d_G(u)}{d_G(u)+2}\right] + d_G(S(u))[m_2 - d_H(S(u))]\left[\frac{d_H(u)^2}{2d_H(u)}\right] \\ &+ 2d_G(S(u))\left[\frac{d_G(u)(2+d_H(v))}{d_G(u)+d_H(v)+2}\right] + d_G(S(u))d_H(S(u))\left[\frac{d_H(v)(2+d_H(u))}{2(1+d_H(u))}\right] \\ ISI[G \bullet_{nv} H] &\leq 2[m_1 - \Delta_G]\left[\frac{2\Delta_G}{\Delta_G + 2}\right] + \frac{\Delta_G\Delta_H}{2}[m_2 - \Delta_H] + 2\Delta_G\left[\frac{\Delta_G(2+\Delta_H)}{\Delta_G + \Delta_H + 2}\right] \\ &+ \Delta_G\Delta_H\left[\frac{\Delta_H(2+\Delta_H)}{2(1+\Delta_H)}\right]. \end{aligned}$$

One can analogously compute the following,

$$ISI[G \bullet_{nv} H] \geq 2[m_1 - \delta_G]\left[\frac{2\delta_G}{\delta_G + 2}\right] + \frac{\delta_G\delta_H}{2}[m_2 - \delta_H] + 2\delta_G\left[\frac{\delta_G(2+\delta_H)}{\delta_G + \delta_H + 2}\right] + \delta_G\delta_H\left[\frac{\delta_H(2+\delta_H)}{2(1+\delta_H)}\right].$$

□

Theorem 2.8. *Let G and H are two simple connected graphs. Then the bounds for the EM_1 index of $G \bullet_{nv} H$ are given by*

$$EM_1[G \bullet_{nv} H] \leq 2\Delta_G^2[m_1 - \Delta_G] + 4\Delta_G[m_2 - \Delta_H][\Delta_H - 1]^2 + 2\Delta_G[\Delta_G + \Delta_H]^2 + 4\Delta_G\Delta_H^3.$$

and

$$EM_1[G \bullet_{nv} H] \geq 2\delta_G^2[m_1 - \delta_G] + 4\delta_G[m_2 - \delta_H][\delta_H - 1]^2 + 2\delta_G[\delta_G + \delta_H]^2 + 4\delta_G\delta_H^3.$$

Proof. Consider,

$$
\begin{aligned}
EM_1[G \bullet_{nv} H] &= \sum_{\substack{uv \in E[G\bullet_{nv}H] \\ u \in V[G], v\in I[G]}} [d_G(u)+2-2]^2 + \sum_{\substack{uv \in E[G\bullet_{nv}H] \\ u,v \in V[H]}} [d_H(u)+d_H(v)-2]^2 \\
&+ \sum_{\substack{uv \in E[G\bullet_{nv}H] \\ u \in Mr[G\bullet_{nv}H], v\in V[G]}} [d_G(u)+d_H(v))+2-2]^2 + \sum_{\substack{uv \in E[G\bullet_{nv}H] \\ u \in Mr[G\bullet_{nv}H], v\in V[H]}} [2+d_H(u)+d_H(v)-2]^2 \\
&= 2[m_1 - d_G(S(u))][d_G(u)]^2 + d_G(S(u))[m_2 - d_H(S(u))][d_H(u)+d_H(v)-2]^2 \\
&+ 2d_G(S(u))[d_G(u)+d_H(v)]^2 + d_G(S(u))d_H(S(u))[d_H(u)+d_H(v)]^2. \\
&\leq 2\Delta_G^2[m_1-\Delta_G] + \Delta_G[m_2-\Delta_H][\Delta_H+\Delta_H-2]^2 + 2\Delta_G[\Delta_G+\Delta_H]^2 + \Delta_G\Delta_H[\Delta_H+\Delta_H]^2.
\end{aligned}
$$

$$EM_1[G \bullet_v H] \leq 2\Delta_G^2[m_1-\Delta_G] + 4\Delta_G[m_2-\Delta_H][\Delta_H-1]^2 + 2\Delta_G[\Delta_G+\Delta_H]^2 + 4\Delta_G\Delta_H^3.$$

