Proceedings of the Jangjeon Mathematical Society
23 (2020), No. 2, pp. 193 - 208
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2020.23.2.193

CENTROIDAL MEAN LABELING OF GRAPHS-II

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Abstract. In this paper the Centroidal mean labeling of graphs such as triangular snake $T_n \bigodot K_1$, double triangular snake $D_n(T_n) \bigodot K_1$, $TL_n \bigodot K_1$, the graph obtained by attaching pendent edges to both sides of each vertex of a path P_n, attaching paths of lengths $0, 1, 2, 3, \ldots, n-1$ on both sides of each vertex of P_n, $D_2(P_n)$, Middle graph of path P_n, Total graph of path P_n, Splitting graph of path P_n and Duplicating each vertex by an edge in path P_n are discussed.

1. Introduction and preliminaries

The mean defined as a midway between the value of other quantities or as an average which is important for the researchers of Mathematics and Statistics for their investigations and justifications. The arithmetic mean, geometric mean and harmonic mean are the three classical means among ten Greek means which are defined on the basis of proportions. These means were studied by Pythagoreans and later developed by Greek Mathematicians because of their importance in geometry and music.

Mean:[1] A mean is defined as a function

$$M : R_+^2 \to R_+$$

which has the property

$$min(x_1, x_2) \leq M(x_1, x_2) \leq max(x_1, x_2)$$

where x_1 and x_2 are positive real numbers.

Centroidal mean: [5, **?**, 7, 11]For any two positive real numbers a and b, a centroidal mean is defined as

$$C(a, b) = \frac{2(a^2 + ab + b^2)}{3(a + b)}.$$

Recently, the researchers have effectively utilized Mathematical means for labeling the graphs, for processing the digital images. According to Wang, Bin Yao and Ming Yao, graph labelings are used for incorporating redundancy in disks, designing drilling machines, creating layouts for circuit boards and configuring resistor networks. Labeled graphs serve as useful models for a broad range of applications such as coding theory, X-ray crystallography, astronomy, circuit design, communication network addressing, database management, secret sharing schemes and models for constraint programming over finite domains.

In [2, 8, 9, 12, 13], the notion of mean labeling was introduced by Somasundaram and Ponraj and also studied the concept of a geometric mean and contra harmonic mean labeling of a graph G with p vertices and q edges and proved paths, cycles,

2010 *Mathematics Subject Classification.* Primary 26D10; Secondary 26D15.
Key words and phrases. Graph, Centriodal mean, Triangular snake, Double triangular snake, Ladder.

combs, ladders are geometric mean graphs and $K_n(n > 4)$ and $K_{1,n}(n > 5)$ are not geometric mean graphs. Authors proved $C_m \cup P_n$; $C_m \cup C_n$; nK_3; $nK_3 \cup P_n$; $nK_3 \cup C_m$; P_n^2 and crowns are geometric mean graphs. Also investigated geometric mean labelings in the context of duplication of graph elements in cycle C_n and path P_n.

According to Beineke and Hegde, graph labeling serves as a frontier between number theory and structure of graphs. Authors in [4] proved that some disconnected graphs are harmonic mean graphs. In [14], harmonic mean labeling of a graph is introduced and also investigated that for a polygonal chain, square of the path and dragon are harmonic mean graph of order atmost 5.

Abundant literature exists as of today concerning the structure of graphs admitting a variety of function assigning real numbers to their elements so that given conditions are satisfied. Vertex functions is defined as $f : V(G) \to A, A \subseteq N$ for which the induced edge function $f^* : E(G) \to N$ is defined as $f^*(uv) = \left\lceil \frac{f(u)+\sqrt{f(u)f(v)}+f(v)}{3} \right\rceil$ or $f^*(uv) = \left\lfloor \frac{f(u)+\sqrt{f(u)f(v)}+f(v)}{3} \right\rfloor$ for every $f(u)f(v) \in E(G)$ are all distinct such graphs are known as Heron mean graphs. In [10] Heron mean labeling of path, cycle, $K_{1,n}$, if and only if $n < 3, C_m \cup P_n, C_m \cup C_n, nK_3, nK_3 \cup P_m, nK_3 \cup C_m, mC_4$, crown $C\Theta K_1$, Dragons $C_n@P_m$, Square graph of path P_n^2, polygonal chain $G_{m,n}$ are discussed.

Definition 1.1. A graph G is a pair (V, E), where V is a nonempty set and E is a set of unordered pairs of elements taken from the set V. A graph which does not contain loops and multiple edges is a simple graph, a finite number of vertices and edges in a graph is a finite graph and undirected with p vertices and q edges. The cardinality of vertex set V of a graph is the order and the cardinality of edge set E is called the size of the graph G. The graph $G - e$ is obtained from G by deleting an edge e. For other terminology and notations refer [3].

Definition 1.2. The middle graph $M(G)$ of a graph G whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent if and only if either they are adjacent edges of G (or) one is a vertex of G and the other is an edge incident on it.

Definition 1.3. The Total graph $T(G)$ of graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent whenever they are either adjacent (or) incident in G.

Definition 1.4. The splitting graph $S^{'}(G)$ is obtained by adding new vertex $V^{'}$ corresponding to each vertex V of G such that $N(V) = N(V^{'})$, where $N(V)$ and $N(V^{'})$ are the neighbourhood sets of V and $V^{'}$ respectively.

Definition 1.5. Let G be a connected graph and $G^{'}$ be the copy of G. Then shadow graph $D_2(G)$ is obtained by joining each vertex u in G to the neighbours of the corresponding vertex $u^{'}$ in $G^{'}$.

Definition 1.6. Duplication of a vertex V_k by a new edge $e = v_k^{'}v_k^{''}$ in a graph G produces a new graph $G^{'}$ such that $N(V_k^{'}) \cap N(V_k^{''}) = V_k$.

2. Centroidal mean labeling of a graph

In this section the centroidal mean labeling of graphs containing cycles such as triangular snake $T_n \bigodot K_1$, double triangular snake $D_n(T_n) \bigodot K_1$ and $TL_n \bigodot K_1$ are discussed using the following definition. Further the centroidal mean labeling of graphs are obtained by attaching pendant edges to both sides of each vertex of a path P_n, graph obtained by attaching paths of lengths $0, 1, \dots, n-1$ on both sides of each vertex of P_n, graph $D_2(P_n)$, middle graph of path P_n, total graph of path P_n, splitting graph of path P_n and the graph obtained by duplicating each vertex by an edge in path P_n are discussed using the centroidal mean labeling technique.

