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Entire Zagreb index of vertex and edge F-join of graphs

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Abstract

Graph theoretical operations are used to obtain combinatorial properties of large graphs in terms of smaller graphs. In this article, we deal with one of the new operations related to the vertex and edge F-join of graphs and calculate the first entire Zagreb index of the vertex and edge F-join of graphs.

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Keywords: F-index, entire Zagreb index, F-join graph, graph operations.

Dedicated to Professor Chandrashekar Adiga on his 62^{nd} Birthday.

1 Prelude

A topological index is a numerical quantity which is basically derived from mathematics and particular manner from the structural graph of a molecule. Several aspects of a chemical compound are closely associated to some topological indices of its molecular graph. The Wiener index is the first and the most studied distance-based topological index in chemical graph theory. This index was first coined by the chemist H. Wiener in 1947 to demonstrate correlations between physio-chemical properties of organic compounds and the topological structure of their molecular graphs. Topological indices are oftenly used for correlation with physical properties in quantitative structure-property/activity relationship (QSPR/QSAR) [2].

The zagreb indices have been evolved extensively due to their ease of calculation and their numerous applications in place of the existing chemical methods which required more time and high costs [3, 6, 7, 12].

The first and second Zagreb indices were introduced more than three decades ago by Gutman and Trinajstic [5] which are defined as,

$$M_1(G) = \sum_{u\in V(G)} d(u)^2 = \sum_{uv\in E(G)} [d(u) + d(v)].$$

and

$$M_2(G) = \sum_{uv \in E(G)} [d(u)d(v)].$$

The F-index [4] of a graph G is defined as,

$$F(G) = \sum_{u \in V(G)} d(u)^3.$$

The first entire Zagreb index is defined by

$$M_1^{\varepsilon}(G) = \sum_{u \in V(G) \cup E(G)} (deg(u))^2$$

Due to many reasons recently, many innovations have been made on Zagreb indices. Anwar alwardi et al., [1] have coined the entire Zagreb indices by adding incidence of edges and vertices to the adjacency of the vertices. The main moto of investigation of entire Zagreb indices was: the intermolecular forces do not only exist between the atoms, but also between the atoms and bonds, Therefore one must also take into account the relations between edges and vertices in addition to the relations between vertices to obtain better approximations to intermolecular forces. They got exact values of these entire Zagreb indices for several graph families and even for some graph operations and also established few essential properties of the entire Zagreb indices.

2 Results

Motivated by Anwar [1], we derived some closed formula of the first entire Zagreb index of graphs based on new operations related to the vertex and edge F-join of graphs in terms of different topological indices.

We recall some of the results of P. Sarkar et al., [10] as a Lemmas it will be essential to establish our results.

Lemma 1. *If G and H be two connected graphs with $|V(G)| = p_1$, $|V(H)| = p_2$ and $|E(G)| = q_1$, $|E(H)| = q_2$ then*

$$M_1(GV_SH) = M_1(G) + M_1(H) + 4p_2q_1 + 4p_1q_2 + p_1^2p_2 + p_1p_2^2 + 4q_1$$
$$M_2(GV_SH) = 2M_1(G) + p_1M_1(H) + M_2(H) + 4q_1(q_2 + p_2) + 2p_1p_2(q_1 + q_2) + p_1^2(q_2 + p_2^2).$$

Lemma 2. *[1] For any graph G the first entire Zagreb index is*

$$M_1^{\varepsilon}(G) = 4q - 3M_1(G) + 2M_2(G) + \sum_{u \in V(G)} (d_G(u))^3.$$

Theorem 1. *Consider two graphs G and H, the first entire zagreb index of GV_SH is*

$$\begin{aligned} M_1^{\varepsilon}(GV_SH) = {} & M_1^{\varepsilon}(G) + M_1^{\varepsilon}(H) + 4M_1(G) - 2M_2(G) + 5p_1M_1(H) + 3p_2M_1(G) \\ & + 4p_1p_2(q_1 + q_2 + 1) + 2p_2q_1(3p_2 - 2) + 8p_1^2q_2 + p_1p_2(p_1^2 + p_2^2) \\ & - 3p_1p_2(p_1 + p_2) + 8q_1q_2 + 2p_1^2p_2^2 - 12p_1q_2. \end{aligned}$$

Proof. Let G and H be two graphs then by Lemma 2.

$$M_1^{\varepsilon}(GV_SH) = 4q_{GV_SH} - 3M_1(GV_SH) + 2M_2(GV_SH) + \sum_{u\in V(GV_SH)} (d_{GV_SH}(u))^3. \quad (1)$$

