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ON FRACTIONAL n-ABSORBING IDEALS OF INTEGRAL DOMAINS

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Abstract. Let R be an integral domain and n a positive integer. In this paper, we introduce the concept of a fractional n-absorbing ideal of R which is a generalization of a strongly prime ideal. Various ring theoretic properties of fractional n-absorbing ideals are studied. In particular, some conditions under which a strongly primary ideal is a fractional n-absorbing ideal are considered.

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1. Introduction

Let R be a commutative ring with a non-zero identity. A prime ideal P of R is a proper ideal of R with the property that for $a, b \in R$, $ab \in P$ implies $a \in P$ or $b \in P$. In the literature, there are several different generalizations of prime ideals (see for example [3, 4, 9]). One useful generalization is the notion of n-absorbing ideal which was firstly investigated by Badawi [4] for n=2, and then it has extensively studied for each positive integer n by Anderson and Badawi [1]. In recent years, 2-absorbing ideals have been generalized in several directions (see for example [6, 7, 8, 11, 12]). For a positive integer n, a proper ideal I of a commutative ring R is called an n-absorbing ideal if whenever $a_1 \dots a_{n+1} \in I$ for $a_1, \dots, a_{n+1} \in R$, then there are n of the a_i's whose product is in I. Prime ideals have also been generalized to strongly n-absorbing ideals [1]. A proper ideal I of a ring R is said to be a strongly n-absorbing ideal if whenever $I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, \dots, I_{n+1}$ of R, then the product of some n of the I_j's is contained in I. It is evident that a 1-absorbing ideal and a strongly 1-absorbing ideal are just a prime ideal. Clearly, a strongly n-absorbing ideal of R is also an n-absorbing ideal of R, and it has been conjectured that these two concepts are equivalent. It has been shown that they agree in every commutative ring for n = 2 [4, Theorem 2.13], and in Prüfer domains for any positive integer n [1, Corollary 6.9].

Another generalization of prime ideals is the concept of strongly prime ideals introduced by Hedstrom and Houston [10]. In fact, a non-zero proper ideal P of a domain R with quotient filed K is called a strongly prime ideal of R if for all $a, b \in K$, $ab \in P$ implies that $a \in P$ or $b \in P$.

In this paper, we introduce the concept of fractional n-absorbing ideal of

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an integral domain R, which is a generalization of strongly prime ideals on the one hand and a generalization of n-absorbing ideals on the other.

Definition 1.1. *Let R be an integral domain with quotient filed K. For a positive integer n, a proper ideal I of a commutative ring R is called a fractional n-absorbing ideal if whenever $a_1 \dots a_{n+1} \in I$ for $a_1, \dots, a_{n+1} \in K$, then there are n of the a_i's whose product is in I.*

With this definition, fractional 1-absorbing ideals are just strongly prime ideals. Naturally, it would have been better for us to name fractionally n-absorbing ideal by strongly n-absorbing ideal, but due to the use of this term by Anderson and Badawi in another sense we prefer that to use "fractional n-absorbing" for our definition.

It is clear that every fractional n-absorbing ideal of an integral domain R is an n-absorbing ideal. However, as Example 1.2 shows, the converse need not be true. It is easily seen that a fractional n-absorbing ideal is fractional m-absorbing for all $m \geq n$. Now, if I is a fractional n-absorbing ideal for some positive integer n, then define

$$\mu_R(I) = \min\{n \mid I \text{ is a fractional } n\text{-absorbing ideal of } R\};$$

otherwise, set $\mu_R(I) = \infty$ (when the context is clear we just write $\mu(I)$). As in [1], $\omega(I) = \min\{n \mid I \text{ is an } n\text{-absorbing ideal of } R\}$. Since every fractional n-absorbing ideal of an integral domain R is also an n-absorbing ideal, we have $\omega(I) \leq \mu(I)$. This inequality may be strict as the following example shows.