Centroidal mean labeling of graph:[10] A Graph G with n vertices and m edges is called a centroidal mean graph if the function $f : V(G) \to A \subseteq N$ to label the vertices $x \in V(G)$ with distinct labels $f(x)$, and each edge $e = x_i x_j$ is labeled with

$$f^*(x_i x_j) = \left\lceil \frac{2[(f(x_i))^2 + f(x_i)f(x_j) + (f(x_j))^2]}{3(f(x_i) + f(x_j))} \right\rceil$$

or

$$f^*(x_i x_j) = \left\lfloor \frac{2[(f(x_i))^2 + f(x_i)f(x_j) + (f(x_j))^2]}{3(f(x_i) + f(x_j))} \right\rfloor$$

for every $x_i, x_j \in V(G)$ and $x_i \neq x_j$ are all distinct.

Theorem 2.1. *A Triangular snake $T_n \bigodot K_1$ is centroidal mean graph.*

Proof. Consider a triangular snake T_n. Let u_i, v_i be the vertices of a triangular snake. Join $u_i v_i$ and $u_{i+1} v_i$. Let w_i, x_i be the pendent vertices. Join $u_i w_i$ and $v_i x_i$ $1 \leq i \leq n$; The function $f : V(T_n \bigodot K_1) \to 1, 2, 3, \dots, (p+q)$ is defined by

$f(u_i) = 9i - 8, \quad 1 \leq i \leq n$

$f(v_i) = 9i - 1, \quad 1 \leq i \leq n-1$

$f(w_i) = 9i - 6, \quad 1 \leq i \leq n$

$f(x_i) = 9i - 3, \quad 1 \leq i \leq n-1$

Edges are labeled with the induced function $f^* : E(G) \to N$ given by

$$f^*(x_i x_j) = \left\lceil \frac{2[(f(x_i))^2 + f(x_i)f(x_j) + (f(x_j))^2]}{3(f(x_i) + f(x_j))} \right\rceil$$

or

$$f^*(x_i x_j) = \left\lfloor \frac{2[(f(x_i))^2 + f(x_i)f(x_j) + (f(x_j))^2]}{3(f(x_i) + f(x_j))} \right\rfloor$$

for every $x_i, x_j \in V(G)$ and $x_i \neq x_j$ as follows:
The edge u_1, u_2 is labeled by

$$f^*(u_1 u_2) = \left\lfloor \frac{2[(f(u_1))^2 + f(u_1)f(u_2) + (f(u_2))^2]}{3(f(u_1) + f(u_2))} \right\rfloor$$

The edges u_i, u_{i+1} are labeled by

$$f^*(u_i u_{i+1}) = \left\lfloor \frac{2[(f(u_i))^2 + f(u_i)f(u_{i+1}) + (f(u_{i+1}))^2]}{3(f(u_i) + f(u_{i+1}))} \right\rfloor = 9i-3; \quad 2 \leq i \leq n-1$$

The edge u_1, v_1 is labeled by

$$f^*(u_1v_1) = \left\lfloor \frac{2[(f(u_1))^2 + f(u_1)f(v_1) + (f(v_1))^2]}{3(f(u_1) + f(v_1))} \right\rfloor$$

The edges u_i, v_i are labeled by

$$f^*(u_iv_i) = \left\lceil \frac{2[(f(u_i))^2 + f(u_i)f(v_i) + (f(v_i))^2]}{3(f(u_i) + f(v_i))} \right\rceil = 9i-4; \quad 2 \leq i \leq n-1.$$

The edges u_{i+1}, v_i are labeled by

$$f^*(u_{i+1}v_i) = \left\lfloor \frac{2[(f(u_{i+1}))^2 + f(u_{i+1})f(v_i) + (f(v_i))^2]}{3(f(u_{i+1}) + f(v_i))} \right\rfloor = 9i; \quad 1 \leq i \leq n-1.$$

The edges u_i, w_i are labeled by

$$f^*(u_iw_i) = \left\lfloor \frac{2[(f(u_i))^2 + f(u_i)f(w_i) + (f(w_i))^2]}{3(f(u_i) + f(w_i))} \right\rfloor = 9i-7; \quad 1 \leq i \leq n.$$

The edges v_i, x_i are labeled by

$$f^*(v_ix_i) = \left\lfloor \frac{2[(f(v_i))^2 + f(v_i)f(x_i) + (f(x_i))^2]}{3(f(v_i) + f(x_i))} \right\rfloor = 9i-2; \quad 1 \leq i \leq n-1,$$

are all distinct. The function f^* admits a centroidal mean labeling. Hence the graph $T_n \bigodot K_1$ is centroidal mean graph. □

Illustration: Consider the triangular snake $T_3 \bigodot K_1$. The following figure shows the centroidal mean labeling of a graph.

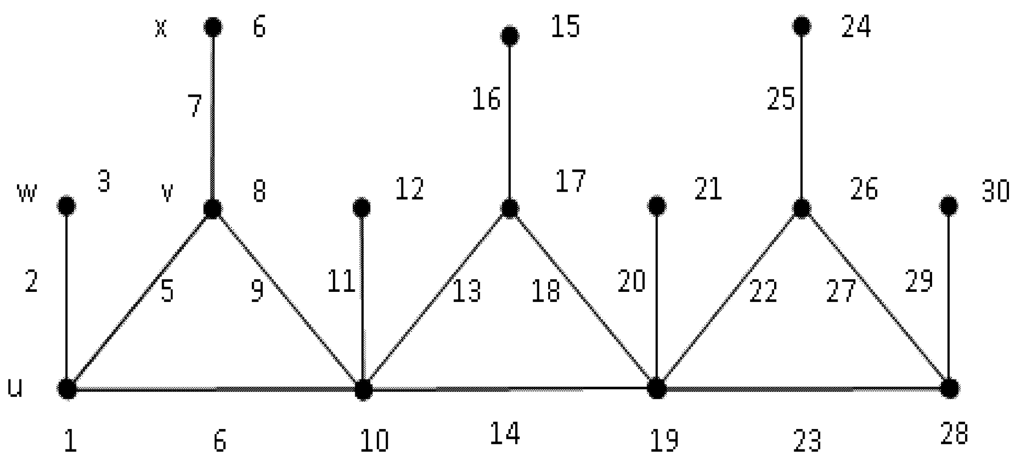

FIGURE 1. Triangular snake $T_3 \bigodot K_1$

Theorem 2.2. *Double triangular snake $D(T_n) \bigodot K_1$ is a centroidal mean graph.*

Proof. Let $D(T_n)$ be a Double triangular snake. Let u_i, v_i, w_i be the vertices of Double triangular snake. Join $u_iv_i, u_iw_i, u_{i+1}w_i, u_{i+1}v_i$. Let x_i, y_i and s_i, t_i be the pendent vertices. Join u_ix_i, v_iy_i, u_is_i and w_it_i, $\quad 1 \le i \le n, \quad 1 \le i \le (n-1)$.
Define a function $f : V(D(T_n) \bigodot K_1) \to 1, 2, 3, \dots, p+q$ by

