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Curling number of Rooted product of Graph

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Abstract:A curling subsequence is a maximal subsequence C of the degree sequence of a simple connected graph G for which the curling number cn(G) corresponds to the curling number of the degree sequence and hence the curling number of the graph G. The curling number of a graph G may be defined as the number of times an element in the degree sequence of G appears the most and compound curling number of G is the product of multiplicities of the degrees of vertices in G. In this paper we establish the bounds for curling number and find Compound curling number of rooted product graph GoH.

Keywords: Curling number,Compound curling number,Rooted product graph
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1 .Introduction: For general notation and basic concepts of graph theory we refer to Harary [7].For different types of product graphs, we refer to [6, 11]. All graphs mentioned in this paper are simple, non-trivial, connected, and finite graphs unless mentioned otherwise.

B. Chaffin [2] introduced curling number of a sequence of integers. Extending the concept of curling number of sequence to the degree sequence of graphs, Kok et al[9]

1.1 Definition of Curling number: Let$S = S_1S_2S_3\cdots\cdots\cdots S_n$ be a finite string. Write S in the form $XYYY\cdots\cdots = XY^k$,consisting of a prefix X,followed by k-copies of a non empty string Y.Pick one with the greatest value of k. Then this integer k is called the curling number of S and is denoted by $cn(S)$.

Kok et al[9] also defined the notion of the curling number and compound curling number of a graph G as follows,

1.2 Definition of Curling number of a graph: Given a finite non -empty graph G with the degree sequence $S = (a_1, a_2, a_3 \cdots\cdots a_n), a_i \in \mathrm{N}_0$. This degree sequence S can be written as a string of subsequences $S = d_1^{k_1}$, $\mathrm{d}_2^{k_2}, d_3^{k_3}, d_4^{k_4} \cdots\cdots\cdots, d_l^{k_l}$, then the curling number of G is defined to be $cn(G) = max\{k_i\}$,where $1 \leq i \leq l$

In other words the curling number of a graph G is defined as the number of times an element in the degree sequence of G appears the most [9].

1.3 Definition of Compound curling number of a graph : Let the degree sequence of the graph G be written as a string of identity curling subsequence say,$d_1^{k_1}, d_2^{k_2}, d_3^{k_3} \cdots\cdots\cdots, d_l^{k_l}$.The compound curling number of G,denoted $cn^c(G)$ is defined as $cn^c(G)= \prod_{i=1}^{l} k_i$where $1 \leq i \leq l$

1.4 Definition of Rooted product graph : The rooted product of two graphs of G_1 and G_2 denoted by $G_1 o G_2$,is the graph obtained by choosing one vertex of G_2 as root and then attaching the root vertex of copy of G_2 to each of the vertices of G_1.

In this paper, we investigate the curling number of rooted product of general graph with bi-regular graph.

2. Curling Number of Rooted product graph

Let$V(G) = \{v_1, v_2, v_3, ...v_m\}$ and $V(H) = \{u_1, u_2, u_3, ...u_n\}$be the vertex sets of G and H respectively.Let $d\,(v_i) = d_i$ be the degree of a vertex of graph G.Let $r_i = d\,(u_i)$ be the degree of the root vertex of u_i of H which is attached to every vertex of G. Then all
m vertices of G in GoH has degree $(r_i + d_i)$ and the degree sequence of GoH is
$\{(d_1 + r_i), (d_2 + r_i), (d_3 + r_i), \cdots (d_n + r_i), (r_1)^m, (r_2)^m, \cdots (r_{i-1})^m, (r_{i+1})^m, \cdots (r_n)^m\}$

Lemma 2.1: Let G be any graph on n vertices with degree sequence $\{r_1^{n_1}, r_2^{n_2}, r_3^{n_3}, \cdots\cdots\cdots r_t^{n_t}\}$ where $n_1 + n_2 + n_3 + ---- n_t = n$ and $r_1 > r_2 > r_3 \cdots\cdots > r_t$ and H is a d-regular graph on m vertices then $cn\,(GoH) = |V(G)|\,(cn(H) - 1)$

Proof: Let G be any graph on n vertices with degree sequence $\{r_1^{n_1}, r_2^{n_2}, r_3^{n_3}, \cdots\cdots r_t^{n_t}\}$ where $n_1 + n_2 + n_3 + ---- n_t = n$ and $r_1 > r_2 > r_3 \cdots\cdots > r_t$.

Suppose H is a regular graph on m vertices with degree d. Let us choose one root vertex of H and attach the root vertex of each copy of H to every vertex of G to get the rooted product graph GoH.