Since $q_{GV_SH} = 2q_1 + q_2 + p_1p_2$ and $M_1(GV_SH)$, $M_2(GV_SH)$ are obtained from Lemma 1, it remain to calculate

$$\begin{aligned}
\sum_{u\in V(GV_SH)} (d_{GV_SH}(u))^3 &= \sum_{u\in V(G)} (d_G(u)+p_2)^3 + \sum_{u\in V(H)} (d_H(u)+p_1)^3 + \sum_{uv\in E(G)} (2)^3 \\
&= \sum_{u\in V(G)} (d_G(u))^3 + \sum_{u\in V(H)} (d_H(u))^3 + p_1p_2^3 + 3p_2M_1(G) \\
&+ 6p_2^2q_1 + p_1^3p_2 + 3p_1M_1(H) + 6p_1^2q_2 + 8q_1 \\
&= F(G) + F(H) + p_1p_2(p_1^2+p_2^2) + 3\big(M_1(G)p_2 + M_1(H)p_1\big) \\
&+ 6(p_1^2q_2 + p_2^2q_1) + 8q_1.
\end{aligned}$$

Therefore from the equation (1),

$$\begin{aligned}
M_1^{\varepsilon}(GV_SH) &= 8q_1 + 4q_2 + 4p_1p_2 - 3M_1(G) - 3M_1(H) - 12p_2q_1 - 12p_1q_2 - 3p_1^2p_2 - 3p_1p_2^2 \\
&- 12q_1 + 4M_1(G) + 2p_1M_1(H) + 2M_2(H) + 8q_1q_2 + 8q_1p_2 + 4p_1p_2(q_1+q_2) \\
&+ 2p_1^2(q_2+p_2^2) + F(G) + F(H) + p_1p_2(p_1)(p_1^2+p_2^2) + 3\big(M_1(G)p_2 + M_1(H)p_1\big) \\
&+ 6(p_1^2q_2 + p_2^2q_1) + 8q_1 \\
&= M_1^{\varepsilon}(G) + M_1^{\varepsilon}(H) + 4M_1(G) - 2M_2(G) + 5p_1M_1(H) + 3p_2M_1(G) \\
&+ 4p_1p_2(q_1+q_2+1) + 2p_2q_1(3p_2-2) + 8p_1^2q_2 + p_1p_2(p_1^2 + n_{2^2}) - 3p_1p_2(p_1+p_2) \\
&+ 8q_1q_2 + 2p_1^2p_2^2 - 12p_1q_2.
\end{aligned}$$

□

Lemma 3. *If G and H be two connected graphs with $|V(G)| = p_1$, $|V(H)| = p_2$ and $|E(G)| = q_1$, $|E(H)| = q_2$ then*

$$\begin{aligned}
M_1(GV_RH) &= 4M_1(G) + M_1(H) + 8p_2q_1 + 4p_1q_2 + p_2^2p_1 + p_2p_1^2 + 4q_1 \\
M_2(GV_RH) &= 4M_2(G) + M_2(H) + (2p_2+4)M_1(G) + p_1M_1(H) + p_2^2q_1 \\
&+ p_1^2q_2 + 2p_1p_2(2q_1+q_2) + 8q_1q_2 + p_1^2p_2^2 + 4q_1p_2.
\end{aligned}$$

Theorem 2. *For any graphs G and H, the first entire Zagreb index of GV_RH is*

$$\begin{aligned}
M_1^{\varepsilon}(GV_RH) &= 4M_1^{\varepsilon}(G) + M_1^{\varepsilon}(H) + 5p_1M_1(H) + 8M_1(G) + 4F(G_1) + 4p_1p_2(2q_1+q_2+1) \\
&+ 16p_2M_1(G) + 16q_1(q_2-p_2) - 12q_2p_1 - 8q_1 - 3p_1p_2(p_1+p_2) + 14q_1p_2^2 \\
&+ p_1p_2(p_1^2+p_2^2) + 2p_1^2(p_2^2+4q_2).
\end{aligned}$$

Proof. Let G and H be two graphs then by Lemma 2,

$$M_1^{\varepsilon}(GV_RH) = 4q_{GV_RH} - 3M_1(G_1V_RH) + 2M_2(GV_RH) + \sum_{u\in V(GV_RH)} (d_{GV_RH}(u))^3.$$

(2)

Since $q_{GV_RH} = 3q_1 + q_2 + p_1p_2$ and $M_1(GV_RH)$, $M_2(GV_RH)$ are obtained from Lemma 3, it remain to calculate

$$\begin{aligned}\sum_{u\in V(GV_RH)} (d_{GV_RH}(u))^3 &= \sum_{u\in V(G)} \big(2d_G(u)+p_2\big)^3 + \sum_{v\in V(H)} \big(d_H(v)+p_1\big)^3 + \sum_{uv\in E(G)} (2)^3\\ &= 8\sum_{u\in V(G)} (d_G(u))^3 + \sum_{u\in V(G)} (p_2)^3 + 12p_2 \sum_{u\in V(G)} (d_G(u))^2\\ &+ 6p_2^2 \sum_{u\in V(G)} d_G(u) + \sum_{v\in V(H)} (d_H(v))^3 + \sum_{v\in V(H)} (p_2)^3\\ &+ 3p_1 \sum_{v\in V(H)} (d_H(v))^2 + 3p_1^2 \sum_{v\in V(H)} d_H(v) + 8q_1\\ &= 8F(G) + F(H) + 3\big[p_1M_1(H) + 4(p_2M_1(G))\big] + 12q_1p_2^2\\ &+ 6p_1^2q_2 + p_1p_2(n_1^2 + p_2^2) + 8q_1.\end{aligned}$$