$f(u_i) = 16i{-}15, \quad 1 \le i \le n$

$f(v_i) = 16i{-}7, \quad 1 \le i \le n{-}1$

$f(w_i) = 16i{-}1, \quad 1 \le i \le n{-}1$

$f(x_i) = 16i{-}12, \quad 1 \le i \le n$

$f(y_i) = 16i{-}5, \quad 1 \le i \le n{-}1$

$f(s_i) = 16i{-}11, \quad 1 \le i \le n$

$f(t_i) = 16i{-}3, \quad 1 \le i \le n{-}1$

Edges are labeled with the induced function $f^* : E(G) \to N$ given by

$$f^*(x_ix_j) = \left\lceil \frac{2[(f(x_i))^2 + f(x_i)f(x_j) + (f(x_j))^2]}{3(f(x_i) + f(x_j))} \right\rceil$$

or

$$f^*(x_ix_j) = \left\lfloor \frac{2[(f(x_i))^2 + f(x_i)f(x_j) + (f(x_j))^2]}{3(f(x_i) + f(x_j))} \right\rfloor$$

for every $x_i, x_j \in V(G)$ and $x_i \neq x_j$.
The edges u_i, u_{i+1} are labeled by

$$f^*(u_iu_{i+1}) = \left\lceil \frac{2[(f(u_i))^2 + f(u_i)f(u_{i+1}) + (f(u_{i+1}))^2]}{3(f(u_i) + f(u_{i+1}))} \right\rceil = 16i{-}8; \quad 1 \le i \le n{-}1$$

The edges u_i, v_i are labeled by $f^*(u_1v_1) = \left\lceil \frac{2[(f(u_1))^2 + f(u_1)f(v_1) + (f(v_1))^2]}{3(f(u_1)+f(v_1))} \right\rceil = 6$ and

$$f^*(u_iv_i) = \left\lfloor \frac{2[(f(u_i))^2 + f(u_i)f(v_i) + (f(v_i))^2]}{3(f(u_i) + f(v_i))} \right\rfloor = 16i{-}10; \quad 2 \le i \le n{-}1$$

The edges v_i, u_{i+1} are labeled by

$$f^*(v_iu_{i+1}) = \left\lceil \frac{2[(f(v_i))^2 + f(v_i)f(u_{i+1}) + (f(v_{i+1}))^2]}{3(f(v_i) + f(u_{i+1}))} \right\rceil = 16i{-}3; \quad 1 \le i \le n{-}1$$

The edges u_i, w_i are labeled by

$$f^*(u_1w_1) = \left\lceil \frac{2[(f(u_1))^2 + f(u_1)f(w_1) + (f(w_1))^2]}{3(f(u_1) + f(w_1))} \right\rceil = 10$$

$$f^*(u_iw_i) = \left\lceil \frac{2[(f(u_i))^2 + f(u_i)f(w_i) + (f(w_i))^2]}{3(f(u_i) + f(w_i))} \right\rceil = 16i{-}9; \quad 2 \le i \le n{-}1$$

The edges u_{i+1}, w_i are labeled by

$$f^*(u_{i+1}w_i) = \left\lceil \frac{2[(f(u_{i+1}))^2 + f(u_{i+1})f(w_i) + (f(w_i))^2]}{3(f(u_{i+1}) + f(w_i))} \right\rceil = 16i; \quad 1 \le i \le n{-}1$$

The edges u_i, x_i are labeled by

$$f^*(u_ix_i) = \left\lceil \frac{2[(f(u_i))^2 + f(u_i)f(x_i) + (f(x_i))^2]}{3(f(u_i) + f(x_i))} \right\rceil = 16i{-}14; \quad 1 \le i \le n$$

The edges u_i, s_i are labeled by

$$f^*(u_i s_i) = \left\lceil \frac{2[(f(u_i))^2 + f(u_i)f(s_i) + (f(s_i))^2]}{3(f(u_i) + f(s_i))} \right\rceil = 16i{-}13; \quad 1 \le i \le n$$

The edges v_i, y_i are labeled by

$$f^*(v_i y_i) = \left\lceil \frac{2[(f(v_i))^2 + f(v_i)f(y_i) + (f(y_i))^2]}{3(f(v_i) + f(y_i))} \right\rceil = 16i{-}6; \quad 1 \le i \le n{-}1$$

The edges w_i, t_i are labeled by

$$f^*(w_i t_i) = \left\lceil \frac{2[(f(w_i))^2 + f(w_i)f(t_i) + (f(t_i))^2]}{3(f(w_i) + f(t_i))} \right\rceil = 16i{-}2; \quad 1 \le i \le n{-}1.$$

Therefore f^* is injective and f^* admits centroidal mean labeling on $D(T_n) \odot K_1$. Hence the graph $D(T_n) \odot K_1$ is a centroidal mean graph. □

Illustration: Consider the $D(T_3) \odot K_1$ The labeling is as shown in figure.