The degree sequence of GoH is

$d(GoH) = \left\{(r_1 + d)^{n_1}, (r_2 + d)^{n_2}, \cdots\cdots (r_t + d)^{n_t}, d^{n(m-1)}\right\}$

$cn\,(GoH) = max\,\{\mathrm{n}_1, n_2, -----n_t, n(m-1)$

As $n_1 + n_2 + n_3 + ---n_t = n$ each of $n_1, n_2, n_3, ---n_t \leq n$ so when $m \geq 2$

$cn\,(GoH) = n(m-1) = |V(G)|\,(|V(H)| - 1)$

As H is a d-regular graph on m vertices, degree sequence of H is $\{d^m\}$ giving curling number of H as cn(H) = m, so we can write $cn\,(GoH) = |V(G)|\,(cn(H) - 1)$

Lemma 2.2: G is a regular graph of degree d on n vertices and H is a biregular graph on m vertices with $m_1$vertices of degree d_1 and $m_2$vertices of degree d_2.Then the Curling number of rooted product GoH with k be the number of vertices in H of degree equal to the degree of the root vertex is $cn\,(GoH)$ =$max\,\{n(k-1), n(m-k+s)\}$ where s=0 if $d + d_2 \neq d_1$ and s=1 if $d + d_2 = d_1$ and $n(cn(H) - 1) \leq cn(GoH) \leq n(cn(H) + 1)$

Proof: Let G be a d- regular graph on n vertices. Let H be a bi regular graph on m vertices with m_1 vertices of degree d_1 and m_2 vertices of degree d_2, where $m_1 + m_2 = m$ and $d_1 > d_2$, with degree sequence of H as $\{d_1^{m_1}, d_2^{m_2}\}$ then cn(H) = max$\{m_1, m_2\}$.

There are two possibilities of the root vertex for the rooted product graph GoH

Case (i): The root vertex is of degree d_1

The root vertex of degree d_1 is attached to every vertex of G in GoH,hence degree of all n vertices of G in GoH is $\{d + d_1\}$ and there are n copies of H in GoH giving $(m_1 - 1)$ vertices of degree d_1 and m_2 vertices of degree d_2 in each copy.

Therefore, the degree sequence of GoH is

$d\left(GoH\right) = \left\{\left(d+d_1\right)^n, d_1^{n(m_1-1)}, d_2^{nm_2}\right\}$ as $d_1 > d_2$ so $d+d_1 > d_2$

Curling number of GoH as $cn\left(GoH\right)$=max$\{n, n(m_1-1), nm_2\}$.

Case(ii) : The root vertex is of degree d_2

The root vertex of degree d_2 is attached to every vertex of G, now the degree of all n vertices of G in GoH becomes($d+d_2$) and n copies of H in GoH giving (m_2-1) vertices of degree d_2 and m_1 vertices of degree d_1.

Therefore, the degree sequence of GoH

$d\left(GoH\right) = \left\{\left(d+d_2\right)^n, d_2^{n(m_2-1)}, d_1^{nm_1}\right\}$

Curling number of GoHas

$cn\left(GoH\right)$=max$\{n, n(m_2-1), nm_1\}$ if d+d$_2 \neq d_1$

$cn\left(GoH\right) = max\left\{n(m_2-1), n(m_1+1)\right\}$if $d+d_2 = d_1$

Considering then combining of results of case(i) and (ii) we can write

As d(root)=d_1 let k =m_1 and m-k =m_2 gives

$cn\left(GoH\right)$=$max\left\{n, n(k-1), n(m-k)\right\}$

As d(root)=d_2 let k =m_2 and m-k =m_1 gives

$$\text{cn}(GoH)= \max\begin{cases} \text{n,n(k-1),n(m-k),} & \text{if } d+d_2 \neq d_1 \\ \text{n(k-1),n(m-k)+n,} & \text{if } d+d_2 = d_1 \end{cases}$$

Hence,

$cn\left(GoH\right)$=$max\left\{n(k-1), n(m-k+s)\right\}$

when s=0 if $d+d_2 \neq d_1$ and s=1 if $d+d_2 = d_1$

Suppose $m_1 \leq m_2$ then cn(H) = $max\left\{m_1, m_2\right\}$=$m_2$ then in either case we can write

$n(m_2-1)=n(cn(H)-1)\leq cn\,(GoH)$

Suppose $m_2 \leq m_1$ then cn(H)=$max\,\{m_1, m_2\}$=m_1 then in either case we can write

$cn\,(GoH)$≤n(m_1+1) = n(cn(H)+1)

We can combine the two results to get the bound on curling number of rooted product as,

n(cn(H)-1)≤cn(GoH)≤ $n(cn(H)+1)$

Corollary 2.3 : For a regular graph G and Bi regular graph H , the Compound curling number is

$cn^{(c)}\,(GoH)$ =$n^2(k-1)(m-k+s)$ s=0 if $d+d_2\neq d_1$and s=1 if $d+d_2=d_1$ and k is the number of vertices in H of degree equal to the degree of the root.

