Proceedings of the Jangjeon Mathematical Society
23 (2020), No. 2, pp. 159 - 168
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2020.23.2.159

Eccentric Harmonic Index for the Subdivision of Some Graphs

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Abstract

Subdivision is an important aspect in graph theory which allows one to calculate properties of some complicated graphs in terms of some easier graphs. Recently, the notion of r-subdivision was similarly defined as a quite useful generalization by adding r new vertices to each edge. Topological graph indices have become popular due to their applications in chemistry or related areas due to their advantages over time and money consuming laboratory experiments. In this paper, we calculate one of the topological graph indices, namely the eccentric harmonic index for the subdivision, r-subdivision graphs of some graphs.

Keywords: Eccentricity of a vertex, harmonic index, eccentric harmonic index, subdivision of graphs.
AMS Subject Classification: Primary 05C76, 05C07, 92E10.

1 Introduction

Throughout this paper, we assume that all the graphs are finite simple connected graphs. A graph $G = (V, E)$ is a simple graph that is having no loops, no multiple and directed edges. As usual, we denote n to be the order and m to be the size of the graph G. For a vertex $v \in V$, the open neighborhood of v in a graph G, denoted $N(v)$, is the set of all vertices that are adjacent to v and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. The degree of a vertex v_i in G is $d_i = d(v_i) = |N(v_i)|$. A vertex of degree one is called pendant vertex. A graph G is said to be k-regular graph if $d(v) = k$ for every $v \in V(G)$. The distance $d(u, v)$ between any two vertices u and v in a graph G is the length of the shortest path connecting them. The eccentricity of a vertex $v \in V(G)$ is $e(v) = max\{d(u, v) : u \in V(G)\}$. The radius of G is $r = r(G) = min\{e(v) : v \in V(G)\}$ and the diameter of G is $diam(G) = max\{e(v) : v \in V(G)\}$. Hence $r(G) \leq e(v) \leq diam(G)$, for every $v \in V(G)$. A vertex v in a connected graph G is a central vertex if $e(v) = r(G)$, while a vertex v in a connected graph G is a peripheral vertex if $e(v) = diam(G)$. A graph G is called a self-centered graph if $e(v) = r(G) = diam(G)$. If G is a regular graph with $e(v) = r(G) = diam(G)$, then G is a regular self-centered graph. We denote the eccentricity of a vertex v_i by $e(v_i) = e_i$. A graph G is called a double star $B_{n,m}$ if it can be constructed from K_2 by joining n pendent edges to one end and m pendent edges to the other end of K_2. Bistar $B_{n,n}$ is a graph obtained from K_2 by joining n- pendent edges to both the ends of K_2. As usual we use the characters P_n, C_n, $K_{a,b}$, $K_{1,n-1}$, $B_{n,m}$, $B_{n,n}$, K_n for the path, cycle, complete bipartite, star, double-stars, bistars and complete graph respectively. All the graph related definitions and terminologies used in this paragraph are taken from [6].

There are some fixed invariant numbers which do not change for the isomorphic graphs and give information about the graph under consideration and these are called topological graph indices. These indices are defined and used in many areas to study several properties of different real life objects such as atoms and molecules in chemistry. These indices can mainly be classified into three groups according to their definitions: those defined by means of matrices, those by means of vertex degrees and those by means of distances etc. Some of the most well-known vertex degree-based topological indices are the first and second Zagreb indices, first and second multiplicative Zagreb indices, atom-bond connectivity index, Narumi–Katayama index, geometric-arithmetic indices, harmonic index and sum-connectivity index.
Topological graph indices have found many uses in several areas including molecular graph theory, due to their advantages over the existing experimental methods. In recent years, a large number of such indices have been defined and utilized for chemical documentation, isomer discrimination, study of molecular complexity, chirality, similarity/dissimilarity, QSAR/QSPR, drug design, database selection and lead optimization etc. The pharmaceutical industry contributed towards increased interest in molecular descriptors because of the necessity to reduce the expenditure involved in synthesis, in vitro, in vivo or clinical testing of new medicinal compounds.

