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PENDANT DOMINATION IN DOUBLE GRAPHS

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ABSTRACT. Let G be any graph. A subset S of vertices in G is called a dominating set if every vertex in $V-S$ is adjacent to atleast one vertex in S. A dominating set S is called a pendant dominating set if the induced subgraph of S contains at least one pendant vertex. The minimum cardinality of a pendant dominating set is called the pendant domination number denoted by γ_{pe}. In this paper, we are considering special types of graphs called double graphs obtained through a graph operation. We study the pendant domination for these graphs. We calculate the exact value of pendant domination number in double graphs of some standard class of graphs. Further, the bounds are estimated for these parameter in terms of order and degree of a graph.

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1. INTRODUCTION

Let G be a connected graph. A subset S of vertices is called a dominating set of G if every vertex not in S is adjacent to at least one vertex in S. The minimum cardinality of a dominating set is called the domination number, denoted by $\gamma(G)$. The concept of pendant domination in graphs is defined and studied in [3]. A dominating set S in G is called a pendant dominating set if induced subgraph of S contains at least one pendant vertex. The minimum cardinality of a pendant dominating set is called the pendant domination number, denoted by $\gamma_{pe}(G)$. In [3], authors have obtained fundamental results related to pendant domination parameter including exact values for standard graphs and bounds in terms of order and domination number.

Let G and H be any two graphs. The direct product of G and H is a graph denoted by $G \times H$ with the vertex set $V(G) \times V(H)$ such that two vertices (v_1, w_1) and (v_2, w_2) are adjacent in $G \times H$ if and only if v_1 and v_2 are adjacent in G and w_1 and w_2 are adjacent in H. The total graph T_n of order n is the graph associated to the total relation. In fact, T_n can be obtained from the complete graph K_n by adding a loop to every vertex. Given a simple graph G, the double graph of G is a simple graph denoted by $\mathfrak{D}(G)$ and is defined by $\mathfrak{D}(G) = G \times T_2$. In the double graph $\mathfrak{D}(G)$, two vertices (v_1, w_1) and (v_2, w_2) are adjacent if and only if v_1 and v_2 are adjacent in G.

From the definition of a double graph [2], it follows that if G is a graph of order n and size m then the double graph is a graph of order $2n$ and size $4m$. The cycle graph C_4 and its double graph $\mathfrak{D}(C_4)$ are shown in the Figure 1. The double graph $\mathfrak{D}(G)$ always decomposes into two subgraphs G_0 and G_1

such that $G_0 \cap G_1 = \emptyset$ and $G_0 \cup G_1$ is a spanning subgraph of $\mathfrak{D}(G)$. Then $\{G_0, G_1\}$ is called the decomposition of $\mathfrak{D}(G)$. The double graph operation is defined for any graph G, throughout this paper, by a graph G, we mean a graph without loops and multiple edges.

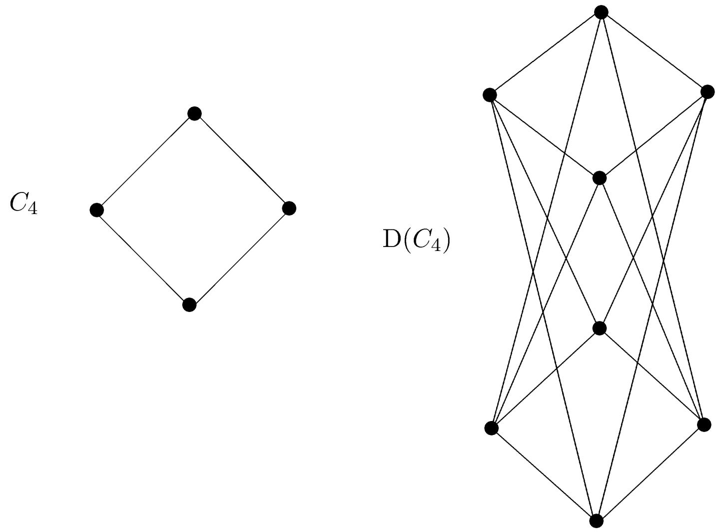


Fig.1. A cycle and its double

Theorem 1.1. *Let G be a path with n vertices. Then*

$$\gamma_{pe}(\mathfrak{D}(G)) = \begin{cases} 3, & \text{if } n=5; \\ 2\lfloor \frac{n}{3} \rfloor, & \text{otherwise.} \end{cases}$$

Proof. Let G be a path of order n. Then double graph of G is a $\{2,4\}-$regular graph of order $2n$. If $n = 5$, then any minimum pendant dominating set of a copy of G will be a minimum pendant dominating set of $\mathfrak{D}(G)$. Therfore, $\gamma_{pe}(\mathfrak{D}(G)) = 3$.

