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SLAVERY ACCESSIBILITY INTEGRITY NUMBER OF GRAPHS

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ABSTRACT. The slavery accessibility integrity number of G is the minimum cardinality of a set $E_1(G)$ of edges whose removal from $E(G)$ results a graph with accessibility integrity greater than accessibility integrity of G, and denoted by $S_{AI}(G)$. $S_{AI}(G) = 0$ for some graphs, but $S_{AI}(G) \neq q$ for any (p, q) graph. First we discuss various properties of slavery accessibility integrity number of graphs. We calculate the slavery accessibility integrity number for several classes of graphs. Next we prove that for every tree T , $S_{AI}(T) \leq 2$, and we characterize all trees with $p \leq 10$, that are AI- stellar and $S_{AI}(T) = 2$. Finally, we discuss accessibility integrity of some graphs.

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1. INTRODUCTION

By a graph, we mean a finite, undirected graph with neither loops nor multiple edges. For a graph G, we denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. We use p to denote the number of vertices and q to denote the number of edges of a graph G. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The degree of a vertex v, denoted by $degv$, is the cardinality of its neighborhood. The distance between the vertices v_i and v_j is the length of the shortest path joining v_i and v_j. The shortest v_iv_j path is often called a geodesic. The diameter of a connected graph G is the length of any longest geodesic, denoted by $diam(G)$. By a pendant vertex we mean a vertex of degree one, while a support vertex is a vertex adjacent to a pendant vertex. Given any vertex $v \in V(G)$, the graph obtained from G by removing the vertex v and all of its incident edges is denoted by $G - v$. The reader follow [8], for graph-theoretical terminology and notation not defined here. There are various measures of the accuracy of a communication network, a stylish one is called the integrity of the network. Barefoot, Entringer and Swart in 1987 [1], introduced this concept as a useful measure of the vulnerability of the graph, and is defined as follows:

$$I(G) = \min\{|S| + m(G - S) : S \subseteq V(G)\},$$

where $m(G-S)$, denotes the order of the largest component of $G-S$. The motivation is as follows. Model the network as a graph, to damage the network a terrorist attempts to remove a small set of vertices such that the remaining connected components are small. There is a substantial literature on integrity, but most of the papers are concerned with calculating the integrity of particular graphs. There are results on the interrelations between integrity and other graph parameters [6].

In [3], Moazzami et al. compared the integrity, connectivity, binding number, toughness, and tenacity for several classes of graphs. For more details on the integrity-parameters see [11, 12, 13, 15].

The complement $\overline{G}$ of a graph G has $V(G)$ as its vertex set, two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G [8]. The double star graph $S_{n,m}$ is a tree with exactly two vertices that are not terminal vertices, with one adjacent to n terminal vertices and the other to m terminal vertices. [7]. The helm H_p is the graph obtained from the wheel $W_{1,p-1}$ by attaching a pendant edge at each vertex of the cycle C_{p-1}. The symbols $\delta(G), \Delta(G)$, denote the minimum, maximum degree of G respectively, $\lceil x \rceil$ denotes the smallest integer number that is greater than or equal to x, $\lfloor x \rfloor$ the greatest integer number that is smaller than or equal to x. The average degree of G is $\overline{\delta}(G) = \frac{\sum_{1\leq i\leq p} deg(v_i)}{p}$.

In [14], Sultan et al. introduced the concept of accessibility integrity (AI) of graphs. Motivated by this, we introduce the concept of slavery accessibility integrity number of a graph G. A vertex of G is called AI-poor if it does not belong to any AI-set of G. A graph G is called AI-stellar if every vertex in $V(G)$ is AI-useful [14]. A vertex of G is called AI-useful if it is contained in some AI-set of G.

In the present work, we introduce the slavery accessibility integrity number of some graphs, some properties of slavery accessibility integrity number are established. Finally the accessibility integrity of corona of some graphs and a flower graph Fl_p are determined. We use the following results for our later results.

Proposition 1.1. [9] *For any connected graph G, there exist two vertices x and y with distance at most two and, with the property that $deg(x) + deg(y) \leq 2\overline{\delta}(G)$.*

Proposition 1.2. [14] *P_p is AI-stellar if and only if $p = 4$, or $p \equiv 0 \ (mod\ 3), 6 \leq p \leq 18$, or 22, or $p \equiv 1 (mod\ 5), p \geq 25$.*

Proposition 1.3. [14] *For the complete bipartite graph $K_{n,m}$,*

$$AI(K_{n,m}) = min\{n, m\} + 1.$$

2. Slavery accessibility integrity number of graphs

Definition 2.1. [4] *A subset S of $V(G)$ is called an accessible set of the graph G if each vertex $v \in V(G)-S$ is adjacent to $N[S]$, where $N[S]$ is the closed neighborhood of S. The accessibility number of G is defined as the minimum number of vertices over all accessible sets of G and is denoted by $\eta(G)$.*

Definition 2.2. [14] *The accessibility integrity of a graph G is defined as $AI(G) = \min\{|S| + m(G-S)$, where S is an accessible set and $m(G-S)$ is the order of a maximum component of $G-S\}$.*

Definition 2.3. [14] *An AI-set of G is any subset S of $V(G)$ for which $AI(G) = |S| + m(G-S)$.*

We define the slavery accessibility integrity number of a graph G.