Lemma 2.4:G is a bi regular graph on n vertices with n_1 vertices of degree r_1 and n_2 vertices of degree r_2 with $n_1\leq n_2$ and H is a biregular graph on m vertices with m_1 vertices of degree d_1 and m_2 vertices of degree d_2.Then the Curling number of rooted product GoH with k be the number of vertices in H of degree equal to the degree of the root vertex is

$cn\,(GoH)=max\,\{n(k-1), n(m-k+s)\}$

s=0 if $r_i+d_H(root)\neq d_H(non-root)$

s=n_i if $r_i+d_H(root)=d_H(non-root)$ for i=1,2

and n(cn(H)-1)≤cn(GoH)≤(n cn(H)+n_2)

Proof: Let G be a bi regular graph on n vertices with n_1 vertices of degree r_1 and n_2 vertices of degree r_2 with $n_1+n_2=n$ and $r_1>r_2$.Let H be bi regular graph on m vertices with $m_1$vertices of degree d_1 and $m_2$vertices of degree d_2 with $m_1+m_2=m$ and $d_1>d_2$ then there are two possibilities of the root vertex .

Case (i) : The root vertex is of degree d_1

In GoH,the root vertex of degree d_1 is attached to all vertices of G ,so the degree of n_1 vertices of G in GoH becomes $(r_1 + d_1)$ and degree of n_2 vertices becomes $(r_2 + d_1)$.As there are n copies of H in GoH with each copy having $(m_1 - 1)$ vertices of degree d_1 and m_2 vertices of degree d_2

We get the degree sequence of GoH as, Degree sequence = $\left\{(r_1 + d_1)^{n_1}, (r_2 + d_1)^{n_2}, d_1^{n(m_1-1)}, d_2^{nm_2}\right\}$

Curling number of GoH

$cn\,(GoH) = max\,\{n_1, n_2, n(m_1 - 1), nm_2\} = max\,\{n(m_1 - 1), nm_2\}$ as $n_1, n_2 < n$

Case(ii) : The root vertex is of degree d_2

As explained in case (i) for root vertex of degree d_2 we get the degree sequence of GoH as

$d(GoH) = \left\{(r_1 + d_2)^{n_1}, (r_2 + d_2)^{n_2}, d_1^{nm_1}, d_2^{n(m_2-1)}\right\}$

Curling number of GoH as

$cn\,(GoH) = max\,\{n_1, n_2, n(m_2 - 1), nm_1\} = max\,\{n(m_2 - 1), nm_1\}$ as $n_1, n_2 < n$

Considering k is equal to number of vertices in H of degree equal to the degree of the root then combining of results case(i) and (ii) we can write

As d(root) = d_1 , let k = m_1 and m-k = m_2 gives

$cn\,(GoH) = max\,\{n(k - 1), n(m - k)\}$

As d(root) = d_2 , let k = m_2 and m-k = m_1 gives

$$cn(GoH) = \begin{cases} \text{n(k-1),n(m-k)} & \text{if } r_i + d_2 \neq d_1 fori = 1, 2 \\ \text{n(k-1),n(m-k)}+n_i & \text{if } r_i + d_2 = d_1 fori = 1, 2 \end{cases}$$

Hence,

$cn\,(GoH)$ $=max\,\{n(k-1), n(m-k+s)\}$

s=0 if $r_i + d_H(root) \neq d_H(non-root)$

s=n_i if $r_i + d_H(root) = d_H(non-root)$ for i=1,2

Suppose $m_1 \leq m_2$ then cn(H) $=$ $max\,\{m_1, m_2\}$ $=m_2$ then in either case we can write

$n(m_2-1)$=n(cn(H)-1)≤cn(GoH)

Suppose $m_2 \leq m_1$ then cn(H) $=$ $max\,\{m_1, m_2\}$ $=m_1$ and assume $n_1 \leq n_2$ then in either case we can write

$cn\,(GoH)$≤(nm$_1 + n_2$) $=$ ($n$cn(H)+n_2)

We can combine the two results to get the bound on curling number of rooted product as,

n(cn(H)-1)≤cn(GoH) ≤(n$cn(H)$+n$_2$).