Now, we consider the following three possible cases here. First, suppose if $n = 3k$ for some positive integer k. Then the graph $\mathfrak{D}(G)$ consist of two copies of P_n, choose a minimum pendant dominating set S' of one copy, which dominates $\mathfrak{D}(G)$ except the vertices corresponding to the vertices of S'. So that the set S is obtained by taking the vertices not dominated by S' together with S', will be a pendant dominating set of $\mathfrak{D}(G)$. Further if v is any vertex in S, then the set $S - \{v\}$ will not be a pendant dominating set in G and so in $\mathfrak{D}(G)$. Therefore $\gamma_{pe}(\mathfrak{D}(G)) = 2\lfloor \frac{n}{3} \rfloor$.

Next, suppose $n = 3k + 1$. As in the previous case, choose a minimum pendant dominating set S' of a copy G and then select the corresponding vertices of S' from another copy of G. This will be a pendant dominating

set but not minimal as the vertices $v_2^{'}$ and $v_3^{'}$ have two neighbor in the set. Hence $S' - \{v_2^{'}, v_3^{'}\}$ will be a minimum pendant dominating set in G. Therefore $\gamma_{pe}(\mathfrak{D}(G)) =\mid S' - \{v_2^{'}, v_3^{'}\} \mid= 2\lfloor \frac{n}{3} \rfloor$.
Finally, if $n = 3k + 2$ for some positive integer $k > 1$. If $k = 2$, then for any minimum pendant dominating set of a copy of G, in which the vertex u of S is replaced by corresponding vertex u', will be a minimum pendant dominating set and so $\gamma_{pe}(\mathfrak{D}(G)) = 2\lfloor \frac{n}{3} \rfloor$. Further, if $k > 2$ let choose a minimum pendant dominating set of one copy of G which dominates $\mathfrak{D}(G)$ except $k-2$ number of vertices in another copy of G. So that, the minimum pendant dominating set of $\mathfrak{D}(G)$ is obtained by taking the $k-2$ number of vertices which is not dominated by S together with S. Therefore $\gamma_{pe}(\mathfrak{D}(G)) =\mid S \mid +(k-2) = 2\lfloor \frac{n}{3} \rfloor$.

□

Theorem 1.2. *Let G be a cycle with n vertices. Then*

$$\gamma_{pe}(\mathfrak{D}(G)) = \begin{cases} 3, & \text{if } n=5; \\ 2\lceil \frac{n}{3} \rceil & \text{if } n \equiv 0 \text{ (mod 3)}; \\ 2\lfloor \frac{n}{3} \rfloor, & \text{if } n \equiv 1 \text{ or } 2 \text{ (mod 3)}. \end{cases}$$

Proof. Let G be a cycle of order $n \geq 3$. Then $\mathfrak{D}(G)$ is a 4 regular graph of order $2n$. If $n = 5$, then any minimum pendant dominating set of a copy of G, in which a vertex v of S replaced by the corresponding vertex v' will be a minimum pendant dominating set and so $\gamma_{pe}(\mathfrak{D}(G)) = 3$.
Now, we may consider the following two possible cases here. First, suppose $n = 3k$ for some positive integer k. The double graph of G consists of two copies of C_n, choose a minimum pendant dominating set S' of one copy of C_n, which dominates $\mathfrak{D}(G)$ except the vertices corresponding to the vertices of S'. So that, the minimum pendant dominating set S of $\mathfrak{D}(G)$ is obtained by taking the vertices not dominated by S' along with the set S'. Therefore $\gamma_{pe}(\mathfrak{D}(G)) = 2\lceil \frac{n}{3} \rceil$.
Next, suppose $n = 3k + 1 \, or \, 3k + 2$ for some positive integer k. As in the above case choose a minimum pendant dominating set S' of a copy G and then select the corresponding vertices of S' from another copy of G. This will be a pendant dominating set but not minimal as the vertices $v_1^{'}$ and $v_2^{'}$ have two neighbors in the set. Hence, $S' - \{v_1^{'}, v_2^{'}\}$ will be a minimum pendant dominating set in G. Therefore, $\gamma_{pe}(\mathfrak{D}(G)) =\mid S' - \{v_1^{'}, v_2^{'}\} \mid= 2\lfloor \frac{n}{3} \rfloor$. □