Example 1.2. Let $R = \mathbb{Z}$ be the ring of integers and $K = \mathbb{Q}$ be the filed of rational numbers. Consider the ideal $I = 4\mathbb{Z}$ of R. Then by [1, Theorem 2.1 (d)], I is a 2-absorbing ideal of R. In particular, by [1, Theorem 2.1 (b)], I is an n-absorbing ideal for all $n \geq 2$. However, it is not a fractional 2-absorbing ideal of R, since $\frac{2}{3} \cdot \frac{3}{2} \cdot 4 \in I$, but $\frac{2}{3} \cdot \frac{3}{2} \notin I$, $\frac{2}{3} \cdot 4 \notin I$ and $\frac{3}{2} \cdot 4 \notin I$. In fact, there is no positive integer n such that $I = 4\mathbb{Z}$ is a fractional n-absorbing ideal of R. Because for every positive integer n, we can choose n distinct prime integers $p_1, \dots, p_n$. Now, if $a_1 = \frac{p_1}{p_2}, a_2 = \frac{p_2}{p_3}, \dots, a_{n-1} = \frac{p_{n-1}}{p_n}, a_n = \frac{p_n}{p_1}, a_{n+1} = 4$, then clearly $a_1 \cdots a_{n+1} \in I$, but no product of any n of the a_i's is in I. Hence I is not a fractional n-absorbing ideal of R for each positive integer n and so $\mu(I) = \infty$. Note that $\omega(I) = 2$.

2. Basic properties of fractional n-absorbing ideals

In this section, we give some basic properties of fractional n-absorbing ideals, and investigate the stability of fractional n-absorbing ideals with respect to some usual ring constructions. We recall that every fractional n-absorbing ideal of an integral domain R is an n-absorbing ideal, but the converse is not true in general (Example 1.2). We start by giving conditions under which these concepts are equivalent.

Theorem 2.1. *Let R be an integral domain with quotient field K, and I an ideal of R. Assume that for each $x \in K \setminus R$, $x^{-1}I \subseteq I$. Then I is a 2-absorbing ideal of R if and only if I is a fractional 2-absorbing ideal of R.*

Proof. Let I be a 2-absorbing ideal of R, and $abc \in I$ for $a, b, c \in K$. If $a, b, c \in R$, then there is nothing to prove. Hence, we may assume that $a \notin R$. Thus by the assumption $bc = a^{-1}(abc) \in I$, which shows I is a fractional 2-absorbing ideal. The converse is clear. $\square$

Theorem 2.2. *Let R be a valuation domain and I a proper ideal of R. Then I is an n-absorbing ideal of R if and only if I is a fractional n-absorbing ideal of R.*

Proof. Let I be an n-absorbing ideal of R. Assume that $a_1 \cdots a_{n+1} \in I$ for $a_1, \ldots, a_{n+1} \in K$. If $a_1, \ldots, a_{n+1} \in R$, then there is noting to prove. Assume that $a_j \in K \setminus R$ for some $1 \leq j \leq n+1$. Since R is a valuation domain, we must have $a_j^{-1} \in R$. Hence $a_1 \cdots a_{j-1}a_{j+1} \cdots a_{n+1} = a_j^{-1}(a_1 \cdots a_{n+1}) \in I$. Thus I is a fractional n-absorbing ideal. The converse is clear. $\square$

Corollary 2.3. *Let R be a valuation domain with quotient field K and n a positive integer. Then the following statements are equivalent for an ideal I of R:*

1. *I is a fractional n-absorbing ideal of R;*
2. *I is a P-primary ideal of R for some prime ideal P of R and $P^n \subseteq I$;*
3. *$I = P^m$ for some prime ideal $P(= \mathrm{rad}(I))$ of R and integer m with $1 \leq m \leq n$. Moreover, $\mu(P^n) = n$ for a non-idempotent prime ideal P of R.*