Corollary 2.5: For a Bi regular graph G and Bi regular graph H , the compound curling number is

$cn^{(c)}\,(GoH)$ $=n^2(k-1)[(m-k)+s]$

Theorem 2.6: Curling number of rooted product of any graph G with bi-regular graph H is $cn^{(c)}\,(GoH)$= $max\,\{n(k-1), n(m-k)+n_i)\}$ if $r_i + d_H(root) = d_H(non-root)$and $n(cn(H)-1)$≤cn(GoH) ≤(n cn(H)+n$_s$).

Proof: Let G be a graph on n vertices with degree sequence $\{r_1^{n_1}, r_2^{n_2}, r_3^{n_3}, \cdots\cdots r_t^{n_t}\}$ with $n_1 + n_2 + n_3 + \cdots\cdots + n_t = n$and $r_1 > r_2 > r_3 \cdots\cdots r_t$

Suppose H is a biregular graph on m vertices with degree sequences $\{d_1^{m_1}, d_2^{m_2}\}$ with $m_1 + m_2 = m$ and $d_1 > d_2$ there are two possibilities of the root vertex ,

Case (i): The root vertex is of degree d_1

The root vertex of degree d_1 is attached to every vertex of G in GoH, hence degree of all n vertices of G in GoH is $(r_i + d_1)$, $1 \leq i \leq t$ and there are n copies of H in GoH giving $(m_1 - 1)$ vertices of degree d_1 and m_2 vertices of degree d_2 in each copy.

Therefore the degree sequence of GoH is

d(GoH) = $\left\{(r_1+d_1)^{n_1},(r_2+d_1)^{n_2},\cdots\cdots(r_t+d_1)^{n_t},d_1^{n(m_1-1)},d_2^{nm_2}\right\}$

$cn\,(GoH) = max\,\{n(m_1-1), nm_2\}$ as $d_1 > d_2$ so $r_i + d_1 \neq d_2$

Case (ii): The root vertex is of degree d_2

The root vertex of degree d_2 is attached to every vertex of G in GoH ,hence degree of all n vertices of G in GoH is $(r_i + d_2)$, $1 \leq i \leq t$ and there are n copies of H in GoH giving $(m_2 - 1)$ vertices of degree d_2 and m_1 vertices of degree d_1 in each copy.

Therefore the degree sequence of GoH is

d(GoH) = $\left\{(r_1+d_2)^{n_1},(r_2+d_2)^{n_2},\cdots\cdots(r_t+d_2)^{n_t},d_2^{n(m_2-1)},d_1^{nm_1}\right\}$

$cn\,(GoH) = max\,\{n(m_2-1), nm_1\}$ if $r_i + d_2 \neq d_1$

$cn\,(GoH) = max\,\{n(m_2-1), nm_1 + n_i\}$ if $r_i + d_2 = d_1$

Considering k to be the number of vertices in H of degree equal to the degree of the root then combining results of case(i) and(ii) we can write

As d(root) = d_1 , let k = m_1 and m-k = m_2 gives

$cn\,(GoH) = max\,\{n(k-1), n(m-k)\}$ for $r_i + d_1 \neq d_2$

As d(root) = d_2 , let k = m_2 and m-k = m_1 gives

$$cn(GoH) = \begin{cases} \text{n(k-1),n(m-k)} & \text{if } r_i + d_2 \neq d_1 \\ \text{n(k-1),n(m-k)+ } n_i & \text{if } r_i + d_2 = d_1 \text{ for some i=1,2...t} \end{cases}$$

Hence,

$cn\,(GoH) = max\,\{n(k-1), n(m-k) + s\}$

s=0 if $r_i + d_H(root) \neq d_H(non-root)$

s=n_i if $r_i + d_H(root) = d_H(non-root)$ for i=1 to t

Suppose $m_1 \leq m_2$ then cn(H) $= max\{m_1, m_2\}$ $=m_2$ then in either case we can write

$n(m_2-1)$=n(cn(H)-1)≤cn(GoH)

Suppose $m_2 \leq m_1$ then cn(H) $= max\{m_1, m_2\}$ $=m_1$ and assume $n_{max} = max\{n_1, n_2, n_3, ...n_t\}$ then in either case we can write

$cn\,(GoH)$≤(nm$_1 + n_s$) $= (ncn(H) + n_{max})$

We can combine the two results to get the bound on curling number of rooted product as,

n(cn(H)-1)≤cn(GoH)≤(n cn(H)+n$_{max}$).