Theorem 1.3. *Let G be a complete graph with n vertices. Then $\gamma_{pe}(\mathfrak{D}(G)) = 2$.*

Proof. Let G be a complete graph of order n. Then double graph of G will be a regular graph of order $2n-2$. The set $S = \{v_1, v_2\}$ is a minimum pendant dominating set of one copy of G. Then the set S itself a minimum pendant dominating set of $\mathfrak{D}(G)$, where v_1 and v_2 are two adjacent vertices in any one copy of $\mathfrak{D}(G)$. Therefore $\gamma_{pe}(\mathfrak{D}(G)) =\mid S \mid= 2$. □

Definition 1.4. *The crown graph S_n for $n \geq 3$ is the graph with vertex set $V = \{u_1, u_2, u_3, ...u_n, v_1, v_2, v_3, ...v_n\}$ and an edge from $\{u_iv_j : 1 \leq i, j \leq n; i \neq j\}$. Therefore S coincides with the complete bipartite graph $K_{m,n}$ with horizontal edges removed.*

Lemma 1.5. *Let G be a crown graph with $n \geq 2$. Then $\gamma(\mathfrak{D}(G)) = 4$*

Theorem 1.6. *Let $G \cong S_n$ be a crown graph with $2n$ vertices. Then $\gamma_{pe}(\mathfrak{D}(G)) = \gamma(\mathfrak{D}(G))$.*

Proof. Let G be a crown graph with $2n$ vertices. Then $\mathfrak{D}(G)$ is a $\{2n-2\}$-regular graph of order $4n$. The double graph of G consists of two copies of S_n, choose a minimum pendant dominating set of S' of one copy of S_n, which dominates the $\mathfrak{D}(G)$ except one vertex corresponding to the vertex of S'. So that, the minimum pendant dominating set of $\mathfrak{D}(G)$ is obtained by taking the vertex is not dominated by S' along with S'. Therefore, $\gamma_{pe}(\mathfrak{D}(G)) =\mid S' \mid +1 = \gamma(\mathfrak{D}(G))$. □

Definition 1.7. *The helm graph H_n is the graph obtained from an n-wheel graph by adjoining a pendant edge at each node of the cycle. The helm graph H_n has $2n+1$ vertices and $3n$ edges.*

Theorem 1.8. *Let $G \cong H_n$ be a helm graph. Then $\gamma(\mathfrak{D}(G)) = n$.*

Proof. Let $G \cong H_n$ be a helm graph and let (XY) be a bipartition of G, with $X = \{v_1, v_2, \ldots, v_n\}$ and $Y = \{u_1, u_2, \ldots, u_n\} \cup \{v\}$, where v is the centre vertex of G and the vertices in Y are the leaves of the helm graph. The set $S = \{v_1, v_2, \ldots, v_n\}$ will be a minimum dominating set in G. As every vertex in S covers all the leaves and corresponding vertices in another copy, it follows that the set S itself a minimum dominating set in $\mathfrak{D}(G)$. Therefore, $\gamma(\mathfrak{D}(G)) =\mid S \mid= n$. □

Lemma 1.9. [4] *Let G be a helm graph. Then $\gamma_{pe}(G) = n$.*

Theorem 1.10. *Let $G \cong H_n$ be a helm graph. Then $\gamma_{pe}(\mathfrak{D}(G)) = 2n-2$.*