One can analogously compute the following,

$$EM_1[G \bullet_v H] \geq 2\delta_G^2[m_1-\delta_G] + 4\delta_G[m_2-\delta_H][\delta_H-1]^2 + 2\delta_G[\delta_G+\delta_H]^2 + 4\delta_G\delta_H^3.$$

□

Theorem 2.9. *Let G and H are two simple connected graphs. Then the bounds for the Atom-bond connectivite index and SK_1 index of $G \bullet_{nv} H$ are given by*

$$
\begin{aligned}
ABC[G \bullet_{nv} H] &\leq \sqrt{2}[m_1-\Delta_G] + \frac{\Delta_G}{\Delta_H}[m_2-\Delta_H]\sqrt{2(\Delta_H-1)} + 2\Delta_G\sqrt{\frac{\Delta_G+\Delta_H}{\Delta_G(2+\Delta_H)}} \\
&+ \Delta_G\Delta_H\sqrt{\frac{2}{2+\Delta_H}}.
\end{aligned}
$$

$$ABC[G \bullet_{nv} H] \geq \sqrt{2}[m_1-\delta_G] + \frac{\delta_G}{\delta_H}[m_2-\delta_H]\sqrt{2(\delta_H-1)} + 2\delta_G\sqrt{\frac{\delta_G+\delta_H}{\delta_G(2+\delta_H)}} + \delta_G\delta_H\sqrt{\frac{2}{2+\delta_H}}.$$

$$SK_1[G \bullet_{nv} H] \leq [m_1-\Delta_G][\Delta_G+2] + \Delta_G\Delta_H[m_2-\Delta_H] + \Delta_G[\Delta_G+\Delta_H+2] + \Delta_G\Delta_H[\Delta_H+1].$$
$$SK_1[G \bullet_{nv} H] \geq [m_1-\delta_G][\delta_G+2] + \delta_G\delta_H[m_2-\delta_H] + \delta_G[\delta_G+\delta_H+2] + \delta_G\delta_H[\delta_H+1].$$

Proof. The proof technique is identical to the proof of Theorem 2.8. □

2.4. **Subdivision-edge neighbourhood splice graph:** [8] Let $p_1 \in I(G)$ be the inserted vertex of $S(G)$ and $y_1 \in V(H)$. Then the S-edge neighbourhood splice of G and H is denoted by $G \bullet_{ne} H$ that is obtained from $S(G)$ and two copies of H identifying the vertices p_1. For $y_1 \in V(H)$, the union of the corresponding neighbourhood separated vertices p_1 of $S(G)$ (FIGURE 5).

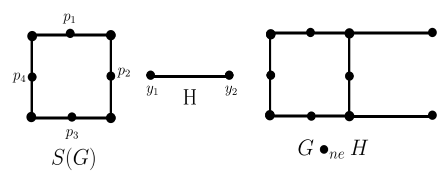

FIGURE 5. S-edge neighbourhood splice

Theorem 2.10. *Let G and H are two simple connected graphs. Then the bounds for the Inverse sum indeg index of $G \bullet_{ne} H$ are given by*

$$ISI[G \bullet_{ne} H] \leq \frac{4\Delta_G[m_1 - 2(\Delta_G - 1)]}{\Delta_G + 2} + \Delta_H[m_2 - \Delta_H] + \frac{8(\Delta_G + \Delta_H)(\Delta_G - 1)}{\Delta_G + \Delta_H + 2} + \frac{2\Delta_H^2(\Delta_G + \Delta_H)}{\Delta_G + 2\Delta_H}.$$

and

$$ISI[G \bullet_{ne} H] \geq \frac{4\delta_G[m_1 - 2(\delta_G - 1)]}{\delta_G + 2} + \delta_H[m_2 - \delta_H] + \frac{8(\delta_G + \delta_H)(\delta_G - 1)}{\delta_G + \delta_H + 2} + \frac{2\delta_H^2(\delta_G + \delta_H)}{\delta_G + 2\delta_H}.$$