Proof. It follows from Theorem 2.2 and [1, Theorem 5.5]. $\square$

Corollary 2.4. *Let R be a valuation domain with quotient field K, and M be its unique maximal ideal. Then M^n is an n-absorbing ideal for all positive integer n.*

Proof. Use (2) $\Rightarrow$ (1) of Corollary 2.3. $\square$

Proposition 2.5. *Let R be an integral domain with quotient field K, I is a fractional 2-absorbing ideal of R and $x \in K \setminus R$. Then for each $a \in I$, either $x^{-1}a \in I$ or $xa \in I$.*

Proof. Let $x \in K \setminus R$ and $a \in I$. Then we have $a = xx^{-1}a \in I$. Hence, either $xx^{-1} \in I$ or $x^{-1}a \in I$ or $xa \in I$, since I is a fractional n-absorbing ideal of R. But I is a proper ideal and so we must have either $xa \in I$ or $x^{-1}a \in I$. $\square$

Proposition 2.6. *Let R be an integral domain with quotient field K. If $P_1, \ldots, P_n$ are strongly prime ideals of R, then $P_1 \cap \cdots \cap P_n$ is a fractional n-absorbing ideal of R. Moreover, $\mu(P_1 \cap \cdots \cap P_n) \leq n$.*

Proof. We proceed by induction on n, the number of strongly prime ideals. Assume that $n = 2$, and $a_1a_2a_3 \in P_1 \cap P_2$ for some $a_1, a_2, a_3 \in K$. Since P_1 is a strongly prime ideal of R, we may assume that $a_1 \in P_1$. Now $a_1a_2a_3 \in P_2$ implies that either $a_1 \in P_2$ or $a_2 \in P_2$ or $a_3 \in P_2$. Hence, either $a_1 \in P_1 \cap P_2$ or $a_1a_2 \in P_1 \cap P_2$ or $a_1a_3 \in P_1 \cap P_2$. Thus $P_1 \cap P_2$ is a fractional 2-absorbing ideal of R. Now, assume that $n > 2$ and the result holds for $n-1$. Let $a_1 \cdots a_{n+1} \in P_1 \cap \cdots \cap P_n$ for some $a_1, \ldots, a_{n+1} \in K$. Since P_1 is a strongly prime ideal of R, we may assume that $a_1 \in P_1$. By induction hypothesis $P_2 \cap \cdots \cap P_n$ is a fractional $(n-1)$-absorbing ideal of R and so

$a_1 \cdots a_{n+1} \in P_2 \cap \cdots \cap P_n$ implies that there are $n-1$ of a_i's whose product is in I. If a_1 is one of these a_i's, then we are done. Otherwise, we may assume that $a_2 \cdots a_n \in P_2 \cap \cdots \cap P_n$, and therefore $a_1 a_2 \cdots a_n \in P_1 \cap \cdots \cap P_n$, which completes the proof.

The "Moreover" statement is clear. □

Proposition 2.7. *Let R be an integral domain with quotient field K and P a prime ideal of R. If I is a fractional n-absorbing ideal of R containing I, then I/P is a fractional n-absorbing ideal of R/P.*

Proof. First note that the filed of fractions of the domain R/P is isomorphic to R_P/PR_P. Let $\overline{a_1}, \ldots, \overline{a_{n+1}} \in R_P/PR_P$ such that $\overline{a_1} \cdots \overline{a_{n+1}} \in I/P$. Then $a_1 \cdots a_{n+1} \in I$. Since $a_1, \ldots, a_{n+1} \in R_P \subseteq K$ and I is a fractional n-absorbing ideal of R, we conclude that $\hat{a}_j = a_1 \cdots a_j a_{j-1} \cdots a_{n+1} \in I$ for some $1 \leq j \leq n+1$. Thus $\overline{a_1} \cdots \overline{a_{j-1}}\ \overline{a_{j+1}} \cdots \overline{a_{n+1}} \in I/P$. Hence I/P is a fractional n-absorbing ideal of R/P. □