Proof. Let G be helm graph of order $2n+1$. Clearly, $\gamma_{pe}(G) = n$. Consider the minimum pendant dominating set S of G. Then double graph of G consists of two copies of helm graph and every vertex in S covers all the vertices in one copy of G and $n-2$ number of vertices which is not dominated by the set S. Therefore the minimum pendant dominating set of $\mathfrak{D}(G)$ is obtained by taking the vertices in S along with the vertices not dominated by S. Hence, $\gamma_{pe}(\mathfrak{D}(G)) =\mid S \mid +(n-2) = 2n-2$ □

Definition 1.11. *For $m \geq 3$, Jahangir graph $J_{n,m}$ is a graph of order $nm+1$, consisting of a cycle of order nm with one vertex adjacent to exactly m vertices of C_{nm} at a distance n to each other. Jahangir graph $J_{2,8}$ is shown in $Fig.2$.*

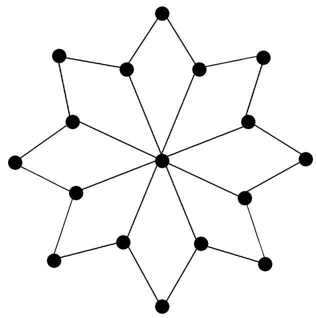


Fig.2. Jahangir graph $J_{2,8}$

Theorem 1.12. [4] *Let $G \cong J_{n,m}$ be a Jahangir graph with $m, n \geq 3$. Then*

$$\gamma_{pe}(G) = \begin{cases} \frac{m(n-1)}{3} + 2, & \text{if } n \equiv 1 \text{ (mod 3)}; \\ \lceil \frac{mn}{3} \rceil + 1, & \text{if } n \equiv 0 \text{ or } 2 \text{ (mod 3)}. \end{cases}$$

Theorem 1.13. *Let $G \cong J_{n,m}$ be a Jahangir graph with $m \geq 3$ and $n \cong 0\, or\, 1 (mod 3)$, then*

$$\gamma_{pe}(\mathfrak{D}(J_{n,m})) = \begin{cases} 2\lceil \frac{nm}{3} \rceil, & \text{if } m=3; \\ 2\lceil \frac{nm}{3} \rceil - 2, & \text{otherwise.} \end{cases}$$

Proof. Let $J_{n,m}$ be a Jahangir graph with $m \geq 3$ and $n \cong 0\, or\, 1(mod3)$. If $m = 3$, then choose a set S' be a minimum pendant dominating set of one copy of $J_{n,m}$. The set S' dominates the double graph $\mathfrak{D}(J_{n,m})$ except some of the corresponding vertices of S' in the other copy of $J_{n,m}$. So that, the minimum pendant dominating set S of $\mathfrak{D}(J_{n,m})$ is obtained by taking the vertices not dominated by S' together with the set S'. Therefore $\gamma_{pe}(\mathfrak{D}(J_{n,m})) = 2 \mid S \mid = 2\lceil \frac{nm}{3} \rceil$.
Next, suppose $m \geq 4$. Let S' be a minimum pendant dominating set of one copy of $\mathfrak{D}(J_{n,m})$ which dominates the $\mathfrak{D}(J_{n,m})$ except $\lceil \frac{nm}{3} \rceil - 2$ number of vertices corresponding to the vertices of S'. So that, the minimum pendant dominating set S of $\mathfrak{D}(J_{n,m})$ is obtained by taking all the vertices which is not dominated by the set S' together with the set S'. Therefore $\gamma_{pe}(\mathfrak{D}(J_{n,m})) = \mid S' \mid + \lceil \frac{nm}{3} \rceil - 2 = 2\lceil \frac{nm}{3} \rceil - 2$.

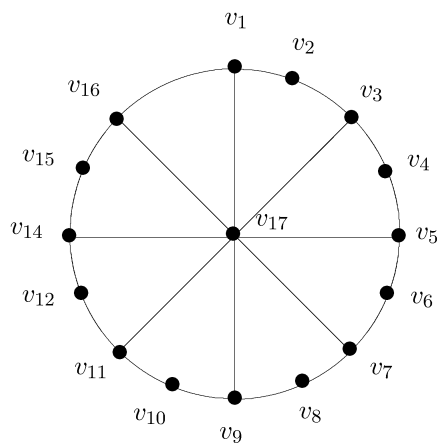
□

Theorem 1.14. *Let $G \cong J_{n,m}$ be a Jahangir graph with $m \geq 3$ and $n \cong 2(mod3)$, then*

$$\gamma_{pe}(\mathfrak{D}(G)) = \begin{cases} 2\lceil \frac{nm}{3} \rceil, & \text{if } m \cong 0 \text{ or } 1 \text{ (mod3)}; \\ 2\lfloor \frac{nm}{3} \rfloor, & \text{if } m \cong 2\text{(mod3)}. \end{cases}$$

