Proceedings of the Jangjeon Mathematical Society
23 (2020), No. 1. pp. 81 - 90

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2020.23.1.81

ON REAL CONTINUED FRACTIONS

AMRAN DALLOUL

Abstract. In this paper, we give some improvements to results obtained by G.Nettler, [2] and T.Okano, [4] related to the irrationality and transcendence of continued fractions.

1. Introduction

The Theory of continued fractions plays a central role in Number Theory. It firstly appeared to get good rational approximations to irrational numbers. Recently, it is widely used to get integer solutions to Pell's equation and other related topics.
In 1973, G.Nettler gave wonderful formulas for sum, difference, quotient, product and exponentiation of two simple continued fractions. His brilliant work had led him to obtain sufficient conditions on the elements of two simple continued fractions $A = a_1 + \frac{1}{a_2+\frac{1}{a_3+...}}, B = b_1 + \frac{1}{b_2+\frac{1}{b_2+...}}$ in order that the six numbers $A, B, A \pm B, A.B^{\pm 1}$ be all irrationals and transcendentals as follows:

Theorem 1.1 (Nettler,[2], Theorem 4.15)**.** *Consider the continued fractions*

$$A = a_1 + \frac{1}{a_2 + \frac{1}{a_3+...}}, B = b_1 + \frac{1}{b_2 + \frac{1}{b_3+...}}.$$

If $\frac{a_n}{2} > b_n > a_{n-1}^{5n}$, for all large n, then the six numbers $A, B, A \pm B, A.B^{\pm 1}$ are all irrationals.

Theorem 1.2 (Nettler,[2], Theorem 5.6)**.** *Consider the continued fractions*

$$A = a_1 + \frac{1}{a_2 + \frac{1}{a_3+...}}, B = b_1 + \frac{1}{b_2 + \frac{1}{b_3+...}}.$$

If $\frac{a_n}{2} > b_n > a_{n-1}^{9n^3}$, for all large n, then the six numbers $A, B, A \pm B, A.B^{\pm 1}$ are all transcendentals.

Later, he relaxed the condition $\frac{a_n}{2} > b_n > a_{n-1}^{9n^3}$ to $a_n > b_n > a_{n-1}^{(n-1)^2}$, [3]. Also, T.Okano, improved the last one to $a_n > b_n > a_{n-1}^{\gamma(n-1)}$, where γ is any constant such that $\gamma > 16$, [4].
Many results have been made towards this direction, for example, T.Töpfer proved the following, [5]:

Theorem 1.3. *Let $\alpha = a_0 + \frac{1}{a_1+\frac{1}{a_2+...}}$ be a continued fraction and Q_n be denominator of the n^{th} convergent of α. Suppose that $\epsilon, c \in \mathbb{R}^+$ and $k \in \mathbb{N}$. If the following inequality*

$$a_{n+k} \geq cQ_n^{\epsilon}$$

holds true for infinitely many $n \in \mathbb{N}$, then α is transcendental.

Also, Hanc̃l, Jaroslav, [1], proved for given k continued fractions $A_i = a_{i1} + \frac{1}{a_{i2}+\frac{1}{a_{i3}+...}}, i = 1, 2, .., k$. If the following conditions:

$$\limsup_{n\to\infty} \frac{\log(\log a_{1n})}{n} = \infty$$
$$a_{i+1,n}2^{2^n} > a_{in} > (1 + \frac{1}{n})(a_{in} + 1)$$

hold true for $1 \le i < k$ and $n \ge 1$, then the sequences $(a_{in})_{n\ge1}$ are continued fractional algebraically independent. As a corollary we find for given two continued fractions $A = a_1 + \frac{1}{a_2+\frac{1}{a_3+...}}, B = b_1 + \frac{1}{b_2+\frac{1}{b_3+...}}$. If the following conditions:

$$\limsup_{n\to\infty} \frac{\log(\log b_n)}{n} = \infty$$
$$a_n 2^{2^n} > b_n > (1 + \frac{1}{n})(a_n + 1)$$

hold true for $n \ge 1$, then the six numbers $A, B, A \pm B, A.B^{\pm1}$ are all transcendentals.

Our work is on the road, where we also give some improvements to the above results.
Our results read as the following:

Theorem 1.4. *Consider the continued fractions*

$$A = a_1 + \frac{1}{a_2 + \frac{1}{a_3+...}}, B = b_1 + \frac{1}{b_2 + \frac{1}{b_3+...}}.$$

If $\frac{a_n}{2} > b_n > a_{n-1}^5, \forall n \ge 2$, then the six numbers $A, B, A \pm B, A.B^{\pm1}$ are all irrationals.