Corollary 2.7: Compound curling number of rooted product of any graph G with biregular graph H is

$cn^{(c)}\,(GoH)$ $=n^2(k-1)[(m-k)+s]$

s=0 if $r_i + d_H(root) \neq d_H(non-root)$

s=n_i if $r_i + d_H(root) = d_H(non-root)$ for i=1,2

Theorem 2.8: Curling number of rooted product of a graph G on n vertices with a graph H on m vertices satisfies

$n*[cn(H)-1]$≤cn(GoH)≤ n * cn(H)+n$_{max}$

Proof: Let G be a graph on n vertices with degree sequence $\{r_1^{n_1}, r_2^{n_2}, r_3^{n_3}, \cdots\cdots r_t^{n_t}\}$ where $n_1 + n_2 + n_3 + \cdots\cdots + n_t = n$ with $n_1 \leq n_2 \leq n_3 \leq \cdots\cdots \leq n_t$ and H be a graph on m vertices with degree sequence $\{d_1^{m_1}, d_2^{m_2}, d_3^{m_3} \cdots\cdots d_s^{m_k}\}$ where $m_1 + m_2 + m_3 + \cdots\cdots + m_s = m$ with $m_1 \leq m_2 \leq m_3 \leq \cdots\cdots \leq m_s$

Giving cn(G) $= n_t$ and cn(H)=m_s

Consider rooted product graph GoH,suppose root vertex is such that

Case (i): The root vertex is of degree d_s

The root vertex of degree d_s is attached to every vertex of G in GoH, hence degree of all n vertices of G in GoH is $(r_i + d_s)$, $1 \leq i \leq t$ and there are n copies of H in GoH giving the degree sequence of rooted product as,

d(GoH) =
$\left\{(r_1+d_s)^{n_1},(r_2+d_s)^{n_2},\cdots\cdots(r_t+d_s)^{n_t},d_1^{nm_1},d_2^{nm_2}\cdots\cdots d_s^{n(m_s-1)}\right\}$

Subcase (i): if $m_{s-1} < m_s$ then

$cn\,(GoH) \geq$ n($m_s - 1$) = n * [cn(H)-1] if $r_i + d_s \neq d_j$

$cn\,(GoH) \leq max\ \{n(m_s-1), nm_j + n_i\}\ if\, r_i + d_s = d_j$

Subcase (ii) if $m_{s-1} = m_s$ then

$cn\,(GoH) \geq$ n(m_{s-1}) $= n * [cn(H)]$ if $r_i + d_s \neq d_j$

$cn\,(GoH) \leq max\ \{n(m_s-1), nm_j + n_i\}$ if $r_i + d_s = d_j$

Case (ii): The root vertex is of degree $\neq d_s$

The root vertex of degree d_j is attached to every vertex of G in GoH ,hence degree of all n vertices of G in GoH is $(r_i + d_j)$, $1 \leq i \leq t$ and there are n copies of H in GoH giving the degree sequence of rooted product as,

d(GoH) =
$\left\{(r_1+d_j)^{n_1},(r_2+d_j)^{n_2},\cdots\cdots(r_t+d_j)^{n_t},d_1^{nm_1},d_2^{nm_2}\cdots\cdots d_j^{n(m_j-1)},\cdots\cdots d_s^{n(m_s)}\right\}$

Subcase (i)if $m_{s-1} < m_s$ then

$cn\,(GoH) \geq$ n($m_s - 1$) = n * [cn(H)-1] if $r_i + d_s \neq d_j$

$cn\,(GoH) \leq max\{n(m_s-1), nm_j + n_i\}$ if $r_i + d_s = d_j$

Subcase (ii)if $m_{s-1} = m_s$ then

$cn\,(GoH) \geq$ n(m_{s-1}) $= n * [cn(H)]$ if $r_i + d_s \neq d_j$

$cn\,(GoH) \leq max\ \{n(m_s-1), nm_j + n_i\}$ if $r_i + d_s = d_j$

Combining all above cases we can state that the curling number of rooted product of a graph G on n vertices with n_{max} as the largest number of vertices of one degree with a graph H on m vertices satisfies

$n * [cn(H) - 1] \leq$ cn(GoH) $\leq$ n * cn(H)+n_{max}

Corollary 2.9: Compound curling number of rooted product of any graph G with H is