Theorem 2.8. *Let R and R' be integral domains with the quotient fields K and K' respectively. Assume that $f : K \longrightarrow K'$ is a ring homomorphism with $f(R) \subseteq R'$. Then the following statements hold:*

(1) *If J is a fractional n-absorbing ideal of R', then $f^{-1}(J)$ is a fractional n-absorbing ideal of R. Moreover, $\mu_R(f^{-1}(J)) \leq \mu_{R'}(J)$.*

(2) *If f is surjective and I is an ideal of R containing $\ker(f)$, then $f(I)$ is a fractional n-absorbing ideal of R' if and only if I is a fractional n-absorbing ideal of R. Moreover, $\mu_{R'}(f(I)) = \mu_R(I)$. In particular, this holds if f is an isomorphism.*

Proof. (1) Let $a_1, \ldots, a_{n+1} \in K$ be such that $a_1 \cdots a_{n+1} \in f^{-1}(J)$. Then $f(a_1) \cdots f(a_{n+1}) = f(a_1 \cdots a_{n+1}) \in J$. Since J is a strongly n-absorbing ideal of R', we may assume that $f(a_1) \cdots f(a_n) \in J$. It follows that $f(a_1 \cdots a_n) \in J$ and so $a_1 \cdots a_n \in f^{-1}(J)$.

The "moreover" statement is clear.

(2) Let I be a fractional n-absorbing ideal of R, and $b_1 \cdots b_{n+1} \in f(I)$ for some $b_1, \ldots, b_{n+1} \in K'$. Then for each $1 \leq i \leq n+1$ there exists $a_i \in K$ such that $f(a_i) = b_i$. Thus we have $a_1 \cdots a_{n+1} \in I$, since $\ker(f) \subseteq I$. Since I is a fractional n-absorbing ideal of R we may assume that $a_1 \cdots a_n \in I$, and therefore $b_1 \cdots b_n \in f(I)$. Conversely, assume that $f(I)$ is a fractional n-absorbing ideal of R'. Note that, we have $f^{-1}(f(I)) = I$, since $\ker(f) \subseteq I$. Now, by (1), I is a fractional n-absorbing ideal of R.

The "moreover" and "in particular" statements are clear.

□

Corollary 2.9. *Let $R \subseteq R'$ be an extension of integral domains and J a strongly n-absorbing ideal of R'. Then $J \cap R$ is a fractional n-absorbing ideal of R. Moreover, $\mu_R(J \cap R) \leq \mu_{R'}(J)$.*

Proof. Consider the inclusion map $f : R \longrightarrow R'$. Clearly, f can be extended to a homomorphism $\hat{f} : K \longrightarrow K'$ defined by $\hat{f}(r/s) = f(r)/f(s)$. Now, the result follows from Theorem 2.8(1). □

Theorem 2.10. *Let R be an integral domain with quotient field K and I an n-absorbing ideal (or in particular a fractional n-absorbing ideal) of R.*

Assume that S is a multiplicatively closed subset of R such that $0 \notin S$ and $I \cap S = \emptyset$. Then I_S is a fractional n-absorbing ideal of R_S. Moreover, $\mu_{R_S}(I_S) \leq \omega_R(I)$.

Proof. Note that R_S is an integral domain since $0 \notin S$. Moreover, the quotient field of R_S is K. Let $a_1, \ldots, a_{n+1} \in K$ be such that $a_1 \cdots a_{n+1} \in I_S$. Then there are elements $t \in R \setminus \{0\}$ and $x_1, \ldots, x_{n+1} \in R$ such that

$$a_1 \cdots a_{n+1} = (x_1/t) \cdots (x_{n+1}/t) = x_1 \cdots x_{n+1}/t^{n+1} \in I_S.$$