Theorem 1.5. *Consider the continued fractions*

$$A = a_1 + \frac{1}{a_2 + \frac{1}{a_3+...}}, B = b_1 + \frac{1}{b_2 + \frac{1}{b_3+...}}.$$

If $\frac{a_n}{2} > b_n > a_{n-1}^{12}, \forall n \ge 2$, then the six numbers $A, B, A \pm B, A.B^{\pm1}$ are all transcendentals.

In our results the sequences $(a_n), (b_n)$ tend to infinity slower than the sequences in Jaroslav's result.

2. Preliminaries

Lemma 2.1. *Let*

$$A(n) = a_1 + \frac{1}{a_2+}\frac{1}{a_3+}...\frac{1}{+a_n} = \frac{P_n^a}{Q_n^a}$$
$$B(n) = b_1 + \frac{1}{b_2+}\frac{1}{b_3}.....\frac{1}{+b_n} = \frac{P_n^b}{Q_n^b}$$

be the n^{th} convergents of A and B respectively. Then, we have for all $n \geq 1$ the following:

$$A(n)+B(n)=a_1+b_1+\frac{a_2+b_2}{a_2.b_2+}\frac{a_2b_2F_3}{E_3-F_3+}\frac{E_3F_4}{E_4-F_4+}.....\frac{E_{n-1}F_n}{+E_n-F_n},$$
$$E_n=Q_n^aQ_n^b(Q_{n-1}^aQ_{n-2}^a+Q_{n-1}^bQ_{n-2}^b)$$
$$F_n=Q_{n-2}^aQ_{n-2}^b(Q_n^aQ_{n-1}^a+Q_n^bQ_{n-1}^b)$$

$$B(n)-A(n)=b_1-a_1+\frac{a_2-b_2}{a_2.b_2+}\frac{a_2b_2H_3}{G_3-H_3+}\frac{G_3H_4}{G_4-H_4+}.....\frac{G_{n-1}H_n}{+G_n-H_n},$$
$$G_n=Q_n^aQ_n^b(Q_{n-1}^aQ_{n-2}^a-Q_{n-1}^bQ_{n-2}^b)$$
$$H_n=Q_{n-2}^aQ_{n-2}^b(Q_n^aQ_{n-1}^a-Q_n^bQ_{n-1}^b)$$

$$\frac{A(n)}{B(n)}=\frac{a_1}{b_1}+\frac{b_1b_2-a_1a_2}{a_2.b_1(b_1b_2+1)+}\frac{a_2.b_1(b_1b_2+1)J_3}{I_3-J_3+}\frac{I_3J_4}{I_4-J_4+}.....\frac{I_{n-1}J_n}{+I_n-J_n},$$
$$I_n=Q_n^aP_n^b(Q_n^aP_{n-1}^a-Q_{n-2}^bQ_{n-1}^b)$$
$$J_n=Q_{n-2}^aP_{n-2}^b(Q_n^aP_{n-1}^a-Q_n^bP_{n-1}^b)$$

$$A(n)B(n)=a_1b_1+\frac{a_1a_2+b_1b_2+1}{a_2b_2+}\frac{a_2b_2L_3}{K_3-L_3+}\frac{K_3L_4}{K_4-L_4+}.....\frac{K_{n-1}L_n}{+K_n-L_n},$$
$$K_n=Q_n^aQ_n^b(Q_{n-2}^aP_{n-1}^a+Q_{n-1}^bP_{n-2}^b)$$
$$L_n=Q_{n-2}^aQ_{n-2}^b(Q_n^aP_{n-1}^a+Q_{n-1}^bP_n^b)$$

Proof. see [2]. □

Lemma 2.2. *Let $A=a_1+\frac{1}{a_2+\frac{1}{a_3+\dots}}$ and β be any constant such that $\beta>4$. If $a_n>a_{n-1}^{\beta}, \forall n\geq 2$,then $Q_n^a<a_n^{\frac{\beta}{3}+\delta}$, where δ is any constant such that $\delta>0$. In particular, $Q_n^a<a_n^2$*