$$\begin{aligned}M_1^\varepsilon(GV_RH) &= 4\big[3q_1 + q_2 + p_1p_2\big] - 3\big[4M_1(G) + M_1(H) + 8q_1p_2 + 4p_1q_2 + p_1^2p_2\\ &+ p_1p_2^2 + 4q_1\big] + 2\big[4M_2(G) + M_2(H) + M_2(H) + (2p_2+4)M_1(G) + p_1M_1(H)\\ &+ p_2^2q_1 + p_1^2q_2 + 2p_1p_2(2q_1+q_2) + 8q_1q_2 + p_1^2p_2^2 + 4q_1p_2\big] + 2\big[4M_2(G) + M_2(H)\\ &+ M_2(H) + (2p_2+4)M_1(G) + p_1M_1(H) + p_2^2q_1 + p_1^2q_2 + 2p_1p_2(2q_1+q_2)\\ &+ 8q_1q_2 + p_1^2p_2^2 + 4q_1p_2\big] + 8F(G) + F(H) + 3\big[p_1M_1(H) + 4(p_2M_1(G))\big]\\ &+ 12q_1p_2^2 + 6p_1^2q_2 + p_1p_2(p_1^2+p_2^2) + 8q_1F(G) + F(H) + p_1p_2(p_1)(p_1^2+p_2^2)\\ &+ 3\big[M_1(G)p_2 + M_1(H)p_1\big] + 6(p_1^2q_2 + p_2^2q_1) + 8q_1.\\ &= 4M_1^\varepsilon(G) + M_1^\varepsilon(H) + 5p_1M_1(H) - 8M_1(G) + 4F(G) + 4p_1p_2(2q_1+q_2+1)\\ &+ 16p_2M_1(G) + 16q_1(q_2-p_2) - 12q_2p_1 - 8q_1 - 3p_1p_2(p_1+p_2) + 14q_1p_2^2\\ &+ p_1p_2(p_1^2+p_2^2) + 2p_1^2(p_2^2+4q_2).\end{aligned}$$

□

Lemma 4. *[10] If G and H be two connected graphs then*

$$\begin{aligned}M_1(GV_QH) &= M_1(G) + M_1(H) + HM(G) + 4p_2q_1 + 4p_1q_2 + p_1^2p_2 + p_1p_2^2\\ M_2(GV_QH) &= EM_2(G) + 2EM_1(G) + 2M_1(G) + HM(G) + 2p_2M_1(G) + M_2(H)\\ &+ p_1M_1(H) + p_1^2q_2 + 4q_1q_2 + 2q_1p_1p_2 + 2q_2p_1p_2 + p_1^2p_2^2 - 4q_1.\end{aligned}$$

Theorem 3. *For any graph G and H, the first entire Zagreb index of GV_QH is*

$$\begin{aligned}M_1^\varepsilon(GV_QH) &= M_1^\varepsilon(G) + M_1^\varepsilon(H) + d_G(u) + d_G(v) - 2 + 4EM_1(G) + 2EM_2(G) - HM(G)\\ &+ 4M_1(G) + 5p_1M_1(H) + 7p_2M_1(G) - 2M_2(G) + 6q_1p_2(p_2-2) + 6p_1q_2(p_1-2)\\ &+ 3p_1p_2(p_1+p_2) + 4p_1p_2(q_1+q_2) + (3p_2-2) + 2p_1^2(q_2^2+p_2^2) + p_1^2(p_1^2+p_2^2)\\ &+ 4q_1(2q_2-1) + \sum_{uv\in E(G)} (d_G(u)+d_G(v))^3.\end{aligned}$$

Proof. Let G and H be two graphs with then by Lemma 2,

$$M_1^\varepsilon(GV_QH) = 4q_{GV_QH} - 3M_1(GV_QH) + 2M_2(GV_QH) + \sum_{u\in V(GV_QH)} (d_{GV_QH}(u))^3. \quad (3)$$

Since $q_{GV_QH} = 2q_1q_2 + p_1p_2 + \frac{1}{2}(d_G(u) + d_H(v) - 2)$ and $M_1(GV_QH)$, $M_2(GV_QH)$ are obtained from Lemma 4, then

$$\begin{aligned}\sum_{u\in V(GV_QH)} (d_{GV_QH}(u))^3 &= \sum_{u\in V(G)} (d_G(u)+p_2)^3 + \sum_{v\in V(H)} (d_H(v)+p_1)^3 \\ &+ \sum_{uv\in E(G)} (d_G(u)+d_G(v))^3 \\ &= \sum_{u\in V(G)} (d_G(u)^3 + \sum_{u\in V(G)} p_2^3 + 3p_2 \sum_{u\in V(G)} (d_G(u))^2 + 3p_2^2 \sum_{u\in V(G)} d_G(u) \\ &+ \sum_{v\in V(H)} (d_H(v))^3 + \sum_{v\in V(H)} p_1^3 + 3p_1 \sum_{v\in V(H)} (d_H(v))^2 + 3p_1^2 \sum_{u\in V(H)} d_H(v) \\ &+ \sum_{uv\in E(G)} (d_G(u)+d_G(v))^3 \\ &= F(G) + F(H) + 3p_1M_1(H) + 3p_2M_1(G) + 6p_1^2q_2 + 6q_1p_2^2 \\ &+ p_1p_2(p_1^2+p_2^2) + \sum_{uv\in E(G)} (d_G(u)+d_G(v))^3.\end{aligned}$$