The classical first and second Zagreb indices was introduced by Gutman and

Trinajestic [4] in 1972 and elaborated in [3] are defined as:

$$M_1(G)=\sum_{i=1}^{n} d_i^2 \qquad and \qquad M_2(G)=\sum_{v_iv_j\in E(G)} d_id_j.$$

The harmonic index $H(G)$ of a graph G was introduced by Fajtlowicz [2] and defined as:

$$H(G)=\sum_{v_iv_j\in E}\frac{2}{d_i+d_j}$$

For details about $H(G)$ the authors advice to see [9, 11, 13, 14].

The eccentric harmonic index $H_e(G)$ of a graph G was introduced by M. I. Sowaity et al [12] and defined as:

$$H_e(G)=\sum_{v_iv_j\in E}\frac{2}{e_i+e_j}$$

The ordinary subdivision graph $S(G)$ of the graph G is obtained from G by inserting a new vertex of degree 2 on each edge of G. For $r\geq 1$, the r^{th} subdivision graph $S_r(G)$ is obtained from G by inserting r new vertices of degree 2 on each edge of G [5].

Theorem 1.1. *[12] Let G be a self-centered graph of order n and size m. Then*

$$H_e(G)=\frac{m}{diam(G)}.$$

Corollary 1.2. *[12] For the cycle C_n, the eccentric harmonic index is*

$$H_e(C_n)=\begin{cases}2, & if\ n\ is\ even,\\ \frac{2n}{n-1}, & if\ n\ is\ odd.\end{cases}$$

Theorem 1.3. *[12] Let $G=P_n$ be a path of order n, $n\geq 2$. Then,*

$$\frac{H_e(P_n)}{2}=\begin{cases}\frac{1}{n}+2\sum\limits_{i=\frac{n}{2}}^{n-2}\frac{1}{2i+1}, & if\ n\ is\ even,\\ 2\sum\limits_{i=\frac{n-1}{2}}^{n-2}\frac{1}{2i+1}, & if\ n\ is\ odd.\end{cases}$$

2 Eccentric Harmonic Index for the subdivision of some Graphs

The Subdivision is an important aspect in graph theory which allows one to calculate properties of some complicated graphs in terms of some easier graphs. Recently, the notion of r-subdivision was similarly defined as a quite useful generalization by adding r new vertices to each edge. Topological graph indices have become popular due to their applications in chemistry or related areas due to their advantages over time and money consuming laboratory experiments. In this section, we calculate the eccentric harmonic index for the subdivision and r-subdivision of some graphs.

Theorem 2.1. *For any cycle* $C_n,\ n \geq 3$

$$H_e(S_r(C_n)) = \begin{cases} 2, & if\ n\ is\ even\ or\ n\ is\ odd\ and\ r\ is\ odd, \\ \frac{2n(r+1)}{n(r+1)-1}, & if\ n\ is\ odd\ and\ r\ is\ even. \end{cases}$$

Proof. Let C_n be a cycle with n vertices. Then $S_r(C_n) = C_{n(r+1)}$. So the r^{th} subdivision of C_n has $n(r+1)$ vertices. Thus, the following cases are forward.
Case 1. If n is even, then $n(r+1)$ is even. Hence, by Corollary 1.2 we get

$$H_e(S_r(C_n)) = 2.$$

Case 2. If n is odd, then we have the following subcases.
Subcase 2.1. If r is odd, then $n(r+1)$ is even. Thus, by Corollary 1.2 we get

$$H_e(S_r(C_n)) = 2.$$

Subcase 2.2. If r is even, then $n(r+1)$ is odd. Thus, by Corollary 1.2 we get

$$H_e(S_r(C_n)) = \frac{2n(r+1)}{n(r+1)-1}.$$

□

Corollary 2.2. *For any cycle* $C_n,\ n \geq 3$

$$H_e(S(C_n)) = 2.$$

Proof. It is clear that $S(C_n)$ has $r = 1$, thus by Theorem 2.1

$$S(C_n) = C_{n(1+1)} = C_{2n}$$

which gives a cycle with even order. Hence, by Corollary 1.2 we get the wanted result. □