Proof. Using the same argument in [4](Lemma 3), we find that

$$Q_n^a<\prod_{i=1}^{n}(1+\frac{1}{a_i})\prod_{i=1}^{n}a_i.$$

Therefore, there exists a positive constant $M>0$ such that $\prod_{i=1}^{n}(1+\frac{1}{a_i})<M$.
Using the condition $a_n>a_{n-1}^{\beta}, \forall n\geq 2$, we find that

$$\begin{aligned}Q_n^a &<M(a_n^{\frac{1}{\beta^{n-1}}})(a_n^{\frac{1}{\beta^{n-2}}}).....(a_n^{\frac{1}{\beta}})(a_n)\\ &<Ma_n^{1+\frac{1}{\beta}+\frac{1}{\beta^2}+...+\frac{1}{\beta^{n-1}}}\\ &<Ma_n^{1+\frac{1}{\beta}+\frac{1}{\beta^2}+...+\frac{1}{\beta^{n-1}}+.....}\\ &<Ma_n^{\frac{\beta}{\beta-1}}.\end{aligned}$$

Since $a_n \longrightarrow \infty$, then so is a_n^{δ}, where δ is any constant such that $\delta > 0$. This implies that for all large n, $M < a_n^{\delta}$. Also, the condition $\beta > 4$ implies that $\frac{\beta}{\beta-1} < \frac{\beta}{3}$. Hence, combining these inequalities, we get the desired result. □

Using the same argument, one can prove the following:

Lemma 2.3. *Let* $A = a_1 + \frac{1}{a_2+\frac{1}{a_3+\ldots}}$ *and* β *be any constant such that* $\beta > 11.5$. *If* $a_n > a_{n-1}^{\beta}, \forall n \geq 2$, *then* $Q_n^a < a_n^{\frac{\beta}{10.5}+\delta}$, *where* δ *is any constant such that* $\delta > 0$. *In particular,* $Q_n^a < a_n^2 < a_{n+1}$.

The following well-known theorem is a criterion for the irrationality of generalized continued fractions. It states as the following:

Theorem 2.4. [2] *Let* $e_1, e_2, ..., e_n, ..., d_2, d_3, ..., d_n, ...$ *be integer numbers. If the following conditions are hold*
1)$e_1, e_2, ..., e_n, ..., d_2, d_3, ..., d_n, ...$ *are positive,*
2) $e_n \geq d_n$ *for all large* n,
then the generalized continued fraction $e_1 + \frac{d_2}{e_2+\frac{d_3}{e_3+\ldots+\frac{d_n}{e_n}}}$ *converges to an irrational number.*

The following Theorem will be used to get results concerning the transcendence of continued fractions. It states as the following:

Theorem 2.5. *(Roth Theorem,*[3]*) Let* α *be a real number and let* $\epsilon > 0$ *be a fixed constant. If the inequality*

$$|\alpha - \frac{p}{q}| < \frac{1}{q^{2+\epsilon}}$$

has infinitely many solutions $(p, q) \in \mathbb{Z} \times \mathbb{Z}^+$, *then* α *is a transcendental number.*

Lemma 2.6. *Let* $C = e_1 + \frac{d_2}{e_2+\frac{d_3}{e_3+\ldots+\frac{d_n}{e_n+\ldots}}}$ *be the continued fraction expansion of either* $A \pm B, A.B$ *or* $\frac{A}{B}$ *(as in Lemma 2.1), and let* $\frac{P_n^c}{Q_n^c} = e_1 + \frac{d_2}{e_2+\frac{d_3}{e_3+\ldots+\frac{d_n}{e_n}}}$ *be the* n^{th} *convergent of* C.
If $\frac{a_n}{2} > b_n > a_{n-1}^5, \forall n \geq 2$, *then all of the quantities* E_n, G_n, K_n, I_n *(in Lemma 2.1) are strictly less than* $(Q_n^a)^3$.

Proof. Using Lemma 2.2 and the condition of lemma, we find the following:

$$\begin{aligned}
2Q^a_{n-1}Q^a_{n-2} =& 2\Big(a_{n-1}Q^a_{n-2} + Q^a_{n-3}\Big)Q^a_{n-2}\\
=& 2\Big(a_{n-1}(Q^a_{n-2})^2 + Q^a_{n-2}Q^a_{n-3}\Big)\\
\text{(Using Lemma 2.2) } <& 2\Big(a_{n-1}(Q^a_{n-2})^2 + a^2_{n-3}Q^a_{n-2}\Big)\\
\text{(Using the condition of Lemma) } <& 2\Big(a_{n-1}(Q^a_{n-2})^2 + a_{n-2}Q^a_{n-2}\Big)\\
<& 2Q^a_{n-2}\Big(a_{n-1}Q^a_{n-2} + a_{n-2}\Big)\\
\text{(Using Lemma 2.2 and the condition of Lemma) } <& 2Q^a_{n-2}\Big(a_{n-1}a^2_{n-2} + a^{\frac{1}{5}}_{n-1}\Big)\\
\text{(Using the condition of Lemma) } <& 2Q^a_{n-2}\Big(a_{n-1}.a_{n-1} + a_{n-1}\Big)\\
<& Q^a_{n-2}(4a^2_{n-1})\\
<& Q^a_{n-2}(a^3_{n-1})\\
\text{(Using the condition of Lemma) } <& a_nQ^a_{n-1} < Q^a_n.
\end{aligned}$$