Thus $x_1 \cdots x_{n+1} \in I$. Since I is a fractional n-absorbing ideal of R there are n of the x_i's whose product is in I, and thus there are n of the a_i's whose product is in I_S. □

Proposition 2.11. *Let R be an integral domain with quotient field K. If I is a fractional n-absorbing ideal of R, then $\operatorname{rad}(I)$ is a fractional n-absorbing ideal of R and $a^n \in I$ for all $a \in \operatorname{rad}(I)$.*

Proof. Since I is a fractional n-absorbing ideal, it is an n-absorbing ideal and so $a^n \in I$ for all $a \in \operatorname{rad}(I)$. Let $a_1 \cdots a_{n+1} \in \operatorname{rad}(I)$ for $a_1, \ldots, a_{n+1} \in K$. Then $a_1^n \cdots a_{n+1}^n = (a_1 \cdots a_{n+1})^n \in I$. Since I is a fractional n-absorbing ideal of R, we may assume that $a_1^n \cdots a_n^n \in I$. Thus $a_1 \cdots a_n \in \operatorname{rad}(I)$, and therefore $\operatorname{rad}(I)$ is a fractional n-absorbing ideal of R. The second part is clear. □

3. Fractional n-absorbing, strongly prime and strongly primary ideals

Let R be an integral domain with quotient field K. Recall that $\omega(I) \leq \mu(I)$ for each ideal I of R. In contrast to Example 1.2, $\omega(I) = \mu(I)$ may happen, as the next theorem shows.

Theorem 3.1. *Let R be an integral domain with quotient field K. Let I be a fractional n-absorbing ideal of R such that I has exactly n minimal prime ideals, say $P_1, \ldots, P_n$. Then $P_1 \cdots P_n \subseteq I$. Moreover, $\omega(I) = \mu(I) = n$.*

Proof. Since every fractional n-absorbing ideal is an n-absorbing ideal, by [1, Theorem 2.14], we have $P_1 \cdots P_n \subseteq I$ and $\omega(I) = n$. The "moreover" statement follows from the fact that $\omega(I) \leq \mu(I) \leq n$. □

Corollary 3.2. *Let R be an integral domain with quotient field K. Let I be a fractional n-absorbing ideal of R such that I has exactly n minimal prime ideals, say $P_1, \ldots, P_n$. If the P_i's are comaximal, then $I = P_1 \cdots P_n$. Moreover $\omega(I) = \mu(I) = n$. In particular, this holds if $\dim(R) \leq 1$.*

Proof. By Theorem 3.1, we have $P_1 \cdots P_n \subseteq I \subseteq P_1 \cap \cdots \cap P_n$ and $P_1 \cap \cdots \cap P_n = P_1 \cdots P_n$ since the P_i's are comaximal. Thus $I = P_1 \cdots P_n$.

The "moreover" and "in particular" statements are clear. □

Corollary 3.3. *Let R be an integral domain with quotient field K, and I be a fractional n-absorbing ideal of R such that I has exactly n minimal prime ideals, say $P_1, \ldots, P_n$. Then $I_{P_i} = P_{iP_i}$ (in R_{P_i}) for all $1 \leq i \leq n$.*

Proof. By Theorem 3.1, we have $P_1 \cdots P_n \subseteq I \subseteq P_i$ for all $1 \leq i \leq n$, and we get the result by localizing these inclusions at P_i. □

Recall that, a proper ideal I of an integral domain R with quotient field K is called a strongly primary ideal of R if whenever $ab \in I$ for $a, b \in K$, then $a \in I$ or $b \in \mathrm{rad}(I)$ (see [5]). It is clear that every strongly primary ideal of an integral domain R is a primary ideal, but the converse is not true in general. For instance, if $I = 4\mathbb{Z}$, then I is a primary ideal of the ring of integers $\mathbb{Z}$ but not a strongly primary ideal of $\mathbb{Z}$ (because $3 \cdot \frac{4}{3} \in I$ but $3 \notin \mathrm{rad}(I)$ and $\frac{4}{3} \notin \mathrm{rad}(I)$).