Definition 2.4. *The slavery accessibility integrity number of G is the minimum cardinality of a set $E_1(G)$ of edges for which $AI(G-E_1) > AI(G)$, and is denoted by $S_{AI}(G)$.*

We now discuss various properties of $S_{AI}(G)$.

Remark 2.1.
(1) *For any (p,q) graph G, $0 \leq S_{AI}(G) \leq q-1$. The upper bound is sharp for $G \cong C_4$ or $G \cong C_5$ and the lower bound is sharp for $G \cong K_p$.*
(2) *For every connected graph G, $0 \leq S_{AI}(\overline{G}) \leq q-1$. The lower bound is achieved for $G \cong K_p$, and the upper bound is sharp for $G \cong C_5$.*

Lemma 2.1. *Let G' be a spanning subgraph obtained by removing m edges from a graph G. Then $S_{AI}(G) \leq S_{AI}(G') + m$.*

Proof. Let E_1 be the set of edges removed from G such that $|E_1| = m$, and let $S_{AI}(G)$ be the slavery accessibility integrity number of G such that $AI(G-E_1) > AI(G)$. Suppose $S_{AI}(G) > S_{AI}(G')+m$. If $G \cong K_p$, then $S_{AI}(G) = 0$, since G' is a spanning subgraph of K_p, then $m \geq 1$, so $S_{AI}(G') + 1 \leq 0$, impossible. Now, if $G \neq K_p$, then there exist some graph G such that $S_{AI}(G) = S_{AI}(G')$, but $m \geq 1$, so the relation $S_{AI}(G) > m + S_{AI}(G')$ is not true. Therefore, $S_{AI}(G) \leq S_{AI}(G') + m$. □

Theorem 2.2. *$S_{AI}(G) \leq min_{d(u,v)\leq 2}\{degu + degv + 1$ and $u \neq v\}$. The bound is sharp for $G = W_{1,5}$.*

Proof. Let u and v be two vertices such that $u \neq v$ and $d(u,v) \leq 2$. If $S_{AI}(G) > degu + degv + 1$, then $S_{AI}(G) - 1 > degu + degv$, and this is impossible for any graph G. Therefore, $S_{AI}(G) \leq min_{d(u,v)\leq 2}\{degu + degv + 1$ *and* $u \neq v\}$. □

Corollary 2.3. *Let G be a connected graph. Then $S_{AI}(G) \leq \delta(G) + \Delta(G) + 1$.*

Proof. Let u be a vertex of minimum degree in G, and v be any vertex not adjacent to u such that $d(u, v) \leq 2$, thus, by Theorem 2.2, $S_{AI}(G) \leq degu + degv + 1 = \delta(G) + deg(v) + 1 \leq \delta(G) + \Delta(G) + 1$. □

Proposition 2.4. $S_{AI}(G) = 0$ *if and only if* $G \cong K_p, \overline{K_p}$, *or* $lK_1 \cup K_p, l \geq 1$.

Proof. Suppose that $S_{AI}(G) = 0$, then there does not exist any edge e of G such that $AI(G - e) > AI(G)$, so we have three cases:
Case 1: G has edges and if we delete all edges of G, then $AI(G - E) = AI(G)$, and this is possible only if $AI(G) = p$, and the graph K_p only achieves it.
Case 2: G has no edges, then $G \cong \overline{K_p}$.
Case 3: G has induced subgraph K_p, then $G \cong lK_1 \cup K_p, l \geq 1$.
Conversely, if $G \cong K_p, \overline{K_p}$ *or* $lK_1 \cup K_p$, clearly, $S_{AI}(G) = 0$. □

We now calculate the slavery accessibility integrity number of complete graph, star, double star and complete bipartite graph.

Proposition 2.5.

(1) For any complete graph $K_p, p \geq 2$, $S_{AI}(K_p) = 0$.
(2) For the star $K_{1,p-1}, p \geq 3$, $S_{AI}(K_{1,p-1}) = 1$.
(3) For the double star $S_{n,m}$,

$$S_{AI}(S_{n,m}) = 1.$$

(4) For the complete bipartite graph $K_{n,m}$,

$$S_{AI}(K_{n,m}) = \begin{cases} 2n - 1, \text{ if } n = m \ ; \\ m, \text{ if } m < n. \end{cases}$$