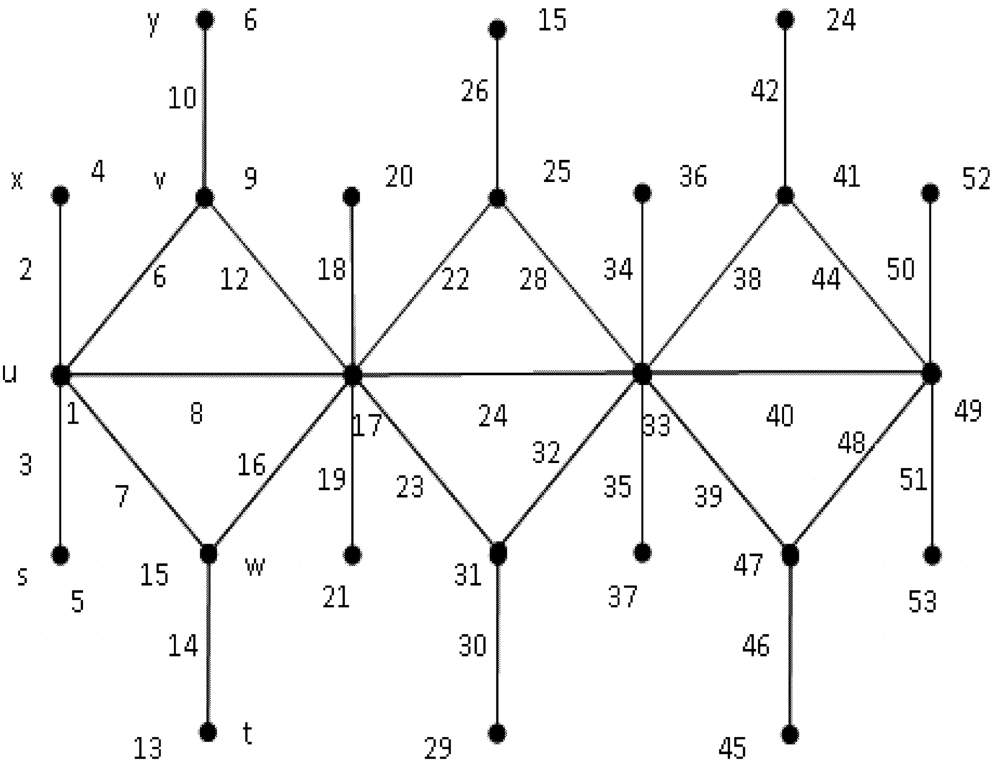

FIGURE 2. The graph $D(T_3) \odot K_1$

Theorem 2.3. *The $TL_n \bigodot K_1$ is a centroidal mean graph.*

Proof. Let TL_n be a triangular ladder. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be two paths of lengths n in the graph $TL_n \bigodot K_1$. Join $u_i v_{i+1}$, $\quad 1 \leq i \leq n-1$. Let w_i and x_i be the pendant vertices. Join $v_i w_i$ and $u_i x_i$.
Define a function $f : V(TL_n \bigodot K_1) \to 1, 2, 3, \ldots, (p+q)$ by

$f(u_i) = 10i{-}3, \quad 1 \leq i \leq n$

$f(v_i) = 10i{-}5, \quad 1 \leq i \leq n$

$f(w_1) = 1, \quad f(w_i) = 10i{-}12, \quad 2 \leq i \leq n$

$f(x_i) = 10i{-}7, \quad 1 \leq i \leq n$

Edges are labeled with the induced function $f^* : E(G) \to N$ given by

$$f^*(x_i x_j) = \left\lceil \frac{2[(f(x_i))^2 + f(x_i)f(x_j) + (f(x_j))^2]}{3(f(x_i) + f(x_j))} \right\rceil$$

or

$$f^*(x_i x_j) = \left\lfloor \frac{2[(f(x_i))^2 + f(x_i)f(x_j) + (f(x_j))^2]}{3(f(x_i) + f(x_j))} \right\rfloor$$

for every $x_i, x_j \in V(G)$ and $x_i \neq x_j$.
The edges u_i, v_i are labeled by

$$f^*(u_i v_i) = \left\lceil \frac{2[(f(u_i))^2 + f(u_i)f(v_i) + (f(v_i))^2]}{3(f(u_i) + f(v_i))} \right\rceil = 10i{-}4; \quad 1 \leq i \leq n$$

The edges u_i, u_{i+1} are labeled by

$$f^*(u_i u_{i+1}) = \left\lceil \frac{2[(f(u_i))^2 + f(u_i)f(u_{i+1}) + (f(u_{i+1}))^2]}{3(f(u_i) + f(u_{i+1}))} \right\rceil = 10i{+}2; \quad 1 \leq i \leq (n{-}1)$$

The edges v_i, v_{i+1} are labeled by

$$f^*(v_i v_{i+1}) = \left\lceil \frac{2[(f(v_i))^2 + f(v_i)f(v_{i+1}) + (f(v_{i+1}))^2]}{3(f(v_i) + f(v_{i+1}))} \right\rceil = 10i; \quad 1 \leq i \leq (n{-}1)$$

The edges v_i, u_{i+1} are labeled by $f^*(v_1 u_2) = \left\lceil \frac{2[(f(v_1))^2 + f(v_1)f(u_2) + (f(u_2))^2]}{3(f(v_1) + f(u_2))} \right\rceil = 12$ and

$$f^*(v_i u_{i+1}) = \left\lfloor \frac{2[(f(v_i))^2 + f(v_i)f(u_{i+1}) + (f(u_{i+1}))^2]}{3(f(v_i) + f(u_{i+1}))} \right\rfloor = 10i{+}2; \quad 2 \leq i \leq (n{-}1)$$

The edges v_i, w_i are labeled by $f^*(v_1 w_1) = 2$.

$$f^*(v_i w_i) = \left\lceil \frac{2[(f(v_i))^2 + f(v_i)f(w_i) + (f(w_i))^2]}{3(f(v_i) + f(w_i))} \right\rceil = 10i - 9; \quad 2 \leq i \leq n,$$

The edges u_i, x_i are labeled by

$$f^*(u_i x_i) = \left\lceil \frac{2[(f(u_i))^2 + f(u_i)f(x_i) + (f(x_i))^2]}{3(f(u_i) + f(x_i))} \right\rceil = 10i{-}5; \quad 1 \leq i \leq n,$$

are all distinct. Therefore the function f^* is admits centroidal mean labeling. Hence the graph $TL_n \bigodot K_1$ is a centroidal mean graph. □

Illustration: The following figure shows the centroidal mean labeling of $TL_5 \bigodot K_1$.

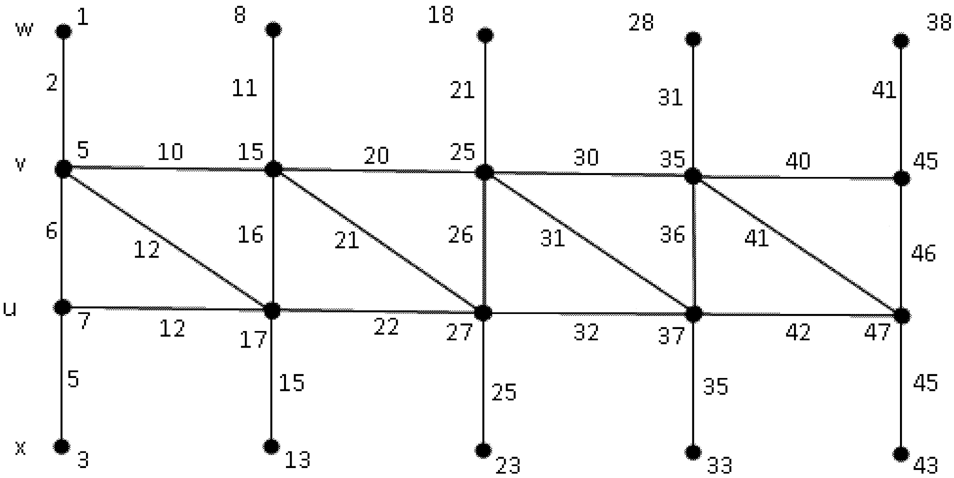

FIGURE 3. Graph $TL_5 \odot K_1$.