Therefore from the equation (3) become

$$\begin{aligned}M_1^\varepsilon(GV_QH) &= 8q_1 + 4q_2 + 4p_1p_2 + 2(d_G(u)+d_H(v)-2) - 3M_1(G) - 3M_1(H) \\ &- 3HM(G) - 12p_2q_1 - 12p_1q_2 - 3p_1p_2^2 - 3p_1^2p_2 + 2EM_2(G) + 4EM_1(G) \\ &+ 4M_1(G) + 2HM(G) + 4p_2M_1(G) + 2M_2(H) + 2p_1M_1(H) + 2p_1^2q_2 \\ &+ 8q_1q_2 + 4q_1p_1p_2 + 4q_2p_1p_2 + 2p_1^2p_2^2 - 4q_1 + F(G) + F(H) + 3p_1M_1(H) \\ &+ 3p_2M_1(G) + 6p_1^2q_2 + 6q_1p_2^2 + p_1p_2(p_1^2+p_2^2) + \sum_{uv\in E(G)} (d_G(u)+d_G(v))^3 \\ &= M_1^\varepsilon(G) + M_1^\varepsilon(H) + 2(d_G(u)+d_G(v)-2) + 4EM_1(G) + 2EM_2(G) - HM(G) \\ &+ 4M_1(G) + 5p_1M_1H + 7p_2M_1(G) - 2M_2(G) + 6q_1p_2(p_2-2) + 6p_1q_2(p_1-2) \\ &+ 3p_1p_2(p_1+p_2) + 4p_1p_2(q_1+q_2) + (3p_2-2) + 2p_1^2(q_2^2+p_2^2) + p_1^2(p_1^2+p_2^2) \\ &+ 4q_1(2q_2-1) + \sum_{uv\in E(G)} (d_G(u)+d_G(v))^3.\end{aligned}$$

□

Lemma 5. *[10] If G and H be two connected graphs with $|V(G)| = p_1$, $|V(H)| = p_2$ and $|E(G)| = q_1$, $|E(H)| = q_2$ then*

$$\begin{aligned}M_1(GV_TH) &= HM(G) + 4M_1(G) + M_1(H) + p_1p_2(p_1+p_2) + 8p_2q_2q_1 + 4q_2p_1 \\ M_2(GV_TH) &= EM_2(G) + 2EM_1(G) + 2HM(G) + (4p_2+2)M_1(G) + 4M_2(G) \\ &+ p_1M_1H + M_2(H) + p_1^2q_2 + 4q_1q_2) + p_2^2q_1 + 8q_1q_2 + 4q_1p_1p_2 \\ &+ 2q_2p_1p_2 + p_1^2p_2^2 - 4q_1.\end{aligned}$$

Theorem 4. *For any graphs G and H, the first entire zagreb index of GV_TH is*

$$\begin{aligned}M_1^\varepsilon(GV_TH) &= M_1^\varepsilon(G) + M_1^\varepsilon(H) + HM(G) + 4EM_1(G) + 2EM_2(G) + 20p_2M_1(G) \\ &+ 5p_1M_1(H) - 5M_1(G) + 6M_2(G) + 7F(G) + p_1p_2(p_1^2+p_2^2) + 8q_2p_1^2 \\ &+ 14q_1p_2^2 + 8q_1p_2(p_1-3) + 4q_2p_1[p_2-3] + p_1p_2[2p_1p_2 + 4 - 3p_1 - 3p_2] + 16q_1q_2 \\ &+ \sum_{uv\in E(G)} (d_G(u)+d_G(v))^3 + 2\sum_{uv\in E(G)} (d_G(u)+d_G(v)-2).\end{aligned}$$

Proof. Consider,

$$M_1^{\varepsilon}(GV_TH) = 4q_{GV_TH} - 3M_1(GV_TH) + 2M_2(GV_TH) + \sum_{u\in V(GV_TH)} (d_{GV_TH}(u))^3. \quad (4)$$