Proof. Consider,

$$\begin{aligned}
ISI[G \bullet_{ne} H] &= \sum_{\substack{uv \in E[G \bullet_{ne} H] \\ u \in V[G], v \in I[G]}} \left[\frac{d_G(u).2}{d_G(u) + 2}\right] + \sum_{\substack{uv \in E[G \bullet_{ne} H] \\ u,v \in V[H]}} \left[\frac{d_H(u).d_H(v)}{d_H(u) + d_H(v)}\right] \\
&+ \sum_{\substack{uv \in E[G \bullet_{ne} H] \\ u \in Mr[G \bullet_{ne} H], v \in I[G]}} \left[\frac{(d_G(u) + d_H(v)).2}{(d_G(u) + d_H(v)) + 2}\right] + \sum_{\substack{uv \in E[G \bullet_{ne} H] \\ u \in Mr[G \bullet_{ne} H], v \in V[H]}} \left[\frac{(d_G(u) + d_H(w)).d_H(v)}{(d_G(u) + d_H(w)) + d_H(v)}\right] \\
&= 2[m_1 - d_G(S(e))]\left[\frac{2d_G(u)}{d_G(u) + 2}\right] + 2[m_2 - d_H(S(u))]\left[\frac{d_H(u).d_H(v)}{d_H(u) + d_H(v)}\right] \\
&+ 2d_G(S(e))\left[\frac{2(d_G(u) + d_H(v))}{d_G(u) + d_H(v) + 2}\right] + 2d_H(S(u))\left[\frac{(d_G(u) + d_H(w)).d_H(v)}{d_G(u) + d_H(w) + d_H(v)}\right]. \\
ISI[G \bullet_{ne} H] &\leq \frac{4\Delta_G[m_1 - 2(\Delta_G - 1)]}{\Delta_G + 2} + \Delta_H[m_2 - \Delta_H] + \frac{8(\Delta_G + \Delta_H)(\Delta_G - 1)}{\Delta_G + \Delta_H + 2} + \frac{2\Delta_H^2(\Delta_G + \Delta_H)}{\Delta_G + 2\Delta_H}.
\end{aligned}$$

One can analogously compute the following,

$$ISI[G \bullet_{ne} H] \geq \frac{4\delta_G[m_1 - 2(\delta_G - 1)]}{\delta_G + 2} + \delta_H[m_2 - \delta_H] + \frac{8(\delta_G + \delta_H)(\delta_G - 1)}{\delta_G + \delta_H + 2} + \frac{2\delta_H^2(\delta_G + \delta_H)}{\delta_G + 2\delta_H}.$$

□

Theorem 2.11. *Let G and H are two simple connected graphs. Then the bounds for the EM_1 index of $G \bullet_{ne} H$ are given by*

$$\begin{aligned}
EM_1[G \bullet_{ne} H] &\leq 2\Delta_G^2[m_1 - 2(\Delta_G - 1)] + 8[m_2 - \Delta_H][\Delta_H - 1]^2 + 4[\Delta_G - 1][\Delta_G + \Delta_H]^2 \\
&+ 2\Delta_H[\Delta_G + 2(\Delta_H - 1)]^2.
\end{aligned}$$

and

$$\begin{aligned}
EM_1[G \bullet_{ne} H] &\geq 2\delta_G^2[m_1 - 2(\delta_G - 1)] + 8[m_2 - \delta_H][\delta_H - 1]^2 + 4[\delta_G - 1][\delta_G + \delta_H]^2 \\
&+ 2\delta_H[\delta_G + 2(\delta_H - 1)]^2.
\end{aligned}$$