Theorem 2.4. *Let G be a graph obtained by attaching pendant edges to both sides of each vertex of a path P_n. Then G is a centroidal mean graph.*

Proof. Consider a graph G obtained by attaching pendant edges to both sides of each vertex of a path P_n. Let $u_1, u_2, \ldots, u_n$ be a path P_n. Let v_i and w_i be the pendant vertices adjacent to u_i, $1 \le i \le n$.
The function $f : V(G) \to 1, 2, 3, \ldots, (q+1)$ is defined by

$f(u_i) = 13i - 7, \quad 1 \le i \le n$

$f(v_i) = 13i - 12, \quad 1 \le i \le n$

$f(w_i) = 13i - 2, \quad 1 \le i \le n$

Edges are labeled with the induced function $f^* : E(G) \to N$ given by

$$f^*(x_i x_j) = \left\lceil \frac{2[(f(x_i))^2 + f(x_i)f(x_j) + (f(x_j))^2]}{3(f(x_i) + f(x_j))} \right\rceil$$

or

$$f^*(x_i x_j) = \left\lfloor \frac{2[(f(x_i))^2 + f(x_i)f(x_j) + (f(x_j))^2]}{3(f(x_i) + f(x_j))} \right\rfloor$$

for every $x_i, x_j \in V(G)$ and $x_i \ne x_j$.
The edges u_i, u_{i+1} are labeled by

$$f^*(u_i u_{i+1}) = \left\lfloor \frac{2[(f(u_i))^2 + f(u_i)f(u_{i+1}) + (f(u_{i+1}))^2]}{3(f(u_i) + f(u_{i+1}))} \right\rfloor = 13i - 1; \quad 1 \le i \le (n-1).$$

The edges u_i, v_i are labeled by

$$f^*(u_i v_i) = \left\lceil \frac{2[(f(u_i))^2 + f(u_i)f(v_i) + (f(v_i))^2]}{3(f(u_i) + f(v_i))} \right\rceil = 13i - 9; \quad 2 \le i \le n.$$

and

$$f^*(u_1v_1) = \left\lfloor \frac{2[(f(u_1))^2 + f(u_1)f(v_1) + (f(v_1))^2]}{3(f(u_1) + f(v_1))} \right\rfloor = 4$$

The edges u_i, w_i are labeled by

$$f^*(u_iw_i) = \left\lfloor \frac{2[(f(u_i))^2 + f(u_i)f(w_i) + (f(w_i))^2]}{3(f(u_i) + f(w_i))} \right\rfloor = 13i{-}5; \quad 1 \le i \le n.$$

are all distinct. The function f^* admits centroidal mean labeling. Hence the graph G is a centroidal mean graph. □

Illustration: A centroidal mean labeling of G with 15 vertices and 14 edges is shown in figure-1.

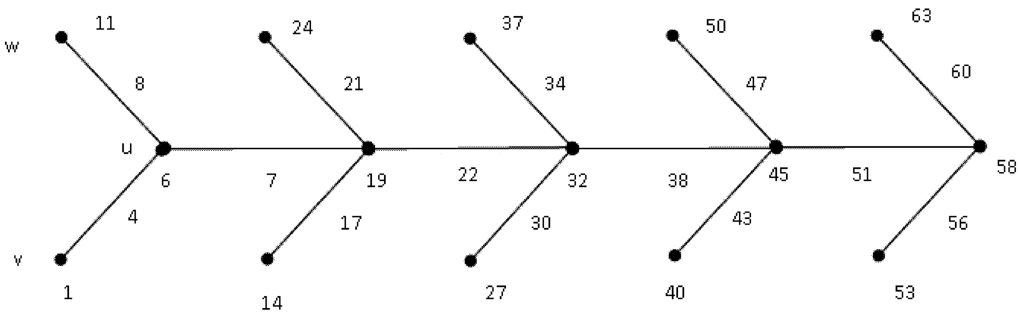

FIGURE 4.

Theorem 2.5. *Let G be a graph obtained by attaching paths of lengths $0, 1, \ldots, n-1$ on both sides of each vertex of P_n. Then G is a centroidal mean graph.*

Proof. Let G be a graph obtained by attaching paths of length $0, 1, 2, \ldots, n-1$ on both sides by each vertex of P_n. Let $u_{11}, u_{22}, u_{33}, \ldots, u_{nn}$ are the vertices of the path P_n. Define a function $f : V(G) \to 1, 2, 3, \ldots, q+1$ by
$f(u_{ij}) = (i-1)^2 + j,\ 1 \le i \le n, \quad 1 \le j \le 2i-1$
From the above vertex labeling pattern, we get distinct edge labels with the induced function $f^* : E(G) \to N$ given by

$$f^*(x_ix_j) = \left\lceil \frac{2[(f(x_i))^2 + f(x_i)f(x_j) + (f(x_j))^2]}{3(f(x_i) + f(x_j))} \right\rceil$$

or

$$f^*(x_ix_j) = \left\lfloor \frac{2[(f(x_i))^2 + f(x_i)f(x_j) + (f(x_j))^2]}{3(f(x_i) + f(x_j))} \right\rfloor$$

for every $x_i, x_j \in V(G)$ and $x_i \neq x_j$.The function f^* provides a centroidal mean labeling. Hence given graph is a centroidal mean graph. □

Illustration: A centroidal mean labeling of $P_6(p_1, 2p_2, 2p_3, 2p_4, 2p_5, 2p_6)$ is as shown in figure-2.