Since $q_{GV_TH} = 3q_1+q_2+p_1p_2+\frac{1}{2}\sum_{uv\in E(G)}(d_G(u)+d_H(v)-2)$ and $M_1(GV_TH)$, $M_2(GV_TH)$ are obtained from Lemma 5,

$$\begin{aligned}\sum_{u\in V(GV_TH)} (d_{GV_TH}(u))^3 &= \sum_{u\in V(G)} \big(2d_G(u)+p_2\big)^3 + \sum_{v\in V(H)} \big(d_H(v)+p_1\big)^3 \\ &+ \sum_{uv\in E(G)} (d_G(u)+d_H(v))^3 \\ &= \sum_{u\in V(G)} [8(d_G(u))^3 + p_2^3 + 6p_2d_G(u)(d_G(u)+p_2)] \\ &+ \sum_{v\in V(H)} [(d_H(v))^3 + p_1^3 + 3p_1d_H(v)(d_H(v)+p_1)] \\ &+ \sum_{uv\in E(G)} (d_G(u)+d_H(v))^3 \\ &= 8F(G) + p_1p_2^3 + 6p_2M_1(G) + 12p_2^2q_1 + F(H) \\ &+ p_2p_1^3 + 3p_1M_1(H) + 6q_2p_1^2 + \sum_{uv\in E(G)} (d_G(u)+d_H(v))^3.\end{aligned}$$

Therefore equation (4) become

$$\begin{aligned}M_1^{\varepsilon}(GV_TH) &= 12q_1 + 4q_2 + 4p_1p_2 + 2\sum_{uv\in E(G)} (d_G(u)+d_H(v)-2) - 3HM(G) - 12M_1(G) \\ &- 3M_1(H) - 3p_1p_2(p_1+p_2) - 24p_2q_1 - 12q_2p_1 + 2EM_2(G) + 4EM_1(G) \\ &+ 4HM(G) + 8p_2M_1(G) + 4M_1(G) + 8M_2(G) + 2p_1M_1(H) + 2M_2(H) + 2p_1^2q_2 \\ &+ 2p_2^2q_1 + 16q_1q_2 + 8q_1p_1p_2 + 4q_2p_1p_2 + 2p_1^2n_2^2 - 8q_1 + 8F(G) + F(H) \\ &+ 12p_2M_1(G) + 3p_1M_1(H) + p_1p_2^3 + p_1^3p_2 + 12q_1p_2^2 + 6p_1^2q_2 \\ &+ \sum_{uv\in E(G)} (d_G(u)+d_H(v))^3 \\ &= M_1^{\varepsilon}(G) + M_1^{\varepsilon}(H) + HM(G) + 4EM_1(G) + 2EM_2(G) + 20p_2M_1(G) \\ &+ 5p_1M_1(H) - 5M_1(G) + 6M_2(G) + 7F(G) + p_1p_2(p_1^2+p_2^2) + 8q_2p_1^2 \\ &+ 14q_1p_2^2 + 8q_1p_2(p_1-3) + 4q_2p_1[p_2-3] + p_1p_2[2p_1p_2+4-3p_1-3p_2] + 16q_1q_2 \\ &+ \sum_{uv\in E(G)} (d_G(u)+d_G(v))^3 + 2\sum_{uv\in E(G)} (d_G(u)+d_G(v)-2).\end{aligned}$$

□

Lemma 6. *If G and H be two connected graphs then*

$$M_1(GV_{-S}H) = M_1(G) + q_1(p_2+2)^2 + M_1(H) + 4q_2 + p_2q_1$$
$$M_2(GV_{-S}H) = (p_2+2)M_1(G) + q_1M_1(H) + M_2(H) + q_1(q_1q_2 + (2q_2+q_1p_2)(2+p_2)).$$

Theorem 5. *For any graph G and H, the first entire Zagreb index of $GV_{-S}H$ is*

$$\begin{aligned}M_1^{\varepsilon}(GV_{-S}H) &= M_1^{\varepsilon}(G) + M_1^{\varepsilon}(H) + 2p_2M_1(G) + 5q_1M_1(H) + 4M_1(G) - 2M_2(G) \\ &+ q_1p_2(q_1^2+p_2^2+q_1+2q_1p_2+3p_2+4) + 4q_1q_2(p_2+2q_1-1).\end{aligned}$$

Proof. Consider,

$$M_1^{\varepsilon}(GV_{-S}H) = 4q_{GV_{-S}H} - 3M_1(GV_{-S}H) + 2M_2(GV_{-S}H) + \sum_{u\in V(GV_{-S}H)} (d_{GV_{-S}H}(u))^3. \quad (5)$$

Since $q_{GV_{-S}H} = 2q_1 + q_2 + q_1p_2$ and $M_1(GV_{-S}H)$, $M_2(GV_{-S}H)$ are obtained from Lemma 6,

$$\begin{aligned}\sum_{u\in V(GV_{-S}H)} (d_{GV_{-S}H}(u))^3 &= \sum_{u\in V(G)} \big(d_G(u)\big)^3 + \sum_{v\in V(H)} \big(d_H(v)+q_1\big)^3 + \sum_{v\in I(G)} (2+p_2)^3\\ &= \sum_{u\in V(G)} (d_G(u))^3 + \sum_{v\in V(H)} (d_H(v))^3 + q_1^3 + 3q_1 d_H(v)(d_H(v)+q_1)\\ &+ \sum_{uv\in E(G)} (8+p_2^2+6p_2(2+p_2))\\ &= F(G)+F(H)+p_2q_1^3+3q_1M_1(H)+6q_1^2q_2+8q_1+q_1p_2^3\\ &+12p_2q_1+6p_2^2q_1.\end{aligned}$$