Proof. (1) Since $AI(K_p) = p$, if we delete one edge or more than one edge, then $AI(K_p - E_1) \leq p$ for any set E_1 of edges of G. Since $1 \leq AI(G) \leq p$ for every graph G, $S_{AI}(K_p) = 0$.
(2) Let $V(K_{1,p-1}) = \{v, v_1, v_2, ..., v_{p-1}\}$. Removal of any edge e incident with any terminal vertex of $K_{1,p-1}$, yields $K_1 \cup K_{1,p-2}$. Since $AI(K_{1,p-2}) = 2$, it follows that $AI(K_{1,p-1} - e) = 3 > AI(K_{1,p-2}) = 2$. Then $S_{AI}(K_{1,p-1}) = 1$.
(3) Let $V(S_{n,m}) = \{v, v_1, v_2, ..., v_n, u, u_1, u_2, ..., u_m\}$. Removal of any edge incident with any terminal vertex, yields $K_1 \cup S_{n-1,m}$, hence $AI(K_1 \cup S_{n-1,m}) > AI(S_{n,m})$. Then $S_{AI}(S_{n,m}) = 1$.
(4) Let $\{v_1, v_2, ..., v_n, u_1, u_2, ..., u_m\}$ be the vertices of $K_{n,m}$, the following two cases are discussed:

Case 1: $n > m$, since $deg(v_i) = m, 1 \leq i \leq n$, then the removal all edges incident with one of the vertices $v_1, v_2, ..., v_n$, yields $S_{AI}(G) = m$. Let $(u_1, v_n), (u_2, v_n), ..., (u_m, v_n) \in E(K_{n,m})$. Let $H = K_{n,m} - \{(u_1, v_n), ..., (u_m, v_n)\}$, then $AI(H) = m + 2$. Since $AI(K_{n,m}) = min\{n, m\} + 1$, by Proposition 1.3, it follows that $AI(H) > AI(K_{n,m})$, then $S_{AI}(K_{n,m}) = m$.
Case 2: $n = m$, since $deg(v_i) = n, 1 \leq i \leq n$, $deg(u_i) = n, 1 \leq i \leq n$, and $AI(K_{n,n}) = n + 1$. Removal of all edges incident with v_n, gives $K_1 \cup K_{n-1,n}$ and $AI(K_1 \cup K_{n-1,n}) = 1 + n = AI(K_{n,n})$, so, we delete all the edges incident with u_n. Since $deg(u_n) = n - 1$, we get $2K_1 \cup K_{n-1,n-1}$ and $AI(2K_1 \cup K_{n-1,n-1}) = n + 2 > AI(K_{n,n})$. Hence, $S_{AI}(K_{n,n}) = 2n - 1$.

□

We now investigate the slavery accessibility integrity number of paths.

Proposition 2.6. *For any path $P_p, 4 \leq p \leq 21$,*

$$S_{AI}(P_p) = \begin{cases} 2, & \text{if } p = 4 \text{ or } p \equiv 0 \ (mod\ 3) \ ; \\ 1, & \text{otherwise.} \end{cases}$$

Proof. Let $V(P_p) = \{v_1, v_2, ..., v_p\}$, and $E(P_p) = \{e_1, e_2, ..., e_{p-1}\}$. The following cases are considered:
Case 1: $p = 4$, if we remove e_1 of $E(P_4)$, then the result graph is $K_1 \cup P_3$ and $AI(K_1 \cup P_3) = 3$, hence, we must delete e_2, so $P_4 - \{e_1, e_2\} = 2K_1 \cup K_2$ and $AI(2K_1 \cup K_2) = 4$, thus $AI(P_4 - \{e_1, e_2\}) = 4 > AI(P_4)$. Then $S_{AI}(P_4) = 2$.
Case 2: $5 \leq p \leq 21$, we have the following cases:
Subcase 2.1: $p \equiv 0 \ (mod\ 3), p = 6, 9, 12, 15, 18, 21$. Removing e_1 from $E(P_p)$ produces $K_1 \cup P_{p-1}$ and $AI(K_1 \cup P_{p-1}) = AI(P_p)$, so, we must remove e_2 and $P_p - \{e_1, e_2\} = 2K_1 \cup P_{p-2}$ such that $AI(2K_1 \cup P_{p-2}) = 2 + AI(P_{p-2})$, thus $AI(P_p - \{e_1, e_2\}) > AI(P_p)$. Then $S_{AI}(P_p) = 2$.
Subcase 2.2: $5 \leq p \leq 21$ and $p \not\equiv 0 (mod\ 3)$. Removal of e_1 results in a graph $K_1 \cup P_{p-1}$ and $AI(K_1 \cup P_{p-1}) > AI(P_p)$, hence, $S_{AI}(P_p) = 1$. □

Proposition 2.7. *For any path $P_p, p \geq 22$,*

$$S_{AI}(P_p) = \begin{cases} 2, & \text{if } p \equiv 1 (mod\ 5) \ ; \\ 1, & \text{otherwise.} \end{cases}$$

Proof. Depending on the number of vertices $v_1, v_2, ..., v_p$ of P_p and edges $e_1, e_2, ..., e_{p-1}$, we discuss the following cases:
Case 1: $p = 22$, since $P_{22} - \{e_1, e_2\} = 2K_1 \cup P_{20}$ and $AI(2K_1 \cup P_{20}) > AI(P_{22})$. Then $S_{AI}(P_{22}) = 2$.