Proof. Consider,

$$EM_1[G \bullet_{ne} H] = \sum_{\substack{uv\in E[G\bullet_{ne}H] \\ u\in V[G], v\in I[G]}} [d_G(u) + 2 - 2]^2 + \sum_{\substack{uv\in E[G\bullet_{ne}H] \\ u,v\in V[H]}} [d_H(u) + d_H(v) - 2]^2$$
$$+ \sum_{\substack{uv\in E[G\bullet_{ne}H] \\ u\in Mr[G\bullet_{ne}H], v\in I[G]}} [(d_G(u) + d_H(v) + 2 - 2]^2$$
$$+ \sum_{\substack{uv\in E[G\bullet_{ne}H] \\ u\in Mr[G\bullet_{ne}H], v\in V[H]}} [(d_G(u) + d_H(w)) + d_H(v) - 2]^2.$$
$$= 2[m_1 - d_G(S(e))][d_G(u)]^2 + 2[m_2 - d_H(S(u))][d_H(u) + d_H(v) - 2]^2$$
$$+ 2d_G(S(e))[d_G(u) + d_H(v)]^2 + 2d_H(S(u))[d_G(u) + d_H(w) + d_H(v) - 2]^2$$
$$EM_1[G \bullet_{ne} H] \leq 2\Delta_G^2[m_1 - 2(\Delta_G - 1)] + 8[m_2 - \Delta_H][\Delta_H - 1]^2 + 4[\Delta_G - 1][\Delta_G + \Delta_H]^2$$
$$+ 2\Delta_H[\Delta_G + 2(\Delta_H - 1)]^2.$$

One can analogously compute the following,

$$EM_1[G \bullet_{ne} H] \geq 2\delta_G^2[m_1 - 2(\delta_G - 1)] + 8[m_2 - \delta_H][\delta_H - 1]^2 + 4[\delta_G - 1][\delta_G + \delta_H]^2$$
$$+ 2\delta_H[\delta_G + 2(\delta_H - 1)]^2.$$

□

Theorem 2.12. *Let G and H are two simple connected graphs. Then the bounds for the Atom-bond connectivite index and SK_1 index of $G \bullet_{ne} H$ are given by*

$$ABC[G \bullet_{ne} H] \leq \sqrt{2}[m_1 - 2[\Delta_G - 1]] + 2\sqrt{2}[m_2 - \Delta_H]\left[\frac{\sqrt{\Delta_H - 1}}{\Delta_H}\right] + 2\sqrt{2}(\Delta_G - 1)$$
$$+ 2\sqrt{\Delta_H}\sqrt{\frac{\Delta_G + 2\Delta_H}{\Delta_G + \Delta_H}}.$$
$$ABC[G \bullet_{ne} H] \geq \sqrt{2}[m_1 - 2[\delta_G - 1]] + 2\sqrt{2}[m_2 - \delta_H]\left[\frac{\sqrt{\delta_H - 1}}{\delta_H}\right] + 2\sqrt{2}(\delta_G - 1)$$
$$+ 2\sqrt{\delta_H}\sqrt{\frac{\delta_G + 2\delta_H}{\delta_G + \delta_H}}.$$
$$SK_1[G \bullet_{ne} H] \leq [m_1 - 1][\Delta_G + 2] + 2\Delta_H[m_2 - \Delta_H] + [\Delta_G + \Delta_H + 2] + \Delta_H[\Delta_G + 2\Delta_H].$$
$$SK_1[G \bullet_{ne} H] \geq [m_1 - 1][\delta_G + 2] + 2\delta_H[m_2 - \delta_H] + [\delta_G + \delta_H + 2] + \delta_H[\delta_G + 2\delta_H].$$

Proof. The proof technique is identical to the proof of Theorem 2.11. □

3. Conclusion

The analysis of graphs and networks plays a significant role to deduce their underlying topologies. As such, it has been extensively used also in biomedicine, cheminformatics and in bioinformatics, where approximations based on graph indices and descriptors have been made available for effectively communicating with the several activities. In this article, we have presented the lower and upper bounds for the inverse sum indeg index, reformulated Zagreb index, atom bond connectivity index and SK_1 index in terms of the graph size and maximum or minimum vertex degrees of special splice graphs are obtained.

4. Acknowledgement

Second author is thankful to National Fellowship and Scholarship for Higher Education 2017-18-NFST-KAR-0083 and also the third author is thankful to University Grant Commission UGC, New Delhi for providing Maulana Azad National Fellowship F1-17.1/2017-18/MANF-2017-18-KAR-77292.

References

[1] A. R. Ashrafi, T. Došlić, A. Hamzeh, *The Zagreb coindices of graph operations*, Discrete Applied Mathematics 158, (2010), 1571-1578.