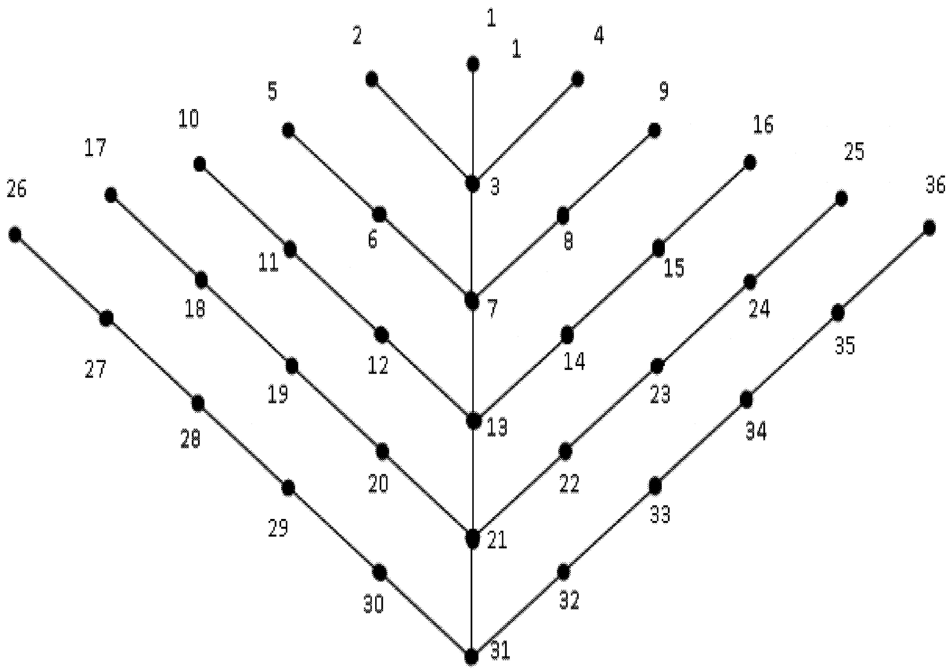

Figure 5. The graph $P_6(p_1, 2p_2, 2p_3, 2p_4, 2p_5, 2p_6)$

Theorem 2.6. *The middle graph of path P_n is a centroidal mean graph.*

Proof. Let $u_1, u_2, \dots, u_n$ be the vertices and $e_1, e_2, \dots, e_{n-1}$ be the edges of path P_n and $G = M(P_n)$ be the Middle graph of the path P_n.
Define a function $f : V(G) \to 1, 2, 3, \dots (q+1)$ by $f(u_1) = 1$, $f(u_i) = 3i - 3$, $2 \le i \le n$ and $f(v_i) = 3i - 1$, $1 \le i \le n-1$.
Edges are labeled with the induced function $f^* : E(G) \to N$ given by

$$f^*(x_i x_j) = \left\lceil \frac{2[(f(x_i))^2 + f(x_i)f(x_j) + (f(x_j))^2]}{3(f(x_i) + f(x_j))} \right\rceil$$

or

$$f^*(x_i x_j) = \left\lfloor \frac{2[(f(x_i))^2 + f(x_i)f(x_j) + (f(x_j))^2]}{3(f(x_i) + f(x_j))} \right\rfloor$$

for every $x_i, x_j \in V(G)$ and $x_i \neq x_j$.
The edges u_i, v_i are labeled with

$$f^*(u_i v_i) = \left\lfloor \frac{2[(f(u_i))^2 + f(u_i)f(v_i) + (f(v_i))^2]}{3(f(u_i) + f(v_i))} \right\rfloor = 3i-2; \quad 1 \le i \le n-1$$

The edges v_i, u_{i+1} are labeled with

$$f^*(v_i u_{i+1}) = \left\lfloor \frac{2[(f(v_i))^2 + f(v_i)f(u_{i+1}) + (f(u_{i+1}))^2]}{3(f(v_i) + f(u_{i+1}))} \right\rfloor = 3i-1; \quad 1 \le i \le n$$

The edges v_i, v_{i+1} are labeled with

$$f^*(v_iv_{i+1}) = \left\lfloor \frac{2[(f(v_i))^2 + f(v_i)f(v_{i+1}) + (f(v_{i+1}))^2]}{3(f(v_i) + f(v_{i+1}))} \right\rfloor = 3i; \quad 1 \le i \le n-2$$

are all distinct. Therefore the function f^* admits centroidal mean labeling. Hence the graph $M(P_n)$ is a centroidal mean graph. □

Illustration: A centroidal mean labeling of middle graph $M(P_n)$ is shown in the following figure.

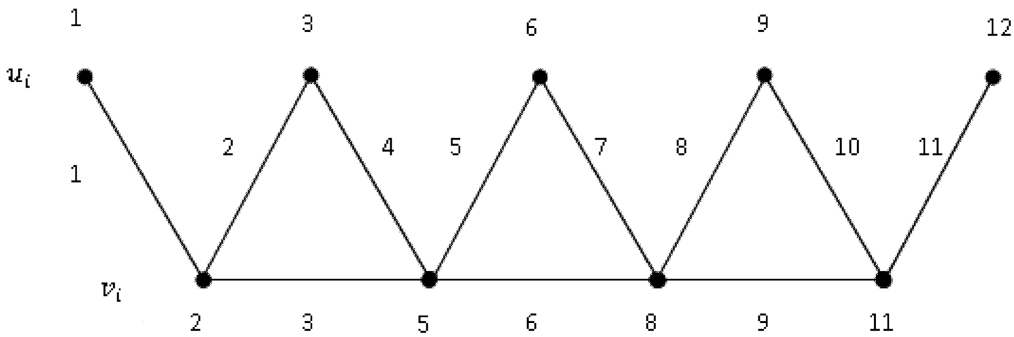

FIGURE 6. The middle graph $M(P_n)$

Theorem 2.7. *The total graph of path P_n is a centroidal mean graph.*

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices and $e_1, e_2, \ldots, e_{n-1}$ be the edges of path P_n and $G = T(P_n)$ be the total graph of the path P_n. Here $|V(G)| = 2n - 1$ and $|E(G)| = 4n - 5$.
Define a function $f : V(G) \to 1, 2, 3, \ldots, (4n - 1)$ by
$f(u_i) = 4i - 1$, $1 \le i \le n$ and $f(v_i) = 4i - 2$, $2 \le i \le n - 1$.
Edges are labeled with the induced function $f^* : E(G) \to N$ given by

$$f^*(x_ix_j) = \left\lceil \frac{2[(f(x_i))^2 + f(x_i)f(x_j) + (f(x_j))^2]}{3(f(x_i) + f(x_j))} \right\rceil$$

or

$$f^*(x_ix_j) = \left\lfloor \frac{2[(f(x_i))^2 + f(x_i)f(x_j) + (f(x_j))^2]}{3(f(x_i) + f(x_j))} \right\rfloor$$

for every $x_i, x_j \in V(G)$ and $x_i \ne x_j$ as follows:
The edges u_i, u_{i+1} are labeled with

$$f^*(u_iu_{i+1}) = \left\lceil \frac{2[(f(u_i))^2 + f(u_i)f(u_{i+1}) + (f(u_{i+1}))^2]}{3(f(u_i) + f(u_{i+1}))} \right\rceil = 4i + 2; \quad 1 \le i \le n - 1$$

The edges u_i, v_i are labeled with

$$f^*(u_iv_i) = \left\lceil \frac{2[(f(u_i))^2 + f(u_i)f(v_i) + (f(v_i))^2]}{3(f(u_i) + f(v_i))} \right\rceil = 4i-1; \quad 1 \le i \le n-1$$

The edges v_i, u_{i+1} are labeled with

$$f^*(v_i u_{i+1}) = \left\lceil \frac{2[(f(v_i))^2 + f(v_i)f(u_{i+1}) + (f(u_{i+1}))^2]}{3(f(v_i) + f(u_{i+1}))} \right\rceil = 4i{+}1; \quad 1 \le i \le n{-}1$$

The edges v_i, v_{i+1} are labeled with

$$f^*(v_i v_{i+1}) = \left\lfloor \frac{2[(f(v_i))^2 + f(v_i)f(v_{i+1}) + (f(v_{i+1}))^2]}{3(f(v_i) + f(v_{i+1}))} \right\rfloor = 4i; \quad 1 \le i \le n{-}2$$

are all distinct. Therefore f^* admits centroidal mean labeling. Hence the graph $T(P_n)$ is a centroidal mean graph. □

Illustration: The total graph of path P_6 and its centroidal mean labeling is shown in the following figure.