Theorem 2.3. *For the complete graph* $K_n,\ n \geq 4$

$$H_e(S(K_n)) = \frac{2n(n-1)}{7}.$$

Proof. Let G be a complete graph with n vertices. Then $S(K_n)$ is a graph that contains $v_1, v_2, ..., v_n, v_{n+1}, ..., v_{\frac{n(n+1)}{2}}$ vertices. Keeping in mind that the vertices $v_1, v_2, ..., v_n$ are the original vertices of the graph and the other are the entering vertices, then the eccentricities of the vertices in $S(K_n)$ are $e(v_i), where\ v_i \in V(S(K_n))$ and results to,

$$e(v_i) = \begin{cases} 3, & if\ i = 1, 2, ..., n, \\ 4, & \text{otherwise.} \end{cases}$$

Also, the edges of $S(K_n)$ are the edges that incident to the entering vertices, thus the number of edges is double the number of the entering vertices, that is

$$m(S(K_n)) = 2\frac{n(n-1)}{2} = n(n-1)$$

Since each edge contains exactly on its both sides, i.e, one entering vertex and the other is the original vertex, then for $v_iv_j \in E(S(K_n))$ the term

$$\frac{2}{e_i + e_j} = \frac{2}{3+4} = \frac{2}{7}$$

So

$$H_e(S(K_n)) = \sum_{v_iv_j \in E(S(K_n))} \frac{2}{7} = \frac{2}{7}n(n-1).$$

□

Theorem 2.4. *For the complete bipartite graph $K_{a,b},\ a, b \geq 2$*

$$H_e(S(K_{a,b})) = \frac{ab}{2}.$$

Proof. It is clear that $S(K_{a,b}),\ a, b \geq 2$, is a self-centered graph with $diam(S(K_{a,b})) = 4$. Thus by Theorem 1.1 we get,

$$H_e(S(K_{a,b})) = \frac{2ab}{4} = \frac{ab}{2}.$$

□

Theorem 2.5. *Let $S_r(S_{1,n-1})$ be the rth subdivision of the star graph $S_{1,n-1},\ n \geq 3$. Then,*

$$H_e(S_r(S_{1,n-1})) = 2(n-1) \sum_{i=r+1}^{2r+1} \frac{1}{2i+1}.$$

Proof. Let $S_r(S_{1,n-1})$ be the rth subdivision of the star graph $S_{1,n-1},\ n \geq 3$. Then

$$H_e(S_r(S_{1,n-1})) = \sum_{v_iv_j \in E(S_r(S_{1,n-1}))} \frac{2}{e_i + e_j}.$$

Since the terms of the summation of H_e are given for the edges of $S_r(S_{1,n-1})$ and by dividing $S_r(S_{1,n-1})$ into $n-1$ path; call it P_r; with each path starting and ending by the central vertex and a pendent vertex of the original star respectively. Since each P_r contains $r+2$ vertices, we get

$$\begin{aligned} H_e(S_r(S_{1,n-1})) &= \sum_{i=1}^{n} \sum_{v_iv_j \in E(P_r)} \frac{2}{e_i+e_j} \\ &= (n-1) \sum_{v_iv_j \in E(P_r)} \frac{2}{e_i+e_j} \\ &= (n-1)H_e(P_r). \end{aligned}$$

By employing Theorem 1.3,keeping in mind that the radii and the diameters of $S_r(S_{1,n-1})$ are $r+1,\ 2(r+1)$ respectively. Then

$$H_e(P_r) = 2\sum_{i=r+1}^{2r+1} \frac{1}{2i+1}$$

Hence,

$$H_e(S_r(S_{1,n-1})) = 2(n-1)\sum_{i=r+1}^{2r+1} \frac{1}{2i+1}$$

. □

Theorem 2.6. *Let $S(S_{1,n-1})$ be the subdivision of the star graph $S_{1,n-1},\ n \geq 3$. Then,*

$$H_e(S(S_{1,n-1})) = \frac{24}{35}(n-1).$$

Proof. Let $S(S_{1,n-1})$ be the subdivision of the star graph $S_{1,n-1},\ n \geq 3$. Then by Theorem 2.5 and using $r=1$, we get

$$\begin{aligned} H_e(S(S_{1,n-1})) &= 2(n-1)\sum_{i=2}^{3} \frac{1}{2i+1} \\ &= 2(n-1)\frac{12}{35} \\ &= \frac{24}{35}(n-1). \end{aligned}$$