[2] M. Azari, H. Divanpour, *Splices,links, and their edge-degree distences*, Transactions on Combinatorics, 6 (4), (2017), 29-42.

[3] A. R. Bindusree, I. N. Cangul, V. Lokesha, A. S. Cevik, *Zagreb polynomial of three graph operators*, Filomat 30 (7), (2016), 19791986.

[4] T. Doslic, *Splices, links and their degree-weighted Wiener polynomials*, **Graph Theory Notes** N. Y., 48, 2005 4755.

[5] E. Estrada, L. Torres, L. Rodriguez, and I. Gutman, *An atom-bond connectivity index: modelling the enthalpy of formation of alkanes*, Indian J. Chem. 37A(10), (1998), 849-855.

[6] C. K. Gupta, V. Lokesha, S. B. Shetty, P. S. Ranjini, *Graph Operations on Symmetric Division Deg Index of Graphs*, Palestine Journal of Mathematics, vol. 6 (1), (2017), 280-286.

[7] V. Lokesha, S. Shetty, P. S. Ranjini, I. N. Cangul, A. S. Cevik, *New bounds for Randic and GA indices*, Journal of Inequalities and Applications, 180, (2013), 1-7.

[8] V. Lokesha, S. Jain, M. Muddalapuram, A. S. Cevik, I. N. Cangul *Bounds for the sum of cubes of vertex degrees of splice graphs*TWMS Journal of Applied and Engineering Mathematics, (Accepted), Preprint.

[9] V. Lokesha and K. Zeba Yasmeen, *SK Indices, forgotten topological indices and hyper Zagreb index of Q operator of carbon nanocone*, TWMS Journal of Applied and Engineering Mathematics, 9(3), (2019), 675−680.

[10] V. Lokesha, K. Zeba Yasmeen and T. Deepika, *Some computational aspects of carbon nanocone using $Q(G)$ operator, hexagonal network and probabilistic neural network*, Creative mathematics and informatics, 28(1), (2019), 69−76.

[11] V. Lokesha, K. Zeba Yasmeen, B. Chaluvaraju and T. Deepika, *Computation of misbalance type degree indices of certain classes of derived-regular graph* , International Journal of Mathematical Combinatorics, 2(1), (2019), 61−69.

[12] V. Lokesha, K. Zeba Yasmeen and T. Deepika, *Edge version of SDD and ISI index for rooted product graphs*, Journal of Discrete Mathematical sciences and cryptography, 22(6), (2019), 1077-1090,

[13] A. Miličević, S. Nikolić and N. Trinjastić, *On reformulated Zagreb indices.* Molecules Divers, 8, (2004), 393-399.

[14] Sakander Hayat, Mehar Ali Malik and Muhammad Imran, *Computing topological indices of honeycomb derived networks*, Romanian J. of Information Science and Technology, 18(2), (2015), 144-165.

[15] V. S. Shegehalli and R. Kanabur, *Arithmetic-Geometric indices of path graph*, J. Comp. and Mathematical Sciences, 6(1), (2015), 19-24.

[16] R. Sharafdini, I. Gutman, *Splice graphs and their topological indices*, Kragujevac J. Sci., 35 , (2013), 89-98.

[17] G. H. Shirdel, H. RezaPour, A. M. Sayadi, *The hyper Zagreb index of graph operations*, Iranian Journal of Mathematical Chemistry, 4 (2), (2013), 213-220.

[18] D. Vukicevic and M. Gasperov, *Bond Additive Modeling 1. Adriatic Indices*, Croatica chemical acta, 83(3), (2010), 243-260.

[19] D. Vukicevic, *Bond Additive Modeling 2. Mathematical Properties of Max-min Rodeg Index*, Croatica chemical acta, 83(3), (2010), 261-273.

(V. Lokesha, M. Manjunath and K. Zeba Yasmeen) Department of Studies in Mathematics, Vijayanagara Sri krishnadevaraya University, Ballari, Karnataka, INDIA.
E-mail address: v.lokesha@gmail.com, manju3479@gmail.com and zebasif44@gmail.com