Proceedings of the Jangjeon Mathematical Society
23 (2020), No. 2, pp. 137 - 140
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2020.23.2.137

ONE-DIMENSIONAL LOCALLY BOUNDED NOT NECESSARILY CONTINUOUS PSEUDOREPRESENTATIONS OF THE GROUP $SL(2,\mathbb{Q}_p)$ WITH SMALL DEFECT ARE IDENTITY REPRESENTATIONS

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Abstract. We prove that every one-dimensional not necessarily continuous locally bounded pseudorepresentation of the group $\mathrm{SL}(2,\mathbb{Q}_p)$ with sufficiently small defect is the ordinary identity representation of the group. Therefore, for every group generated by its subgroups isomorphic to $\mathrm{SL}(2,\mathbb{Q}_p)$, every one-dimensional pseudorepresentation of the group with sufficiently small defect is the identity representation.

§ 1. Introduction

We need some definitions, which are given below in the simplest form sufficient for our purposes. Let G be a group, and let π be a locally bounded (i.e., bounded on some neighborhood of the identity element) mapping of G into the field $\mathbb{C}$ of the complex numbers such that $\pi(e_G)=1\in\mathbb{C}$ (e_G stands for the identity element of G) and

$$|\pi(g_1g_2)-\pi(g_1)\pi(g_2)|\le\varepsilon$$

for all $g_1,g_2\in G$ and some $\varepsilon\ge 0$; then π is said to be a one-dimensional quasirepresentation of G with defect ε. A one-dimensional quasirepresentation is said to be a pseudorepresentation if

$$\pi(g^n)=\pi(g)^n \quad \text{for all} \quad g\in G \quad \text{and all positive integers} \quad n.$$

2010 *Mathematics Subject Classification.* Primary 22A99, Secondary 22A25, 22A10.
Key words and phrases. pseudorepresentation, $\mathrm{SL}(2,\mathbb{Q}_p)$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

A one-dimensional pseudorepresentation π is said to be *pure* if the restriction of π to every amenable subgroup of G (in particular, to every commutative subgroup of G) is an ordinary character of the subgroup (see [1]). For specific features concerning one-dimensional pseudorepresentations, see [2].

§ 2. Preliminaries

Let us consider the group $G = \mathrm{SL}(2, \mathbb{Q}_p)$ of 2×2 matrices over the field $\mathbb{Q}_p$ of p-adic numbers (p is a prime) with determinant one. Let K be the compact subgroup of G formed by the matrices $u = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $ad - bc = 1$ in $\mathbb{Q}_p$ and $a, b, c, d \in \mathbb{O}_p$, where $\mathbb{O}_p$ is the ring of p-adic integers (i.e., if $|\cdot|$ stands for the p-adic valuation, then $|a|, |b|, |c|, |d| \leq 1$). As is well known (and can readily be seen), every element $g \in G$ can be represented as a product $g = ur$ of an element u of K and an element r of the subgroup

$$R = \{r(\lambda, \mu) = \left(\begin{smallmatrix} \lambda & \mu \\ 0 & \lambda^{-1} \end{smallmatrix}\right)\},$$

where $\mu \in \mathbb{Q}_p$ and λ belongs to the multiplicative group $\mathbb{Q}_p^*$ of the field $\mathbb{Q}_p$. This representation is obviously not unique.

§ 3. Main theorem

The following assertion is the main result of the paper.

Theorem 1. *Every one-dimensional continuous pseudorepresentation of the group $G = \mathrm{SL}(2, \mathbb{Q}_p)$ with sufficiently small defect ($\varepsilon < 1/6$) is the one-dimensional identity representation.*

Proof. Let π be a one-dimensional pseudorepresentation of G with a defect $\varepsilon < 1/6$. Since the group R is obviously solvable, it follows from the fundamental property of pseudorepresentations [1] that the restriction of π to R is an ordinary one-dimensional representation of R.

