Proceedings of the Jangjeon Mathematical Society
23 (2020), No. 1. pp. 109 – 112
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2020.23.1.109

ONE-DIMENSIONAL CONTINUOUS PSEUDOREPRESENTATIONS OF THE GROUP $SL(2,\mathbb{Q}_p)$ ARE IDENTITY REPRESENTATIONS

A. I. Shtern

Abstract. We introduce the notion of discontinuity set for a locally relatively compact pure pseudorepresentation of a topological group and establish its simplest properties.

§ 1. Introduction

We need some definitions, which are given below in the simplest form sufficient for our purposes. Let G be a group, and let π be a locally bounded (i.e., bounded on some neighborhood of the identity element) mapping of G into the Banach algebra of continuous linear operators $\mathcal{L}(E)$ on some normed vector space E such that $\pi(e_G) = 1_E$ (e_G stands for the identity element of G and 1_E for the identity operator on E) and

$$\|\pi(g_1g_2) - \pi(g_1)\pi(g_2)\| \le \varepsilon$$

for all $g_1, g_2 \in G$ and some $\varepsilon \ge 0$; then π is said to be a quasirepresentation of G on E with defect ε. A quasirepresentation is said to be a pseudorepresentation if

$$\pi(g^n) = A(n,g)\pi(g)^n A(n,g)^{-1}$$

for all $g \in G$ and all positive integers n, where

$$A(n,g) \in \mathcal{L}(E)$$

2010 *Mathematics Subject Classification.* Primary 22A99, Secondary 22A25, 22A10.
Key words and phrases. pseudorepresentation, $\mathrm{SL}(2,\mathbb{Q}_p)$.

and

$$\|A(n,g) - 1_E\| \le \delta(\varepsilon)$$

for some increasing positive function δ on $(0, +\infty)$ with $\lim_{\varepsilon\to 0}\delta(\varepsilon) = 0$. A pseudorepresentation is said to be *pure* if $A(n,g) \equiv 1_E$ for all $g \in G$ and all positive integers n (see [1]). For specific features concerning one-dimensional pseudorepresentations, see [2].

§ 2. Preliminaries

Let us consider the group $G = \mathrm{SL}(2, \mathbb{Q}_p)$ of 2×2 matrices over the field $\mathbb{Q}_p$ of p-adic numbers (p is a prime) with determinant one. Let K be the compact subgroup of G formed by the matrices $u = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $ad - bc = 1$ in $\mathbb{Q}_p$ and $a, b, c, d \in \mathbb{O}_p$, where $\mathbb{O}_p$ is the ring of p-adic integers (i.e., if $|\cdot|$ stands for the p-adic valuation, then $|a|, |b|, |c|, |d| \le 1$). As is well known (and can readily be seen), every element $g \in G$ can be represented as a product of an element u of K and an element r of the subgroup

$$R = \{r(\lambda, \mu) = \left(\begin{smallmatrix} \lambda & \mu \\ 0 & \lambda^{-1} \end{smallmatrix}\right), \}$$

where $\mu \in \mathbb{Q}_p$ and λ belongs to the multiplicative group $\mathbb{Q}_p^*$ of the field $\mathbb{Q}_p$. This representation is obviously not unique.

§ 3. Main theorem

The following assertion is the main result of the paper.

Theorem 1. *Every one-dimensional continuous pseudorepresentation of the group $G = \mathrm{SL}(2, \mathbb{Q}_p)$ with sufficiently small defect ($\varepsilon < \sqrt{3}/6$) is the one-dimensional identity representation.*

Proof. Let π be a one-dimensional pseudorepresentation of G with a defect $\varepsilon < 1/6$. Since the group R is obviously solvable, it follows from the fundamental property of a one-dimensional pseudorepresentation [2] that the restriction of π to R is an ordinary representation of $R/$

Note that, if π is unbounded, then π is an ordinary unbounded representation of G. One can immediately see that the only ordinary one-dimensional representation of G is the identity representation, and hence π is bounded; this means that the restriction of π to R is an ordinary unitary character of R. Clearly, every mapping of this kind takes the elements of the subgroup

$$N = \{n(\mu) = \left(\begin{smallmatrix} 1 & \mu \\ 0 & 1 \end{smallmatrix}\right), \mu \in \mathbb{Q}_p\}$$

to one.

On the other hand, the group K is (topologically) amenable (as a compact group), and therefore it follows from the continuity assumption for π and from the fundamental property of pseudorepresentations that the continuous restriction of π to K is an ordinary (one-dimensional) representation of K. Therefore, this restriction is equal to one on the commutators. Since the commutators generate the whole group K, it follows that the restriction of π to K is the identity representation of K.

Finally, let us use the well-known formula

$$r(a,0) = wn(a^{-1})wn(a)wn(a^{-1}), \qquad a \in \mathbb{Q}_p^*,$$

where

$$w = \left(\begin{smallmatrix} 0 & 1 \\ - & 0 \end{smallmatrix}\right).$$

Since it is clear from the consideration of the restriction to K and N that $\pi(w) = 1$ and

$$\pi(n(a)) = \pi(a^{-1}) = 1$$

for every $a \in \mathbb{Q}_p^*$, this formula shows that the restriction of the unitary pseudorepresentation π to the group $\{r(a,0), a \in \mathbb{Q}_p^*\}$ belongs to the ball with center at 1 and radius less than $\sqrt{3}$. Therefore, the image of this restriction is equal to $\{1\}$. Thus, the image of π is a union of subgroups belonging to a small neighborhood of 1 in the circle $\mathbb{T}$, and hence this image coincides with 1, which completes the proof.

§ 4. Discussion

The result can be extended to semisimple Chevalley group over $\mathbb{Q}_p$.

Acknowledgments

I thank Professor Taekyun Kim for the invitation to publish this paper in the Proceedings of the Jangjeon Mathematical Society.

Funding

The research was partially supported by the Russian Foundation for Basic Research (grant no. 18-01-00398).

References

1. A. I. Shtern, *Finite-Dimensional Quasirepresentations of Connected Lie Groups and Mishchenko's Conjecture*, Fundam. Prikl. Mat. **13** (2007), no. 7, 85–225; J. Math. Sci. (N.Y.) **159** (2009), no. 5, 653–751.
2. A. I. Shtern, *Specific properties of one-dimensional pseudorepresentations of groups*, Fundam. Prikl. Mat. **21** (2016), no. 1, 247–255; English transl. in: J. Math. Sci. (N.Y.) **233** (2018), no. 5, 770–776.

Department of Mechanics and Mathematics,
Moscow State University,
Moscow, 119991 Russia, and
Scientific Research Institute of System Analysis (FGU FNTs NIISI RAN),
Russian Academy of Sciences,
Moscow, 117312 Russia
E-mail: aishtern@mtu-net.ru, rroww@mail.ru