Case 2: $p \geq 23$, we have the following cases:
Subcase 2.1: $p \equiv 1(mod\ 5)$, by same the method as in Case 1, removing the edges $e_1, e_2 \in E(P_p)$, the resulting graph is $2K_1 \cup P_{p-2}$ and $AI(2K_1 \cup P_{p-2}) > AI(P_p)$. So $S_{AI}(P_p) = 2$.
Subcase 2.2: $p \not\equiv 1(mod\ 5)$. Since $AI(P_p) = AI(P_{p-1}), p \not\equiv 1(mod\ 5)$, $AI(P_p - e_1) = AI(K_1 \cup P_{p-1}) > AI(P_p)$. Therefore, $S_{AI}(P_p) = 1$. □

Proposition 2.8. *For any graph G, if a vertex v of G is adjacent with three or more vertices of degree one, then* $S_{AI}(G) = 1$.

Proof. Since a vertex v is adjacent with three or more vertices of degree one, if we remove any edge between v and any vertex of degree one that is adjacent to v, then we get graph with isolated vertex, so $AI(G - e) > AI(G)$. Then $S_{AI}(G) = 1$. □

Now, we study the relation between $S_{AI}(G)$ and some of parameters.

Remark 2.2. *If* $S_{AI}(G) = \Delta(G)$*, then G is AI-stellar graph. But the converse is not true, for example,* C_4 *is AI-stellar graph but* $S_{AI}(G) = 3 \neq \Delta(C_4)$.

Theorem 2.9. *For any graph G,* $S_{AI}(G) \neq q$.

Proof. Suppose that $S_{AI}(G) = q$, then $AI(G - E_1) > AI(G)$, where $|E_1| = q$, therefore there exists E_2 subset of E_1 such that $AI(G - E_2) = AI(G)$, and $|E_2| = q-1$. So $G - E_2$ is subgraph of G with one edge, so $AI(G - E_2) = p$. Thus $AI(G) = p$. Then $G - E_2 \cong (q-1)K_1 \cup K_2$ and $G \cong K_p$. Since $S_{AI}(K_p) = 0$ and $E(K_p) = \frac{p(p-1)}{2}$, a contradiction. Then $S_{AI}(G) \neq q$. □

Lemma 2.10. *For any connected graph G,* $S_{AI}(G) \leq \lfloor \frac{4q}{p} \rfloor + 1$.

Proof. By Theorem 2.2 and Proposition 1.1, $4q(G) = 2p\overline{\delta}(G) \geq p(S_{AI}(G) - 1)$. $S_{AI}(G) \leq \lfloor \frac{4q}{p} \rfloor + 1$. □

Observation 2.1. *(1)If* $G \cong C_4, C_5$ *or* $W_{1,6}$*, then* $AI(G) = S_{AI}(G)$.
(2) If $G \cong W_{1,4}$*, then* $S_{AI}(G) = p$.

Proposition 2.11. *If G is disconnected, and if every component is complete, then* $S_{AI}(G) = \delta(G)$.

Proof. Let $|E_1| = S_{AI}(G)$. Since G is disconnected, $AI(G) = k + m(G - S)$, where k is the number of components of G. So to find slavery accessibility integrity number of G, it is enough to remove the edges incident with any vertex of a minimum component of G, then $|E_1| = deg(v) = \delta(G)$. □

We prove that the slavery accessibility integrity number of any tree is either one or two.

Theorem 2.12. *For any tree T, $S_{AI}(T) \leq 2$.*

Proof. If T has order 3, by removing any edge of T, we get $K_1 \cup P_2$ graph, and $AI(K_1 \cup P_2) = 3$. So $AI(K_1 \cup P_2) = 3 > AI(T) = 2$. Therefore, $S_{AI}(T) = 1$. Now, if a tree T has vertices of order ≥ 4. We have the following cases:
Case 1: Suppose that T has a vertex v that is adjacent to three or more pendant vertices, then removing any edge incident with v and one of these vertices increases the accessibility integrity of T. So $S_{AI}(T) = 1$.
Case 2: Assume that T has a vertex v adjacent to just one pendant vertex u, also v is adjacent to a vertex w of degree ≥ 4, if we remove any pendant edge incident with w, it gives an isolated vertex. Then $S_{AI}(T) = 1$.
Case 3: We suppose that T has a vertex v adjacent to two pendant vertices u and w and maybe more than two vertices, also v is adjacent to vertex of degree ≥ 2, say z. Let S be AI-set of T such that $AI(T) = |S| + m(T - S)$. Assume S is an accessible set of T and $v \notin S$, then $u, w \in S$ and $S - (u, w) \cup \{v\}$ is an accessible set of T. If we remove (v, u) or (v, w), we get disconnected graph $T - (v, u)$ or $T - (v, w)$ with isolated vertex and hence an accessibility integrity of T increases or stays the same. Then $AI(T - (v, u)) \geq AI(T)$. Therefore, $S_{AI}(T) \in \{1, 2\}$.
Case 4: Suppose that each vertex of T is adjacent with at most one pendant vertex. Then there exists a vertex v of T of degree 2, adjacent with just one pendant vertex u. Let z be other vertex adjacent to v, and S be an accessible set of $T - \{(v, u), (v, z)\}$, it is easy to see that the vertices v and u are in S, and $S - u$ is an accessible set for T. Now, if the edge vu is removed from T, then we get forest T' that is $T \cong T' \cup K_1$, hence $AI(T - \{(v, u)\}) \geq AI(T)$, so $AI(T - \{(v, u)\}) > AI(T)$ in some graphs. Then $S_{AI}(T) = 1$. If we remove the edges $\{(v, u), (v, z)\}$, then resulting graph has a tree $T^\star$ of order $p - 2$ together with two isolated vertices, i.e $T \cong T^\star \cup 2K_1$, thus $AI(T - \{(v, u), (v, z)\}) > AI(T)$. Therefore, $S_{AI}(T) \leq 2$. □