Note that, if π is unbounded, then π is an ordinary unbounded representation of G [1]. One can immediately see that the only ordinary one-dimensional representation of G is the identity representation, and hence π is bounded; this means that the restriction of π to R is an ordinary unitary character of R. Clearly, every mapping of this kind takes the elements of the subgroup

$$N = \{n(\mu) = \left(\begin{smallmatrix} 1 & \mu \\ 0 & 1 \end{smallmatrix}\right), \mu \in \mathbb{Q}_p\}$$

to one.

On the other hand, every element of the group K is a product of elements of the form

$$\mu(a,b,c) = \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right), \quad \text{where} \quad a,b,c \quad \text{are } p\text{-adic integers},$$

$$\nu(a,b,c) = \left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right), \quad \text{where} \quad a,b,c \quad \text{are } p\text{-adic integers},$$

and

$$w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right).$$

Certainly, the restriction of π to the subgroup of $\{\mu(a,b,c)\}$ does not depend on b and the restriction of π to the subgroup of $\{\nu(a,b,c)\}$ does not depend on b either. Let us use the well-known formula

$$(1) \qquad r(a,0) = wn(a^{-1})wn(a)wn(a^{-1}), \qquad a \in \mathbb{Q}_p^*,$$

An elementary computation with the inequality $|1-\pi(f)\pi(f^2)| < \varepsilon < 1/6$ shows that $\pi(f)$, where $f = wn(1)$, is a root of unity of order three which is sufficiently close to $1 \in \mathbb{C}$, and thus is equal to 1. Since $\pi(w)^4 = 1$, it follows that $|\pi(wn(1)) - \pi(w)| < \varepsilon < 1/6$, or $|1-\pi(w)| < 1/6$, which shows that $\pi(w) = 1$. Then (1) shows that $\{\pi(r(a,0))\}$ belongs to a small neighborhood of 1 and is a subgroup in this neighborhood. Thus,

$$(2) \qquad \pi(r(a,0)) = 1 \quad \text{for all} \quad a \in \mathbb{Q}_p.$$

Combining the above observations, we see that the restriction of π to K belongs to a small neighborhood of 1. Since every element of K is conjugate to one of the elements of the form μ, nu, and w, the restriction of π to the subgroups $\{\mu\}$, $\{\nu\}$, and $\{w^n, n \in \mathbb{Z}\}$ is unitary, and the defect of π is sufficiently small, it follows that $\pi|_K$ is identically equal to 1.

Let us use formula (1) again. It is clear from the consideration of the restriction of π to K and N that

$$\pi(n(a)) = \pi(a^{-1}) = 1 \quad \text{for every} \quad a \in \mathbb{Q}_p^*.$$

Moreover, formula (2) shows that the restriction of the unitary pseudorepresentation π to the group $\{r(a,0), a \in \mathbb{Q}_p^*\}$ is equal to $\{1\}$. Thus, the image of π is a union of subgroups $\{\pi(g^n), n \in \mathbb{Z}\}$, belonging to a small neighborhood of 1 in the circle $\mathbb{T}$, and hence this image coincides with 1, which completes the proof.

§ 4. Discussion

Corollary. *Let G be a group generated by its subgroups isomorphic to $\mathbb{Q}_p$. Then every one-dimensional locally bounded (not necessarily continuous) pseudorepresentation of G with a defect $\varepsilon < 1/6$ is an ordinary trivial one-dimensional representation of G.*

Proof. Immediate.

Acknowledgments

I thank Professor Taekyun Kim for the invitation to publish this paper in the Proceedings of the Jangjeon Mathematical Society.

Funding

The research was partially supported by the Russian Foundation for Basic Research (grant no. 18-01-00398).

References

1. A. I. Shtern, *Finite-Dimensional Quasirepresentations of Connected Lie Groups and Mishchenko's Conjecture*, Fundam. Prikl. Mat. **13** (2007), no. 7, 85–225; J. Math. Sci. (N.Y.) **159** (2009), no. 5, 653–751.
2. A. I. Shtern, *Specific properties of one-dimensional pseudorepresentations of groups*, Fundam. Prikl. Mat. **21** (2016), no. 1, 247–255; English transl. in: J. Math. Sci. (N.Y.) **233** (2018), no. 5, 770–776.

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