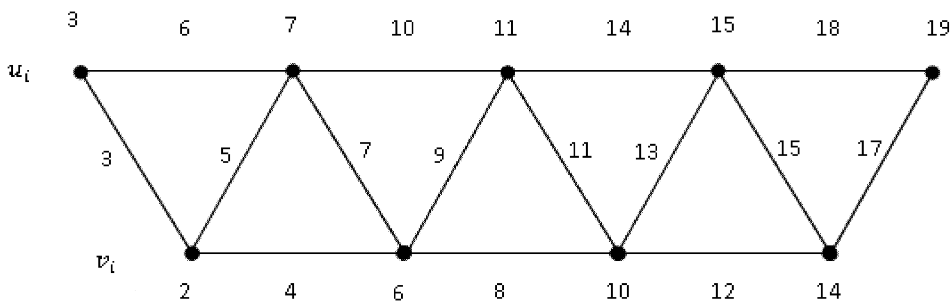

FIGURE 7. The total graph of path P_6

Theorem 2.8. *The splitting graph of path P_n is a centroidal mean graph.*

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices and $e_1, e_2, \ldots, e_{n-1}$ be the edges of path P_n. Let $v_1, v_2, \ldots, v_n$ be the newly added vertices to form the splitting graph of path P_n. Here $|V(G)| = 2n$ and $|E(G)| = 3n - 3$.
Define a function $f : V(G) \to 1, 2, 3, \ldots, (q + 1)$ by $f(u_i) = 3i - 1, \quad 1 \le i \le n$ and $f(v_i) = 3i - 2, \quad 1 \le i \le n$.
Edges are labeled with the induced function $f^* : E(G) \to N$ given by

$$f^*(x_i x_j) = \left\lceil \frac{2[(f(x_i))^2 + f(x_i)f(x_j) + (f(x_j))^2]}{3(f(x_i) + f(x_j))} \right\rceil$$

or

$$f^*(x_i x_j) = \left\lfloor \frac{2[(f(x_i))^2 + f(x_i)f(x_j) + (f(x_j))^2]}{3(f(x_i) + f(x_j))} \right\rfloor$$

for every $x_i, x_j \in V(G)$ and $x_i \neq x_j$.
The edges v_i, u_{i+1} are labeled with

$$f^*(v_i u_{i+1}) = \left\lceil \frac{2[(f(v_i))^2 + f(v_i)f(u_{i+1}) + (f(u_{i+1}))^2]}{3(f(v_i) + f(u_{i+1}))} \right\rceil = 3i + 1; \quad 1 \le i \le n - 1$$

The edges v_i, v_{i+1} are labeled with

$$f^*(v_iv_{i+1}) = \left\lfloor \frac{2[(f(v_i))^2 + f(v_i)f(v_{i+1}) + (f(v_{i+1}))^2]}{3(f(v_i) + f(v_{i+1}))} \right\rfloor = 3i - 1; \quad 1 \leq i \leq n-1$$

The edges u_i, v_{i+1} are labeled with

$$f^*(u_iv_{i+1}) = \left\lfloor \frac{2[(f(u_i))^2 + f(u_i)f(v_{i+1}) + (f(v_{i+1}))^2]}{3(f(u_i) + f(v_{i+1}))} \right\rfloor = 3i; \quad 1 \leq i \leq n-1$$

are all distinct. Therefore the function f^* admits centroidal mean labeling. Hence the splitting graph of path P_n is a centroidal mean graph. □

Illustration: The splitting graph of path P_4 and its centroidal mean labeling is shown in the following figure.

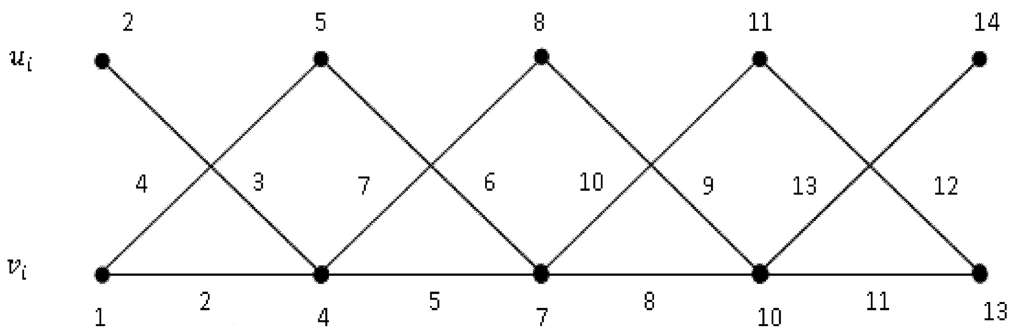

FIGURE 8. The splitting graph of path P_4

Theorem 2.9. *Duplicating each vertex by an edge in path P_n is a centroidal mean graph.*

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of a path P_n. Let G be the graph obtained by duplicating each vertex v_i of P_n by an edge $v_i^{'}v_i^{''}$ at a time $1 \leq i \leq n$. Here $|V(G)| \leq 3n$ and $|E(G)| = 4n - 1$.
Define a function $f : V(G) \to 1, 2, 3, \ldots, (q+1)$ by

$$f(u_i) = 4i - 2, \quad 1 \leq i \leq n$$

$$f(v_i^{'}) = 4i - 3, \quad 1 \leq i \leq n.$$

$$f(v_i^{''}) = 4i - 1, \quad 1 \leq i \leq n.$$

Edges are labeled with the induced function $f^* : E(G) \to N$ given by

$$f^*(x_ix_j) = \left\lceil \frac{2[(f(x_i))^2 + f(x_i)f(x_j) + (f(x_j))^2]}{3(f(x_i) + f(x_j))} \right\rceil$$

or

$$f^*(x_ix_j) = \left\lfloor \frac{2[(f(x_i))^2 + f(x_i)f(x_j) + (f(x_j))^2]}{3(f(x_i) + f(x_j))} \right\rfloor$$

for every $x_i, x_j \in V(G)$ and $x_i \neq x_j$.
The edges u_i, u_{i+1} are labeled with

$$f^*(u_iu_{i+1}) = \left\lfloor \frac{2[(f(u_i))^2 + f(u_i)f(u_{i+1}) + (f(u_{+1}))^2]}{3(f(u_i) + f(u_{+1}))} \right\rfloor = 4i; \quad 1 \leq i \leq (n{-}1)$$

The edges $u_i, v_i^{'}$ are labeled with

$$f^*(u_iv_i^{'}) = \left\lfloor \frac{2[(f(u_i))^2 + f(u_i)f(v_i^{'}) + (f(v_i^{'}))^2]}{3(f(u_i) + f(v_i^{'}))} \right\rfloor = 4i{-}3; \quad 1 \leq i \leq n$$