Observation 2.2. *If T is $AI-$stellar, then there exists $e \in E(G)$ such that $AI(T - \{e\}) = AI(T)$.*

Theorem 2.13. *For any tree T, T is AI-stellar if and only if $S_{AI}(T) = 2$.*

Proof. Let E_1 be a set of edges consisting of two edges, and E_2 be a set that contains just one edge. Suppose that $S_{AI}(T) = 2$, then $AI(T - E_1) > AI(T)$, there exists $E_2(G) \subseteq E(G)$ with $|E_2(G)| < |E_1(G)|$, $AI(T - E_2) = AI(T)$. So by Observation 2.2, T is an AI-stellar graph. Now, suppose that T is an AI-stellar graph, we consider $S_{AI}(T) \neq 2$, and by Theorem 2.12, $S_{AI}(T) = 1$, so there exist trees with $S_{AI}(T) = 1$, and it is not AI-stellar. Therefore, a contradiction, hence we get the result. □

Observation 2.3. *For any tree T with $p \leq 10$, T is $AI-$ stellar and $S_{AI}(T) = 2$ if and only if $T \cong P_p, p = 4, 6, 9$ or $T_1, T_2, T_3, T_4, T_5, T_6$ as in the Figure 1:*

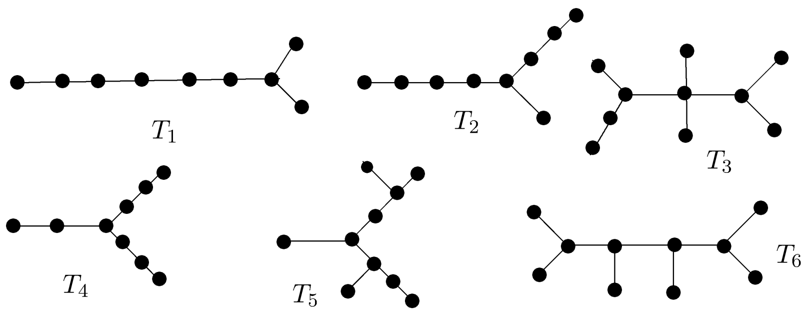

Figure 1: $T_1, T_2, T_3, T_4, T_5, T_6$

Lemma 2.14. *P_p is an AI-stellar graph if and only if $S_{AI}(P_p) = 2$.*

Proof. Suppose that P_p is an AI-stellar graph. By Proposition 1.2, $p = 4$, or $p \equiv 0(mod\ 3), 6 \leq p \leq 18$, or 22, or $p \equiv 1(mod\ 5), p \geq 25$, and Proposition 2.6 and Proposition 2.7 completes the proof. Conversely, suppose that $S_{AI}(P_p) = 2$. Let us consider P_p not an AI-stellar graph, by Proposition 1.2, p is not equal any one of these values, $p = 4$, or $p \equiv 0(mod\ 3), 6 \leq p \leq 18$, or 22, or $p \equiv 1(mod\ 5), p \geq 25$. So $p = 3, 5, 7, 8, 10, 11, 13, 14, 17, 19, 20, 21$ and $p \not\equiv 1(mod\ 5), p \geq 23$. and by Proposition 2.6 and Proposition 2.7, $S_{AI}(P_p) = 1$, a contradiction, hence the result. □

Next we discuss accessibility integrity of some graphs.

Proposition 2.15. *For any connected graph G, if $AI(G) = p$, then $diam(G) = 1$.*

Proof. Let G be a connected graph with $AI(G) = p$. Suppose $diam(G) \geq 2$. In case $diam(G) = 2$, $AI(G) = 2 \neq p$, a contradiction. Also, if $diam(G) > 2$, there does exist graphs such that $AI(G) \neq p$, this completes the proof. □

Remark 2.3. (1) *$AI(G) = 3$ if $G \cong S_{n,m}, P_4, P_5,$ or C_4,*

(2) *$AI(G) = q$ if $G \cong P_3, P_4$ or C_3,*

(3) *$AI(G) = \gamma(G)$ if $G \cong \overline{K_p}$ or graphs G_1, G_2 as shown in Figure 2.*