Proof. Let $G \cong J_{n,m}$ be Jahangir graph with $n \cong 2(mod3)$. First, suppose $m = 3k\, or\, 3k+1$ for some positive integer k. The double graph of Jahangir graph consists of two copies of $J_{n,m}$, choose a minimum pendant dominating set S' of one copy of $J_{n,m}$, therefore $\mid S' \mid = \lceil \frac{nm}{3} \rceil + 1$ which dominates $\mathfrak{D}(G)$ except $\lceil \frac{nm}{3} \rceil - 1$ number of corresponding vertices of S' in the other copy of $\mathfrak{D}(G)$. The minimum pendant dominating set S' of $\mathfrak{D}(G)$ is obtained by taking the vertices which is not dominated by S' together with the set S'. Therefore $\gamma_{pe}(\mathfrak{D}(G)) = \mid S' \mid + \lceil \frac{nm}{3} \rceil - 1 = 2\lceil \frac{nm}{3} \rceil$
Next, suppose if $m = 3k + 2$ for some positive integer k. As in the previous case, choose a minimum pendant dminating set of S' one copy of $J_{n,m}$, therefore $\mid S' \mid = \lfloor \frac{nm}{3} \rfloor + 1$ which dominates the double graph of G except $\lfloor \frac{nm}{3} \rfloor - 1$ number of corresponding vertices of S' in the another copy of $\mathfrak{D}(G)$. So that, the set S is obtained by taking the vertices which is not dominated by S' together with S' will be a pendant dominating set of $\mathfrak{D}(G)$. Further, for any vertex v of S, the set $S - \{v\}$ will not be a pendant dominating set in G and so in $\mathfrak{D}(G)$. Hence, the set S is minimum pendant dominating set of $\mathfrak{D}(G)$. Therefore $\gamma_{pe}(\mathfrak{D}(G)) = \mid S \mid = 2\lfloor \frac{nm}{3} \rfloor$. □

Definition 1.15. *The gear graph is a wheel graph with a vertex added between each pair adjacent graph vertices of the outer cycle. The gear graph G has $2n+1$ vertices and $3n$ edges. In Fig.3. we display G_8.*



$Fig.3$. Gear graph G_8

Theorem 1.16. *Let G_n be a gear graph with $n \geq 3$. Then*

$$\gamma(\mathfrak{D}(G_n)) = \begin{cases} \lceil \frac{4n}{3} \rceil, & \text{if } n \cong 0 \text{ or } 1 \text{ (mod3)}; \\ \lceil \frac{4n}{3} \rceil + 1, & \text{if } n \cong 2 \text{(mod3)}. \end{cases}$$

Proof. Let G_n be a gear graph with $n \geq 3$. Then $\mathfrak{D}(G_n)$ is a $\{4, 6\}$ regular graph of order $4n$. Now, suppose $n = 3k$ *or* $3k+1$ for some positive integer k. The double graph of G_n consists of two copies of G_n, choose minimum dominating set S' of one copy of G_n, which dominates $\mathfrak{D}(G_n)$ except the vertices corresponding to the vertices of S'. So that, the minimum dominating set S of G_n is obtained by taking the vertices not dominated by S' together with S' will be a dominating set of $\mathfrak{D}(G_n)$. Therefore $\gamma(\mathfrak{D}(G_n)) = \mid S \mid = \lceil \frac{4n}{3} \rceil$. Next, suppose $n = 3k + 2$ for some $k > 0$. As in the previous case, choose a minimum dominating set S' of a copy of G_n and then select the corresponding vertices from another copy of G_n. This will be a minimum dominating set of cardinality $\lceil \frac{4n}{3} \rceil + 1$ □

Theorem 1.17. *Let G_n be a gear graph with $n \geq 3$. Then $\gamma_{pe}(\mathfrak{D}(G_n)) = \lceil \frac{n}{2} \rceil + 1$*