Lemma 3.4. *Let R be an integral domain with quotient field K. If I is a strongly primary ideal of R, then* $\mathrm{rad}(I)$ *is a strongly prime ideal of R.*

Proof. It is clear that $\mathrm{rad}(I)$ is a proper ideal of R. Let $ab \in \mathrm{rad}(I)$ and $a \notin \mathrm{rad}(I)$ for $a, b \in K$. Then there exists $n \geq 1$ such that $a^n b^n \in I$; however, no positive power of a^n is in I. It follows that $b^n \in I$, since I is strongly primary. Thus $b \in \mathrm{rad}(I)$. □

If R is an integral domain and I is a strongly primary ideal of R with $\mathrm{rad}(I) = P$, then I is called a strongly P-primary ideal of R. In the following theorem, we consider the relationship between fractional n-absorbing ideals and strongly primary ideals.

Theorem 3.5. *Let P be a strongly prime ideal of an integral domain R with quotient field K, and I a strongly P-primary ideal of R such that $P^n \subseteq I$ for some positive integer n. Then I is a fractional n-absorbing ideal of R. Moreover, $\mu(I) \leq n$. In particular, if P^n is a strongly P-primary ideal of R, then P^n is a fractional n-absorbing ideal of R with $\mu(P^n) \leq n$, and $\mu(P^n) = n$ if $P^{n+1} \subset P^n$.*

Proof. Let $a_1 \cdots a_{n+1} \in I$ for $a_1, \ldots, a_{n+1} \in K$. If one of the a_i's is not in P, then the product of the other a_i's is in I, since I is strongly P-primary. Thus we may assume that every a_i is in P. Since $P^n \subseteq I$, we have $a_1 \cdots a_n \in I$. Hence I is a fractional n-absorbing ideal of R.

The "moreover" and first part of the "in particular" statements are clear. Now suppose that $P^{n+1} \subset P^n$. Then there are $a_1, \ldots, a_n \in P$ such that $a_1 \cdots a_n \in P^n \setminus P^{n+1}$. Thus no product of $n-1$ of the a_i's is in P^n since otherwise $a_1 \cdots a_n \in P^{n+1}$, a contradiction. Hence P^n is not a fractional $(n-1)$-absorbing ideal of R and therefore $\mu(P^n) = n$, since P^n is a fractional n-absorbing ideal of R by the first part. □

Next, we see that Theorem 3.5 fails if the condition $P^n \subseteq I$ for some positive integer n is removed. For this, we shall need to the following lemma.

Lemma 3.6. *Let R be a valuation domain of dimension one, with maximal ideal M and quotient field K. Then every nonzero proper ideal I of R is a strongly M-primary ideal of R.*

Proof. Note that every nonzero proper ideal of R is an M-primary ideal of R. Let I be a nonzero proper ideal of R and $ab \in I$ for $a, b \in K$. If $a, b \in R$, then since I is an M-primary ideal of R, we have $a \in I$ or $b \in \mathrm{rad}(I)$ and so we are done. Thus we may assume that $a \in K \setminus R$. Since R is a valuation domain, we must have $a^{-1} \in R$. So $ab \in I$ implies that $b \in I$. Hence I is a strongly M-primary ideal of R. □

Example 3.7. *Let R be a one-dimensional valuation domain with maximal ideal M and quotient field K. If M is not principal, then $M = M^2$, and hence (0) and M are the only n-absorbing ideals of R for any positive integer n by* Corollary 2.3*. Now if I is an ideal of R such that $(0) \subset I \subset M$, then by* Lemma 3.6*, I is a strongly M-primary ideal of R but not a fractional n-absorbing ideal for all positive integer n.*

Let I be a proper ideal of a ring R. For $x \in R$, let $I_x = \{y \in R \mid yx \in I\} = (I :_R x)$. We next investigate when I_x is a fractional n-absorbing ideal of R.