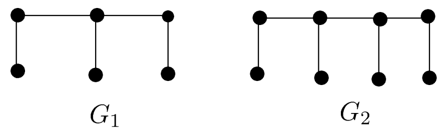

Figure $2: G_1, G_2$

Definition 2.5. [5] *The corona* $G_1 \circ G_2$ *of two graphs* G_1 *and* G_2 *is the graph* G *obtained by taking one copy of* G_1 *(which has* p_1 *vertices) and* p_1 *copies of* G_2*, and then joining the* i^{th} *vertex of* G_1 *to every vertex in the* i^{th} *copy of* G_2*.*

Theorem 2.16. $AI(P_{m_1} \circ P_{m_2}) = \lfloor \frac{m_1}{2} \rfloor + m_2 + 1.$

Proof. Let $V(P_{m_1}) = \{v_1, v_2, ..., v_{m_1-1}, v_{m_1}\}$, and $|V(P_{m_1})| = m_1$, $|V(P_{m_2})| = m_2$. We have two cases:
Case 1: m_1 is even. Consider $S = \{v_2, v_4, ..., v_{m_1}\}$, an accessible set of $P_{m_1} \circ P_{m_2}$ such that $|S| = \lfloor \frac{m_1}{2} \rfloor$ and $m((P_{m_1} \circ P_{m_2}) - S) = m_2 + 1$. Therefore, $AI(P_{m_1} \circ P_{m_2}) \leq |S| + m((P_{m_1} \circ P_{m_2}) - S) = \lfloor \frac{m_1}{2} \rfloor + m_2 + 1$. If we take $S = \{v_1, v_2, ..., v_{m_1}\}$, then $(P_{m_1} \circ P_{m_2}) - S = m_1 P_{m_2}$, hence, $|S| = m_1$ and $m((P_{m_1} \circ P_{m_2}) - S) = m_2$. Then $AI(P_{m_1} \circ P_{m_2}) \leq |S| + m((P_{m_1} \circ P_{m_2}) - S) = m_1 + m_2$. From above, no other accessible set has less than or equal to $\lfloor \frac{m_1}{2} \rfloor + m_2 + 1$ elements. This completes the proof.
Case 2: m_1 is odd. Consider $S = \{v_2, v_4, ..., v_{m_1-1}\}$, an accessible set of $P_{m_1} \circ P_{m_2}$, then $|S| = \lfloor \frac{m_1}{2} \rfloor$ and $m(P_{m_1} \circ P_{m_2} - S) = m_2 + 1$. Then $AI(P_{m_1} \circ P_{m_2}) \leq |S| + m((P_{m_1} \circ P_{m_2}) - S) = \lfloor \frac{m_1}{2} \rfloor + m_2 + 1$. □

Theorem 2.17. $AI(C_{m1} \circ C_{m2}) = \lceil \frac{m_1}{2} \rceil + m_2 + 1.$

Proof. Let $V(C_{m_1}) = \{v_1, v_2, v_3, ..., v_{m_1}\}$. We have two cases:
Case 1: m_1 is odd, consider $S = \{v_1, v_3, v_5, v_7, ..., v_{m_1}\}$, $|S| = \lceil \frac{m_1}{2} \rceil$, and hence $m(C_{m1} \circ C_{m2} - S) = m_2 + 1$. Therefore, $AI(C_{m_1} \circ C_{m_2}) \leq |S| + m(C_{m_1} \circ C_{m_2} - S) = \lceil \frac{m_1}{2} \rceil + m_2 + 1$.
Case 2: m_1 is even, consider $S = \{v_1, v_3, v_5, v_7, ..., v_{m_1-1}\}$, $|S| = \lceil \frac{m_1}{2} \rceil$, $m(C_{m_1} \circ C_{m_2} - S) = m_2 + 1$. Then, $AI(C_{m_1} \circ C_{m_2}) \leq |S| + m(C_{m_1} \circ C_{m_2} - S) = \lceil \frac{m_1}{2} \rceil + m_2 + 1$. □

Definition 2.6. [10] *A flower graph* Fl_p *is the graph obtained from a helm graph by joining each pendant vertex to the central vertex of the helm graph.*

Theorem 2.18.

$$AI(Fl_p) = \begin{cases} \lceil \frac{p}{2} \rceil + 3, & \text{if } 4 \leq p \leq 16; \\ \lceil \frac{p}{3} \rceil + 5, & \text{if } 17 \leq p \leq 27; \\ \lceil \frac{p}{4} \rceil + 7, & \text{if } 28 \leq p \leq 44; \\ \lceil \frac{p}{5} \rceil + 9, & \text{if } p \geq 45. \end{cases}$$

Proof. Let Fl_p be a flower graph as in Figure 3. Let $V(Fl_p) = \{v, v_1, v_2, v_3, ..., v_p, u_1, u_2, u_3, ..., u_p\}$, so $|V(Fl_p)| = 2p + 1$. Let S be an accessible set of Fl_p. We have the following cases:

Case 1: p is even and $4 \leq p \leq 16$. Consider $S = \{v, v_1, v_3, ..., v_{p-3}, v_{p-1}\}$, an accessible set of Fl_p such that $|S| = \lceil \frac{p}{2} \rceil + 1$, if we remove the set S from Fl_p, then there exist $\lfloor \frac{p}{2} \rfloor$ components each containing only two vertices, hence $m(Fl_p - S) = 2$. Therefore, $AI(Fl_p) = \lceil \frac{p}{2} \rceil + 3$.