$cn^{(c)}(GoH) \leq \prod_{i=1}^{t} n_i \times \mathrm{n}^m \prod_{j=1}^{s} m_j$

3. Results and Conclusion :

The following results are established for the curling number of rooted product of two graphs

G-regular, H-Bi regular			
GoH	d(root)	degree sequence	curling number
C_noP_m	1	$1^n2^{n(m-2)}3^n$	n(m-2)
C_noP_m	2	$1^{2n}2^{n(m-3)}4^n$	n(m-3)
K_noP_m	1	$n^n2^{n(m-2)}1^m$	n(m-2)
K_noP_m	2	$(n+1)^m2^{n(m-3)}1^{2n}$	n(m-3)
C_noS_{m+1}	m	(m+2)$^n1^{nm}$	nm
C_noS_{m+1}	1	$1^{n(m-1)}3^n4^n$	n(m-1)
G-Biregular, H- regular			
GoH	d(root)	degree sequence	curling number
$K_{m,n}oS_{b+1}$	b	$(m+b)^n(n+b)^m1^{b(m+n)}$	$(m+n)b$
$K_{m,n}oS_{b+1}$	1	$[(m+1)^n(n+1)^mb^{(m+n)}1^{(m+n)(b-1)}]$	$(m+n)(b-1)$
$K_{m,n}oP_a$	1	$[(m+1)^m(n+1)^m2^{(a-2)(m+n)}1^{(m+n)}]$	$(m+n)(a-2)$
$K_{m,n}oP_a$	1	$[(m+1)^n(n+1)^m2^{(a-3)(m+n)}1^{2(m+n}]$	$(m+n)(a-3)$

Sudev and Sushanth in their paper some new results of curling number of graphs [14] , have given results for curling number of standard graphs as, $cn\,(C_n)$ = n , $cn\,(P_n)$ = n-2 , $cn\,(K_n)$ = n , $cn\,(S_{n+1})$ = n , $cn\,(K_{m,n}) = max\,\{m, n\}$. From these results and the above table we observe following results.

Let G be a graph and H be a bi-regular graph on m vertices with m_1 vertices of degree k_1 and m_2 vertices of degree k_2 ,where $m_1 + m_2 = m$ and curling number of b-regular graph H is $cn\,(H)$ =$max\,\{m_1, m_2\}$,Suppose in GoH the root vertex chosen is of degree k_i, with m_i not equal to $cn\,(H)$ then in such cases from above table we have

1)$cn(C_n o P_m)$= $|V(C_n)|$ ×cn(P$_m$)

2)$cn(K_n o P_m)$= $|V(K_n)|$ ×cn(P$_m$)

3)$cn(C_n o S_{m+1})$=$cn(C_n o K_{1,m})$ = $|V(C_n)|$ ×cn(P$_m$)

4)$cn(K_{m,n} o S_{b+1})$= $|V(K_{m,n})|$ ×cn(S$_{b+1}$)

5)$cn(K_{m,n} o P_a)$=$|V(K_{m,n})|$ ×cn(P$_a$)

Thus in general for such choice of root we get $cn\,(GoH)$ =$|V(G)|$ ×cn(H)

Next suppose in GoH the root vertex chosen is of degree k_i ,with m_i not equal to $cn\,(H)$ then in such cases from above table we have

1)$cn(C_n o P_m)$= $|V(C_n)|$ ×[cn(P$_m$) − 1]

2)$cn(K_n o P_m)$= $|V(K_n)|$ ×[cn(P$_m$) − 1]

3)$cn(C_n o S_{m+1})$=$cn(C_n o K_{1,m})$ = $|V(C_n)|$ ×[cn(P$_m$) − 1]

4)$cn(K_{m,n} o S_{b+1})$= $|V(K_{m,n})|$ ×$[cn(S_{b+1}) - 1]$

5)$cn(K_{m,n} o P_a)$=$|V(K_{m,n})|$ ×[cn(P$_a$) − 1]

Thus in general for such choice of root we get $cn\,(GoH)$ =$|V(G)|$ ×[cn(H)-1]

Hence for any graph G and bi-regular graph H the curling number of GoH depends on the choice of root vertex being the one with number of such vertices equal to $cn\,(H)$

Thus for any graph G with n_{max} as the maximum number of vertices of same degree and a bi-regular graph H we can write the bounds for curling number GoH as

$|V(G)| \times$[cn(H)-1]$\leq$cn(GoH) $\leq |V(G)| \times$cn(H)+n$_{max}$

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