$$\begin{aligned}M_1^{\varepsilon}(GV_{-S}H) &= 4(2q_1+q_2+q_1p_2) - 3\big[M_1(G)+M_1(H)+q_1\big[(p_2+2)^2+4q_2+p_2q_1\big]\\ &+ M_1(G)+M_2(H)+q_1(p_2+2)^2+4q_2+p_2q_1\big] + 2\big[(p_2+2)M_1(G)+q_1M_1(H)\\ &+ M_2(H)+q_1(q_1q_2+(2q_2+q_1p_2)(2+p_2))\big] + F(G)+F(H)+p_2q_1^3\\ &+3q_1M_1(H)+6q_1^2q_2+8q_1+q_1p_2^3+12p_2q_1+6p_2^2q_1\\ &= M_1^{\varepsilon}(G)+M_1^{\varepsilon}(H)+2p_2M_1(G)+5q_1M_1(H)+4M_1(G)-2M_2(G)\\ &+q_1p_2(q_1^2+p_2^2+q_1+2q_1p_2+3p_2+4)+4q_1q_2(p_2+2q_1-1).\end{aligned}$$

□

Lemma 7. *If G and H be two connected graphs then Zagerb indices are*

$$\begin{aligned}M_1(GV_{-R}H) &= 4M_1(G)+M_1(H)+4q_1q_2+q_1^2p_2+(p_2+2)^2q_1\\ M_2(GV_{-R}H) &= 4M_2(G)+M_2(H)+2(p_2+2)M_1(G)+q_1M_1(H)\\ &+q_1(2+p_2)(q_1p_2+2q_2)+q_1^2q_2.\end{aligned}$$

Theorem 6. *For any graphs G and H, the first entire Zagreb index of $GV_{-R}H$ is*

$$\begin{aligned}M_1^{\varepsilon}(GV_{-R}H) &= M_1^{\varepsilon}(G)+M_1^{\varepsilon}(H)-M_1(G)+7F(G)+6M_2(G)+4p_2M_1(G)+5q_1M_1(G)\\ &+4q_1+4q_1p_2-3q_1p_2(q_1+p_2)-4q_1q_2+2q_1^2(2p_2+q_2)+p_2q_1(q_1^2+p_2^2)\\ &+2q_1p_2^2(q_1+3)+2q_1q_2(3q_1+2p_2).\end{aligned}$$

Proof. Consider,

$$M_1^{\varepsilon}(GV_{-R}H) = 4q_{GV_{-R}H} - 3M_1(GV_{-R}H) + 2M_2(GV_{-R}H) + \sum_{u\in V(GV_{-R}H)} (d_{GV_{-R}H}(u))^3. \quad (6)$$

Since $q_{GV_{-R}H} = 3q_1 + q_2 + q_1p_2$ and $M_1(GV_{-R}H)$, $M_2(GV_{-R}H)$ are obtained from Lemma 7,

$$\sum_{u\in V(GV_{-R}H)}(d_{GV_{-R}H}(u))^3=\sum_{u\in V(G)}\big(2d_G(u)\big)^3+\sum_{v\in V(H)}\big(d_H(v)+q_1\big)^3+\sum_{uv\in E(G)}(2+p_2)^3$$
$$=8\sum_{u\in V(G)}d_G(u)^3+\sum_{v\in V(H)}(d_H(v)+q_1^3+3q_1d_H(u)^2+3q_1^2d_H(u))$$
$$+\sum_{uv\in E(G)}(p_2^3+6p_2^2+12p_2+8)$$
$$=8F(G)+F(H)+p_2q_1^3+3q_1M_1H+6q_1^2q_2+8q_1+p_2^3q_1$$
$$+12p_2q_1+6p_2^2q_1.$$

$$M_1^{\varepsilon}(GV_{-R}H)=4(3q_1+q_2+q_1p_2)-3\big[4M_1(G)+M_1(H)+4q_1q_2+q_1^2p_2+(p_2+2)^2q_1\big]$$
$$+2\big[4M_2(G)+M_2(H)+2(p_2+2)M_1(G)+q_1M_1(H)+q_1(2+p_2)(q_1p_2+2q_2$$
$$+q_1^2q_2\big]+8F(G)+F(H)+p_2q_1^3+3q_1M_1(H)+6q_1^2q_2+8q_1+q_1p_2^3$$
$$+12p_2q_1+6p_2^2q_1$$
$$=M_1^{\varepsilon}(G)+M_1^{\varepsilon}(H)-M_1(G)+7F(G)+6M_2(G)+4p_2M_1(G)+5q_1M_1(H)$$
$$+4q_1+4q_1p_2-3q_1p_2(q_1+p_2)-4q_1q_2+2q_1^2(2p_2+q_2)$$
$$+p_2q_1(q_1^2+p_2^2)+2q_1p_2^2(q_1+3)+2q_1q_2(3q_1+2p_2).$$