□

Proposition 2.7. *Let $S_r(P_n)$ be the r^{th} subdivision of the path $P_n,\ n \geq 2$. Then*

$$S_r(P_n) = P_{n+(n-1)r}. \tag{2.1}$$

In case of computing $H_e(S_r(P_n))$, it is easy to employ Proposition 2.7 in Theorem 1.3. Also, if we substitute $r = 1$ in 2.1, we get $S(P_n) = P_{2n-1}$, which can easily be used to derive $H_e(S(P_n))$ after employing Theorem 1.3.

Theorem 2.8. *For any double star $B_{n,m}$, the eccentric harmonic index is*

$$H_e(B_{n,m}) = \frac{1}{2} + \frac{2(n+m)}{5}.$$

Proof. Let $B_{n,m}$ be the a double star. By the definition of H_e we have,

$$H_e(B_{n,m}) = \sum_{v_iv_j \in E(B_{n,m})} \frac{2}{e_i + e_j}$$

Since the number of edges in $B_{n,m}$ is equals to $n + m + 1$, then by computing the terms we get $n + m$ terms equals to $\frac{2}{5}$ and one term equals to $\frac{1}{2}$. Thus

$$H_e(B_{n,m}) = \frac{1}{2} + \frac{2(n+m)}{5}.$$

□

Corollary 2.9. *For any bistars $B_{n,n}$, the eccentric harmonic index is*

$$H_e(B_{n,n}) = \frac{1}{2} + \frac{4n}{5}.$$

Theorem 2.10. *Let $S_r(B_{n,m})$ be the r^{th} subdivision of the double-stars $B_{n,m}$. Then*

$$H_e(S_r(B_{n,m})) = \begin{cases} 2(n+m)\displaystyle\sum_{i=2(r+1)}^{3r+2} \frac{1}{2i+1} + \frac{2}{3r+4} + 4\sum_{i=\frac{3r+4}{2}}^{2r+1} \frac{1}{2i+1}, & if\ r\ is\ even, \\ 2(n+m)\displaystyle\sum_{i=2(r+1)}^{3r+2} \frac{1}{2i+1} + 4\sum_{i=2(\frac{3}{2}(r+1))}^{2r+1} \frac{1}{2i+1}, & if\ r\ is\ odd. \\ . \end{cases}$$

Proof. Let $S_r(B_{n,m})$ be the rth subdivision of the double-stars $B_{n,m}$ as in Figure 1.

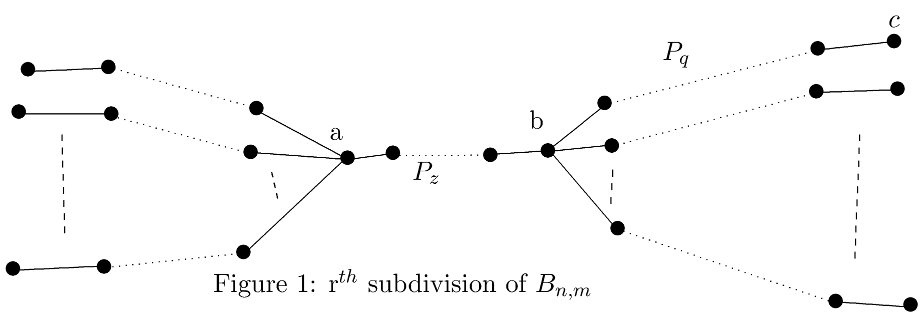

Figure 1: rth subdivision of $B_{n,m}$

Denote the path from a to b by P_z and the path from b to c by P_q, then it is clear that we can divide $S_r(B_{n,m})$ into $n+m$ copies of P_q and one copy of P_z as shown in Figure 1. Thus,

$$\begin{aligned} H_e(S_r(B_{n,m})) &= \sum_{i=1}^{n+m} \sum_{v_iv_j \in E(P_q)} \frac{2}{e_i + e_j} + \sum_{v_iv_j \in E(P_z)} \frac{2}{e_i + e_j} \\ &= (n+m)H_e(P_q) + H_e(P_z). \end{aligned} \tag{2.2}$$

Hence, we have the following two cases.