Case 2: p is odd and $4 \leq p \leq 16$. Assume that $S = \{v, v_1, v_3, ..., v_{p-2}, v_p\}$, an accessible set of Fl_p and $|S| = \lceil \frac{p}{2} \rceil + 1$. If S is removed from Fl_p, we get $\lceil \frac{p}{2} \rceil - 1$ components of order 2, i.e $m(Fl_p - S) = 2$. Then $AI(Fl_p) = \lceil \frac{p}{2} \rceil + 3$.

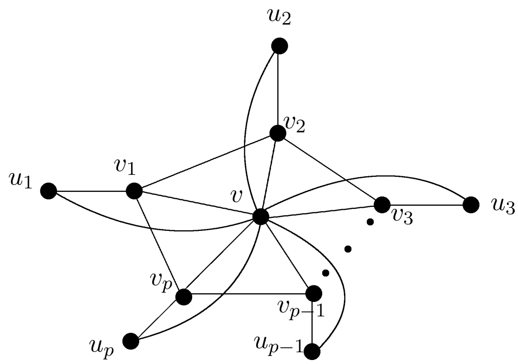

Figure 3: Flower graph

Case 3: $p \equiv 0 \ (mod\ 3), 17 \leq p \leq 27$. Let $S = \{v, v_1, v_4, v_7, ..., v_{p-5}, v_{p-2}\}$ be an accessible set. $|S| = \lceil \frac{p}{3} \rceil + 1$. If we remove the vertices of S from Fl_p, then we get $\lceil \frac{2p-3}{3} \rceil$ components which are P_4, so $m(Fl_p - S) = 4$. Then $AI(Fl_p) = |S| + m(Fl_p - S) = \lceil \frac{p}{3} \rceil + 5$.

Case 4: $p \equiv 1 \ (mod\ 3), 17 \leq p \leq 27$. Let $S = \{v, v_1, v_4, v_7, ..., v_{p-3}, v_p\}$, an accessible set of Fl_p, $|S| = \lceil \frac{p}{3} \rceil + 1$. If S is removed from Fl_p, we have $p - \lceil \frac{p}{3} \rceil - 1$ components, each component being P_4. So $AI(Fl_p) = |S| + m(Fl_p - S) = \lceil \frac{p}{3} \rceil + 5$.

Case 5: $p \equiv 2 \ (mod\ 3), 17 \leq p \leq 27$. Let $S = \{v, v_1, v_4, v_7, ..., v_{p-4}, v_{p-1}\}$, an accessible set of Fl_p such that $|S| = \lceil \frac{p}{3} \rceil + 1$, removing all vertices of S, we get $p - \lceil \frac{p}{3} \rceil - 1$ components, which are P_4 and $m(Fl_p - S) = 4$. Therefore,