Proof. Let G_n be a gear graph of order $2n + 1$ and let $V(G) = \{v_1, v_2, \ldots v_{2n}, v_{2n+1}\}$, where v_{2n+1} is the centre vertex of G_n. From the definition, the vertex v_{2n+1} is adjacent to n vertices of G_n. Since G contains a unique minimum pendant dominating set $S = \{v_1, v_5, v_9, \ldots, v_{2n-1}\} \cup v_{2n+1}$ and the set S dominates the double graph of G_n. Therefore the set S itself a minimum pendant dominating set of $\mathfrak{D}(G_n)$. Hence, $\gamma_{pe}(\mathfrak{D}(G_n)) = \mid S \mid = \lceil \frac{n}{2} \rceil + 1$ □

2. Bounds for $\gamma_{pe}(\mathfrak{D}(G))$

Proposition 2.1. *Let G be any connected graph of order n. Then $1 \leq \gamma(G) \leq \gamma_{pe}(G) \leq \gamma(\mathfrak{D}(G)) \leq \gamma_{pe}(\mathfrak{D}(G)) \leq 2n$. Further, $\gamma_{pe}(\mathfrak{D}(G)) = 2$ if and only if G contains an edge of degree atleast $n - 2$.*

Proof. let G be any connected graph with n vertices. The inequalities are obvious. Suppose $\gamma_{pe}(\mathfrak{D}(G)) = 2$. Let $S = \{v_1, v_2\}$ be a minimum pendant dominating set in G and S dominates the double graph $\mathfrak{D}(G)$. Then v_1 and v_2 must be adjacent vertices and every vertex in $V - S$ must be adjacent to either v_1 or v_2. Thus, the degree of the edge $e = v_1v_2$ must be atleast $n - 2$. Converse is obvious. □

Proposition 2.2. *Let G be any graph . If $diam(G) = 2$ then $\gamma_{pe}(\mathfrak{D}(G)) \leq \delta(G) + 1$. Equality holds if G is a path.*

Proof. Let G be any connected graph and v be a vertex of G of degree $\delta(G)$. As $diam(G) = 2$, any vertex in G will be at a distance atmost 2 from u. Thus $N[u]$ will be a minimum pendant dominating set in $\mathfrak{D}(G)$. Hence,$\gamma_{pe}(\mathfrak{D}(G)) \leq \delta(G) + 1$ □

Proposition 2.3. *Let G be a connected graph of order $n \geq 2$. Then $\gamma_{pe}(\mathfrak{D}(G)) \leq \gamma(\mathfrak{D}(G)) + \delta(\mathfrak{D}(G))$*

Proof. Let G be a connected graph of order $n \geq 2$. Then, $\delta(G) \geq 1$ and let v be a vertex of degree $\delta(G)$. Clearly, any minimum pendant dominating set in $\mathfrak{D}(G)$ must contain either v or a vertex from $N(v)$. Thus $\gamma_{pe}(\mathfrak{D}(G)) \leq \gamma(\mathfrak{D}(G)) + \mid N(v) \mid$. This prove that $\gamma_{pe}(\mathfrak{D}(G)) \leq \gamma(\mathfrak{D}(G)) + \delta(\mathfrak{D}(G))$. □

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"Dedicated to Prof. Chandrashekar Adiga on his 62^{nd} Birthday"

References

[1] Mojdeh, D.A. and Ghameshlou, A.N. Doimnation in Jahangir graph $J_{2,m}$, Int. J.Contemp. Math. Sciences, 2007, vol.2, no.24,pp. 1193-1199.

[2] Munarini, E., Perelli Cippo, C., Scagliola, A. and Zagaglia Salvi, N. *Double Graphs*, Discrete Mathematics, 2008, vol. 308, pp. 242-254.

[3] Nayaka S.R, Puttaswamy and Purushothama S, *Pendant Domination in Graphs*. Journal of Combinatorial Mathematics and Combinatorial computing, Volume 112, February, 2020, pp. 219-230.

[4] Nayaka S.R, Puttaswamy and Purushothama S, *Pendant Domination in some Generalised Graphs*, International Journal of Scientific Engineering and Science. Volume 1, Issue 7, pp. 13-15, 2017.

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