□

Lemma 8. *If G and H be two connected graphs then*

$$M_1(GV_{-Q}H)=M_1(G)+M_1(H)+HM(G)+2p_2M_1(G)+4q_1q_2+q_1^2p_2+p_2^2q_1$$
$$M_2(GV_{-Q}H)=EM_2(G)+(2+p_2)EM_1(G)+(2+p_2)^2\frac{M_1(G)}{2}+HM(G)+p_2M_1(G)$$
$$+2q_2M_1(G)+q_1p_2M_1(G)+M_2(H)+q_1M_1(H)+2p_2q_2q_1+q_1^2p_2^2$$
$$+q_1^2q_2-q_1(2+p_2)^2.$$

Theorem 7. *For any graph G and H, the first entire Zagreb index of $GV_{-Q}H$ is equal to*

$$M_1^{\varepsilon}(GV_{-Q}H)=M_1^{\varepsilon}(G)+M_1^{\varepsilon}(H)-HM(G)+2EM_2(G)+2EM_1(G)(p_2+2)$$
$$+M_1(G)[p_2^2+4q_2+2q_1p_2+4]+5q_1M_1(H)-2M_2(G)+q_1p_2[q_1^2$$
$$-3q_1-3p_2+2q_1p_2-2p_2+4]+q_1q_2[4p_2+8q_1-12]-4q_1$$
$$+2\sum_{uv\in E(G)}(d_G(u)+d_H(v)-2)+\sum_{uv\in E(G)}(d_G(u)+d_H(v)+p_2)^3.$$

Proof. Consider,

$$M_1^{\varepsilon}(GV_{-Q}H)=4q_{GV_{-Q}H}-3M_1(GV_{-Q}H)+2M_2(GV_{-Q}H)+\sum_{u\in V(GV_{-Q}H)}(d_{GV_{-Q}H}(u))^3. \tag{7}$$

Since $q_{GV_{-Q}H}=2q_1+q_2+q_1p_2+\frac{1}{2}\sum_{uv\in E(G)}(d_G(u)+d_H(v)-2)$ and $M_1(GV_{-Q}H)$, $M_2(GV_{-Q}H)$ are obtained from Lemma 8,

$$\sum_{u\in V(GV_{-Q}H)} (d_{GV_{-Q}H}(u))^3 = \sum_{u\in V(G)} \big(d_G(u)\big)^3 + \sum_{v\in V(H)} \big(d_H(v)+q_1\big)^3 + \sum_{uv\in E(G)} (d_G(u)+d_G(v)+p_2)^3$$
$$= F(G) + \sum_{v\in V(H)} (d_H(v))^3 + q_1^3 + 3q_1(d_H(v))^2 + 3q_1^2 d_H(v) + \sum_{uv\in E(G)} (d_G(u)+d_G(v)+p_2)^3$$
$$= F(G) + F(H) + p_2q_1^3 + 3q_1M_1(H) + 6q_1^2q_2 + \sum_{uv\in E(G)} (d_G(u)+d_G(v)+p_2)^3.$$

$$M_1^{\varepsilon}(GV_{-Q}H) = 8q_1 + 4q_2 + 4q_1p_2 + 2\sum_{uv\in E(G)} (d_G(u)+d_H(v)-2) - 3M_1(G) - 3M_1(H)$$
$$- 3HM(G) + 6p_2M_1(G) - 12q_1q_2 - 3q_1^2p_2 - 3p_2^2q_1 + 2EM_2(G) + 4EM_1(G)$$
$$+ 2p_2EM_1(G) + 4M_1(G) + p_2^2M_1(G) + 4p_2M_1(G) + 2HM(G) + 2p_2M_1(G)$$
$$+ 4q_2M_1(G) + 2m_1p_2M_1(G) + 2M_2(H) + 2q_1M_1(H) + 4q_1q_1p_2 + 2p_2^2q_1^2$$
$$+ 2q_1^2p_2^2 + 2q_1^2q_2 - 2p_2^2q_1 - 8q_1 - 8p_2q_1 + F(G) + F(H) + p_2q_1^3$$
$$+ 3q_1M_1(H) + 6q_1^2q_2 + \sum_{uv\in E(G)} (d_G(u)+d_G(v)+p_2)^3$$
$$= M_1^{\varepsilon}(G) + M_1^{\varepsilon}(H) - HM(G) + 2EM_2(G) + 2EM_1(G)(p_2+2) + M_1(G)[p_2^2$$
$$+ 4q_2 + 4] + 5q_1M_1(H) - 2M_2(G) + q_1p_2[q_1^2 - 3q_1 - 3p_2 + 2q_1p_2 - 2p_2 + 4]$$
$$+ q_1q_2[4p_2 + 8q_1 - 12] - 4q_1 + 2\sum_{uv\in E(G)} (d_G(u)+d_H(v)-2)$$
$$+ \sum_{uv\in E(G)} (d_G(u)+d_H(v)+p_2).$$