Case 1. If r is even, then by using Theorem 1.3 and keeping in mind that the radius of P_q is $2(r+1)$ holds for the vertex b and all its synonyms and the eccentricity increases by 1 for the next vertex till it reaches to the diameter that holds at c and all its synonyms and equals to $3r+2$, then

$$H_e(P_q) = 2 \sum_{i=2(r+1)}^{3r+2} \frac{1}{2i+1}. \tag{2.3}$$

For P_z, we know that the radius is $\frac{3r+4}{2}$ and the diameter is $2r+1$, then by using Theorem 1.3, we get

$$H_e(P_z) = \frac{2}{3r+4} + 4 \sum_{i=\frac{3r+4}{2}}^{2r+1} \frac{1}{2i+1}. \tag{2.4}$$

By substituting (2.3) and (2.4) in (2.2) we get,

$$H_e(S_r(B_{n,m})) = 2(n+m) \sum_{i=2(r+1)}^{3r+2} \frac{1}{2i+1} + \frac{2}{3r+4} + 4 \sum_{i=\frac{3r+4}{2}}^{2r+1} \frac{1}{2i+1}.$$

Case 2. If r is odd, then by using Theorem 1.3 and also keeping in mind that the radius of P_q is $2(r+1)$ holds for the vertex b and all its synonyms and the eccentricity increases by 1 for the next vertex till it reaches to the diameter that holds at c and all its synonyms and equals to $3r+2$, then

$$H_e(P_q) = 2 \sum_{i=2(r+1)}^{3r+2} \frac{1}{2i+1}. \tag{2.5}$$

For P_z, since the radius is $\frac{3}{2}(r+1)$ and the diameter is $2r+1$, then by using Theorem 1.3, we get

$$H_e(P_z) = 4 \sum_{i=\frac{3}{2}(r+1)}^{2r+1} \frac{1}{2i+1}. \tag{2.6}$$

By substituting (2.5) and (2.6) in (2.2) we get,

$$H_e(S_r(B_{n,m})) = 2(n+m)\sum_{i=2(r+1)}^{3r+2}\frac{1}{2i+1} + 4\sum_{i=\frac{3}{2}(r+1)}^{2r+1}\frac{1}{2i+1}.$$

Embedding the two cases gives the required result. □

Corollary 2.11. *Let $S_r(B_{n,n})$ be the r^{th} subdivision of the bistars $B_{n,n}$. Then,*

$$H_e(S_r(B_{n,n})) = \begin{cases} 4n\sum_{i=2(r+1)}^{3r+2}\frac{1}{2i+1} + \frac{2}{3r+4} + 4\sum_{i=\frac{3r+4}{2}}^{2r+1}\frac{1}{2i+1}, & if\ r\ is\ even, \\ 4n\sum_{i=2(r+1)}^{3r+2}\frac{1}{2i+1} + 4\sum_{i=2(\frac{3}{2}(r+1))}^{2r+1}\frac{1}{2i+1}, & if\ r\ is\ odd. \\ . \end{cases}$$

Corollary 2.12. *Let $S(B_{n,m})$ be the subdivision of the double-stars $B_{n,m}$. Then,*

$$H_e(S(B_{n,m})) = \frac{40}{99}(n+m) + \frac{4}{7}.$$

Proof. Let $S(B_{n,m})$ be the subdivision of the double-stars $B_{n,m}$. Then by substituting $r = 1$ in Theorem 2.10, we get

$$\begin{aligned} H_e(S(B_{n,m})) &= 2(n+m)(\frac{1}{9} + \frac{1}{11}) + 4 \times \frac{1}{7} \\ &= \frac{40}{99}(n+m) + \frac{4}{7}. \end{aligned}$$

□

Corollary 2.13. *Let $S(B_{n,n})$ be the subdivision of the bistars $B_{n,n}$. Then,*

$$H_e(S(B_{n,n})) = \frac{80}{99}n + \frac{4}{7}.$$

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