The edges $u_i, v_i^{''}$ are labeled with

$$f^*(u_iv_i^{''}) = \left\lceil \frac{2[(f(u_i))^2 + f(u_i)f(v_i^{''}) + (f(v_i^{''}))^2]}{3(f(u_i) + f(v_i^{''}))} \right\rceil = 4i{-}1; \quad 1 \leq i \leq n$$

The edges $v_i^{'}, v_i^{''}$ are labeled with

$$f^*(v_i^{'}v_i^{''}) = \left\lfloor \frac{2[(f(v_i^{'}))^2 + f(v_i^{'})f(v_i^{''}) + (f(v_i^{''}))^2]}{3(f(v_i^{'}) + f(v_i^{''}))} \right\rfloor = 4i{-}2; \quad 1 \leq i \leq n$$

are all distinct. Therefore the function f^* admits centroidal mean labeling. Hence the duplicating each vertex by an edge in path P_n is a centroidal mean graph □

Illustration: Duplicating each vertex by an edge in path P_4 and its centroidal mean labeling is shown in the following figure.

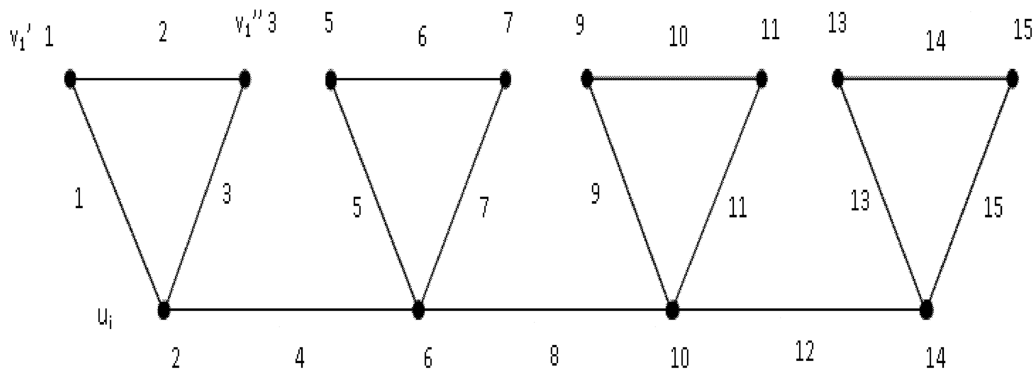

FIGURE 9. Duplicating each vertex by an edge in path P_4

Theorem 2.10. *The graph $D_2(P_n)$ is a centroidal mean graph.*

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of a path P_n and $v_1, v_2, \ldots, v_n$ be the newly added vertices corresponding to the vertices $u_1, u_2, \ldots, u_n$ in order to obtain $D_2(P_n)$. Denoting $G = D_2(P_n)$, then $|V(G)| = 2n$ and $|E(G)| = 4(n-1)$.
Define a function $f : V(G) \rightarrow 1, 2, 3, \ldots (q+1)$ by

$f(u_i) = 5i{-}4, \quad 1 \leq i \leq n \quad and \quad f(v_i) = 5i{-}1, \quad 1 \leq i \leq n$

Edges are labeled with the induced function $f^*: E(G) \to N$ given by

$$f^*(x_i x_j) = \left\lceil \frac{2[(f(x_i))^2 + f(x_i)f(x_j) + (f(x_j))^2]}{3(f(x_i) + f(x_j))} \right\rceil$$

or

$$f^*(x_i x_j) = \left\lfloor \frac{2[(f(x_i))^2 + f(x_i)f(x_j) + (f(x_j))^2]}{3(f(x_i) + f(x_j))} \right\rfloor$$

for every $x_i, x_j \in V(G)$ and $x_i \neq x_j$.
The edges u_i, u_{i+1} are labeled with

$$f^*(u_1 u_2) = \left\lfloor \frac{2[(f(u_1))^2 + f(u_1)f(u_2) + (f(u_2))^2]}{3(f(u_1) + f(u_2))} \right\rfloor = 4$$

and

$$f^*(u_i u_{i+1}) = \left\lceil \frac{2[(f(u_i))^2 + f(u_i)f(u_{i+1}) + (f(u_{i+1}))^2]}{3(f(u_i) + f(u_{i+1}))} \right\rceil = 5i - 1; \quad 2 \leq i \leq n - 1$$

The edges v_i, v_{i+1} are labeled with

$$f^*(v_i v_{i+1}) = \left\lceil \frac{2[(f(v_i))^2 + f(v_i)f(v_{i+1}) + (f(v_{i+1}))^2]}{3(f(v_i) + f(v_{i+1}))} \right\rceil = 5i+2; \quad 1 \leq i \leq n-1$$

The edges u_i, v_{i+1} are labeled with

$$f^*(u_i v_{i+1}) = \left\lfloor \frac{2[(f(u_i))^2 + f(u_i)f(v_{i+1}) + (f(v_{i+1}))^2]}{3(f(u_i) + f(v_{i+1}))} \right\rfloor = 5i+1; \quad 1 \leq i \leq n-1$$

The edges v_i, u_{i+1} are labeled with

$$f^*(v_i u_{i+1}) = \left\lfloor \frac{2[(f(v_i))^2 + f(v_i)f(u_{i+1}) + (f(u_{i+1}))^2]}{3(f(v_i) + f(u_{i+1}))} \right\rfloor = 5i; \quad 1 \leq i \leq n-1$$

are all distinct. Therefore the function f^* admits centroidal mean labeling. Hence the graph $D_2(P_n)$ is a centroidal mean graph. □

Illustration: The graph $D_2(P_5)$ and its centroidal mean labeling is shown in the following figure.

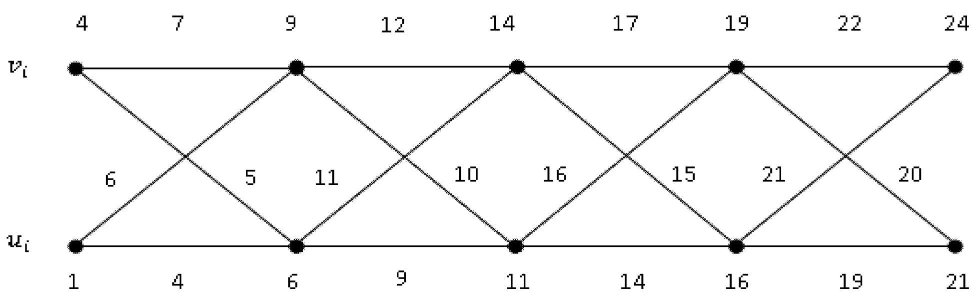

FIGURE 10. The graph $D_2(P_5)$

3. Acknowledgement

The authors are thankful to the referees for their valuable suggestions and fruitful comments.

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