□

Lemma 9. *[10] If G and H be two connected graphs then*

$$M_1(GV_{-T}H) = (2p_2+4)M_1(G) + HM(G) + M_1(H) + q_1^2p_2 + p_2^2q_1 + 4q_2q_1$$
$$M_2(GV_{-T}H) = EM_2(G) + (2+p_2)EM_1(G) + 2HM(G) + \big(4q_2 + 4p_2 + 2q_1p_2 + 4M_2(G) + (2+p_2)^2\big)\frac{M_1(G)}{2} + M_2(H) + q_1M_1(H) + q_1^2(q_2+p_2^2) + 2q_1q_2p_2 - q_1(2+p_2)^2.$$

Theorem 8. *For any graphs G and H, the first entire Zagreb index of $GV_{-T}H$ is*

$$M_1^{\varepsilon}(GV_{-T}H) = M_1^{\varepsilon}(G) + M_1^{\varepsilon}(H) + M_1(G)[2q_1p_2 + p_2^2 + 2p_2 + 4q_2 - 5] + HM(G) + 2EM_2(G)$$
$$+ 2EM_1(G) + 2EM_1(G)(p_2+2) + 5q_1M_1(H) + 6M_2(G) + 7F(G)$$
$$+ q_1p_2[q_1^2 + 2q_1p_2 - 3(q_1+p_2) - 2p_2 - 4] + 4q_1q_2[2q_1 + p_2 - 3]$$
$$+ 2\sum_{uv\in E(G)} (d_G(u)+d_H(v)-2) + \sum_{uv\in E(G)} (d_G(u)+d_H(v)+p_2)^3.$$

Proof. Consider,

$$M_1^{\varepsilon}(GV_{-T}H) = 4q_{GV_{-T}H} - 3M_1(GV_{-T}H) + 2M_2(GV_{-T}H) + \sum_{u\in V(GV_{-T}H)} (d_{GV_{-T}H}(u))^3. \tag{8}$$

Since, $q_{GV_{-T}H} = 3q_1 + q_2 + q_1p_2 + \frac{1}{2}\sum_{uv\in E(G)}(d_G(u) + d_H(v) - 2)$ and $M_1(GV_{-T}H)$, $M_2(GV_{-T}H)$ are obtained from Lemma 9,

$$\begin{aligned}\sum_{u\in V(GV_{-T}H)} (d_{GV_{-T}H}(u))^3 &= \sum_{u\in V(G)} \big(2d_G(u)\big)^3 + \sum_{v\in V(H)} \big(d_H(v)+q_1\big)^3 \\ &+ \sum_{uv\in E(G)} (d_G(u)+d_G(v)+p_2)^3 \\ &= 8F(G) + \sum_{v\in V(H)} (d_H(v))^3 + q_1^3 + 3q_1(d_H(v))^2 + 3q_1^2(d_H(v)) \\ &+ \sum_{uv\in E(G)} (d_G(u)+d_G(v)+p_2)^3 \\ &= 8F(G) + F(H) + p_2q_1^3 + 3q_1M_1(H) + 6q_1^2q_2 \\ &+ \sum_{uv\in E(G)} (d_G(u)+d_G(v)+p_2)^3.\end{aligned}$$

$$\begin{aligned}M_1^{\varepsilon}(GV_{-T}H) &= 12q_1 + 4q_2 + 4q_1p_2 + 2\sum_{uv\in E(G)} (d_G(u)+d_H(v)-2)) - 6p_2M_1(G) - 12M_1(G) \\ &- 3HM(G) - 3M_1(H) - 3q_1^2p_2 - 3p_2^2q_1 - 12q_2q_1 + 2EM_2(G) + 4EM_1(G) \\ &+ 2p_2EM_1(G) + 4HM(G) + 4q_2M_1(G) + 4p_2M_1(G) + 2q_1p_2M_1(G) + 4M_1(G) \\ &+ p_2^2M_1(G) + 4p_2M_1(G) + 8M_2(G) + 2M_2(H) + 2q_1M_1(H) + 2q_1^2q_2 + 2q_1^2p_2^2 \\ &+ 4q_1q_2p_2 - 8q_1 - 2p_2^2q_1 - 8p_2q_1 + 8F(G) + F(H) + p_2q_1^3 + 3q_1M_1(H) \\ &+ 6q_1^2q_2 + \sum_{uv\in E(G)} (d_G(u)+d_G(v)+p_2)^3 \\ &= M_1^{\varepsilon}(G) + M_1^{\varepsilon}(H) + M_1(G)[2q_1p_2 + p_2^2 + 2p_2 + 4q_2 - 5] + HM(G) + 2EM_2(G) \\ &+ 2EM_1(G) + 2EM_1(G)(p_2+2) + 5q_1M_1(H) + 6M_2(G) + 7F(G) \\ &+ q_1p_2[q_1^2 + 2q_1p_2 - 3(q_1+p_2) - 2p_2 - 4] + 4q_1q_2[2q_1 + p_2 - 3] \\ &+ 2\sum_{uv\in E(G)} (d_G(u)+d_H(v)-2) + \sum_{uv\in E(G)} (d_G(u)+d_H(v)+p_2)^3.\end{aligned}$$

□

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