Lemma 3.8. *Let R be an integral domain and I a fractional n-absorbing ideal of R. Then for all $x \in R \setminus I$, $I_x = (I :_R x)$ is a fractional n-absorbing ideal of R containing I. Moreover, $\mu(I_x) \leq \mu(I)$ for all $x \in R$.*

Proof. Let $a_1 \cdots a_{n+1} \in (I :_R x)$ for $a_1, \ldots, a_{n+1} \in K$. Since $(xa_1)a_2 \cdots a_{n+1} \in I$, we have either $a_2 \cdots a_{n+1} \in I$ or product of xa_1 with $n-1$ of the a_i's for $2 \leq i \leq n+1$ is in I. In either case, there is a product of n of the a_i's that is in I_x. Thus I_x is a fractional n-absorbing ideal of R. Clearly $I \subseteq I_x$.

The "moreover" statement is clear if $x \in R \setminus I$ by above. If $x \in I$, then $I_x = R$, and so $\mu(I_x) = 0 \leq \mu(I)$. □

Theorem 3.9. *Let R be an integral domain with quotient field K, $n \geq 2$ and $I \subset \operatorname{rad}(I)$ a fractional n-absorbing ideal of R such that I has exactly n minimal prime ideals, say $P_1, \ldots, P_n$. Suppose that $x \in \operatorname{rad}(I) \setminus I$, and let $m(\geq 2)$ be the least positive integer such that $x^m \in I$. Then every product of $n-m+1$ of the P_i's is contained in $I_{x^{m-1}} = (I :_R x^{m-1})$*

Proof. Note that $m \leq n$, since I is a fractional n-absorbing ideal of R; so $n-m+1 \geq 1$. Let $F = \{Q_1, \ldots, Q_{m-1}\} \subset G = \{P_1, \ldots, P_n\}$ and $D = G \setminus F$. Then D contains exactly $n-m+1$ of the P_i's. Since $x \in \operatorname{rad}(I) \setminus I$, we have $x \in Q_i$ for every $1 \leq i \leq m-1$, and thus $x^{m-1} \in Q_1 \cdots Q_{m-1}$. Moreover, $(\prod_{Q \in F} Q)(\prod_{P \in D} P) = P_1 \cdots P_n \subseteq I$ by Theorem 3.1. Hence, we have $x^{m-1} \prod_{P \in D} P \subseteq I$, and so $\prod_{P \in D} P \subseteq I_{x^{m-1}}$. □

The proof of the following result is similar to that of Theorem 3.9, and so is omitted.

Theorem 3.10. *Let R be an integral domain with quotient field K, $n \geq 2$ and $I \subset \operatorname{rad}(I)$ be a fractional n-absorbing ideal of R such that I has exactly n minimal prime ideals, say $P_1, \ldots, P_n$. If $x \in \operatorname{rad}(I) \setminus I$, then every product of $n-1$ of the P_i's is contained in $I_x = (I :_R x)$.*

Theorem 3.11. *Let I be a strongly P-primary ideal of a domain R with quotient field K such that $P^n \subseteq I$ for some positive integer n (for example, if R is a Noetherian ring), and let $x \in P \setminus I$. If $x^m \notin I$ for some positive integer m, then $(I :_R x^m) = I_{x^m}$ is a fractional $(n-m)$-absorbing ideal of R.*

Proof. First note that $m < n$, since $P^n \subseteq I$ and $x^m \notin I$; so $n - m \geq 1$. It is easy to show that, I_{x^m} is a strongly P-primary ideal of R. Since $P^n \subseteq I$, we have $x^m P^{n-m} \subseteq I$, and thus $P^{n-m} \subseteq I_{x^m}$. Hence I_{x^m} is a fractional $(n-m)$-absorbing ideal of R by Theorem 3.5. □

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