$AI(Fl_p) = |S| + m(Fl_p - S) = \lceil \frac{p}{3} \rceil + 5$.
Case 6: If $p \equiv 0\ (mod\ 4), 28 \le p \le 44$. Consider $S = \{v, v_1, v_5, v_9, v_{13}, ..., v_{p-7}, v_{p-3}\}$, an accessible set of Fl_p, so $|S| = \lceil \frac{p}{4} \rceil + 1$, if we remove all vertices of the set S, we get components of order 6. Then $AI(Fl_p) = |S| + m(Fl_p - S) = \lceil \frac{p}{4} \rceil + 7$.
Case 7: If $p \equiv 1\ (mod\ 4), 28 \le p \le 44$. Consider $S = \{v, v_1, v_5, v_9, v_{13}, ..., v_{p-4}, v_p\}$, an accessible set of Fl_p, such that $|S| = \lceil \frac{p}{4} \rceil + 1$. If S is removed from Fl_p, we get $p - \lceil \frac{p}{4} \rceil - 1$ components of order 6, hence $m(Fl_p - S) = 6$. Then $AI(Fl_p) = \lceil \frac{p}{4} \rceil + 7$.
Case 8: If $p \equiv 2\ (mod\ 4), 28 \le p \le 44$. Let $S = \{v, v_1, v_5, v_9, v_{13}, ..., v_{p-5}, v_{p-1}\}$, an accessible set of Fl_p, and $|S| = \lceil \frac{p}{4} \rceil + 1$. If we remove all vertices of the set S, we have components of order 6. Therefore, $AI(Fl_p) = |S| + m(Fl_p - S) = \lceil \frac{p}{4} \rceil + 7$.
Case 9: If $p \equiv 3\ (mod\ 4), 28 \le p \le 44$. Let $S = \{v, v_1, v_5, v_9, v_{13}, ..., v_{p-6}, v_{p-2}\}$, an accessible set of Fl_p, and $|S| = \lceil \frac{p}{4} \rceil + 1$. If we remove all vertices of the set S, we have $p - \lfloor \frac{p}{4} \rfloor - 1$ components of order 6. Therefore, $AI(Fl_p) = |S| + m(Fl_p - S) = \lceil \frac{p}{4} \rceil + 7$.
Case 10: If $p \equiv 0\ (mod\ 5), p \ge 45$. Consider $S = \{v, v_1, v_5, v_9, v_{13}, ..., v_{p-4}\}$, an accessible set of Fl_p, such that $|S| = \lceil \frac{p}{5} \rceil + 1$ and if we remove the set S of Fl_p, then we get $\frac{4p-5}{5}$ components of order 8. Then $AI(Fl_p) = \lceil \frac{p}{5} \rceil + 9$.
Case 11: If $p \equiv 1\ (mod\ 5), p \ge 45$. Let $S = \{v, v_1, v_5, v_9, v_{13}, ..., v_p\}$, an accessible set of Fl_p. Then $|S| = \lceil \frac{p}{5} \rceil + 1$, if S is removed from Fl_p, we get components of order 8 and therefore $AI(Fl_p) = |S| + m(Fl_p - S) = \lceil \frac{p}{5} \rceil + 9$.
Case 12: If $p \equiv 2\ (mod\ 5), p \ge 45$. Let $S = \{v, v_1, v_5, v_9, v_{13}, ..., v_{p-1}\}$, an accessible set of Fl_p. Then $|S| = \lceil \frac{p}{5} \rceil + 1$, if we remove S from Fl_p, then we get $p - \lceil \frac{p}{5} \rceil - 1$ components of order 8, i.e $m(Fl_p - S) = 8$. Then $AI(Fl_p) = \lceil \frac{p}{5} \rceil + 9$.
Case 13: If $p \equiv 3\ (mod\ 5), p \ge 45$. Let $S = \{v, v_1, v_5, v_9, v_{13}, ..., v_{p-2}\}$, an accessible set of Fl_p. Then $|S| = \lceil \frac{p}{5} \rceil + 1$, removal of S from Fl_p, leads to $p - \lceil \frac{p}{5} \rceil - 1$ components of order 8. It follows that $AI(Fl_p) = \lceil \frac{p}{5} \rceil + 9$.
Case 14: If $p \equiv 4\ (mod\ 5), p \ge 45$. Consider $S = \{v, v_1, v_5, v_9, v_{13}, ..., v_{p-3}\}$, an accessible set of Fl_p. Then $|S| = \lceil \frac{p}{5} \rceil + 1$, by the removal of S from Fl_p, we get $p - \lceil \frac{p}{5} \rceil - 1$ components of order 8. Therefore, $AI(Fl_p) = \lceil \frac{p}{5} \rceil + 9$. □

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References

[1] C. A. Barefoot, R. Entringer and H. Swart, *Vulnerability in graphs - A comparative survey*, J. Combin. Math. Combin. Comput., **1** (1987), 12-22.

[2] L. H. Clark, R. C. Entringer and M. R. Fellows, *Computational complexity of integrity*, J. Combin. Math. Combin. Comput., **2** (1987), 179-191.

[3] M. Cozzens, D. Moazzami and S. Stueckle, *The tenacity of a graph*, Proc. Seventh International Conference on the Theory and Applications of Graphs, New York, USA, 1995, 1111-1122.

[4] P. Dundar, *Accessibility number and the neighbor-integrity of generalised petersen graphs*, Neural network world, **2** (2001), 167-174.

[5] R. Frucht and F. Harary, *On the corona of two graphs*, Aequat. Math., **4** (1970), 322-325

[6] W. Goddard and H. C. Swart, *Integrity in graphs: bounds and basics*, J. Combin. Math. Combin. Comput., **7** (1990), 139-151.

[7] I. Gutman and N. Trinajstic, *Graph theory and molecular orbitals, Total - electron energy of alternant hydrocarbons*, Chem. Phys. Lett., **17** (1972), 535-538.

[8] F. Harary, Graph Theory, Addison Wesley, Reading Mass, 1969.

[9] B. L. Hartnell and D. F. Rall, *A bound on size of a graph with given order and bondage number*, Discrete Math., 197/198 (1999), 409-413.

[10] M. A. Seoud and M. Z. Youssef, *Harmonious labellings of helms and related graphs*, preprint.

[11] Sultan Senan Mahde, Veena Mathad and Ali Mohammed Sahal, *Hub-integrity of graphs*, Bull. Int. Math. Virtual Inst., **5** (2015), 57-64

[12] Sultan Senan Mahde and Venna Mathad, *Distance majorization integrity of graphs*, Proc. Jangjeon Math. Soc., 20(3) (2017), 353-364.

[13] Sultan Senan Mahde and Venna Mathad, *Global domination integrity of graphs*, Math. Sci. Let., **6** (2017), 263-269.

[14] Sultan Senan Mahde, Veena Mathad and Ismail Naci Cangul, *Accessibility integrity of graphs*, submitted.

[15] Veena Mathad and Sultan Senan Mahde, *Vulnerability: vertex neighbor integrity of middle graphs*, J. Comp. Math. Sci., **6** (2015), 43-48.

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