Hence,

$$2Q^a_{n-1}Q^a_{n-2} < Q^a_n \quad \text{for all large } n. \tag{1}$$

Now, we have

$$\begin{aligned}
E_n =& Q^a_nQ^b_n(Q^a_{n-1}Q^a_{n-2} + Q^b_{n-1}Q^b_{n-2})\\
\text{(since } a_n > b_n, \forall n \geq 1,\ Q^a_n > Q^b_n, \forall n \geq 2) <& (Q^a_n)^2(2Q^a_{n-1}Q^a_{n-2})\\
\text{(Using relation (1)) } <& (Q^a_n)^2(Q^a_n) = (Q^a_n)^3
\end{aligned}$$

Applying the same argument we get the same result to the rest by considering the relation $\frac{P^a_n}{Q^a_n} \leq 2a_1$ and $\frac{P^b_n}{Q^b_n} \leq 2b_1$,i.e, $P^a_n \leq 2a_1Q^a_n$ and $P^b_n \leq 2b_1Q^b_n$ □

Using the same argument, one can prove that

Lemma 2.7. *Let* $C = e_1 + \frac{d_2}{e_2+\frac{d_3}{e_3+\ldots+\frac{d_n}{e_n+\ldots}}}$ *be the continued fraction expansion of either* $A \pm B, A.B$ *or* $\frac{A}{B}$ *(as in Lemma 2.1), and let* $\frac{P^c_n}{Q^c_n} = e_1 + \frac{d_2}{e_2+\frac{d_3}{e_3+\ldots+\frac{d_n}{e_n}}}$ *be the* n^{th} *convergent of* C.
If $\frac{a_n}{2} > b_n > a^{12}_{n-1}, \forall n \geq 2$, *then all of the quantities* E_n, G_n, K_n, I_n *(in Lemma 2.1) are strictly less than* $(Q^a_n)^{2.5}$.

Proof. As in the previous Lemma, one can check that

$$2Q^a_{n-1}Q^a_{n-2} < a^{2.1}_{n-1}Q^a_{n-2} < a^{\frac{1}{5}}_nQ^a_{n-2} < (Q^a_n)^{0.5}.$$

We prove the last inequality as follows:
From the assumption, we have

$$a_n > a^{11.75}_{n-1} \Rightarrow Q^a_n < a_n^{\frac{11.75}{10.5}+\frac{0.25}{10.5}} = a_n^{\frac{12}{10.5}}.$$

So,

$$Q_n^a < a_n^{\frac{12}{10.5}} < a_n^{\frac{12}{10}} \tag{2}$$

Thus,

$$\begin{aligned} Q_n^a >& a_n Q_{n-1}^a \\ \text{(using relation (2))} >& (Q_n^a)^{\frac{10}{12}} Q_{n-1}^a \\ >& (Q_{n-1}^a)^{\frac{10}{12}} Q_{n-1}^a = (Q_{n-1}^a)^{\frac{22}{12}} \Rightarrow Q_n^a > (Q_{n-1}^a)^{\frac{22}{12}}. \end{aligned}$$

Since the relation holds true for all large n, it also can be satisfied if we replaced each n by $n-1$. Hence, $Q_{n-1}^a > (Q_{n-2}^a)^{\frac{22}{12}}$. Therefore,

$$Q_n^a > (Q_{n-1}^a)^{\frac{22}{12}} > (Q_{n-2}^a)^{(\frac{22}{12})^2} > (Q_{n-2}^a)^{3.36}.$$

Therefore,

$$Q_n^a > (Q_{n-2}^a)^{3.36}. \tag{3}$$

On the other hand, we have $(Q_n^a)^{1.5} > (Q_n^a)^{\frac{5}{3.36}}$. So,

$$(Q_n^a)^{2.5} > (Q_n^a)(Q_n^a)^{\frac{5}{3.36}} > a_n (Q_n^a)^{\frac{5}{3.36}}.$$

Hence,

$$\begin{aligned} (Q_n^a)^{2.5} >& a_n[(Q_n^a)^{\frac{1}{3.36}}]^5 \\ \text{(Using relation (3))} >& a_n (Q_{n-2}^a)^5. \end{aligned}$$

Thus,

$$(Q_n^a)^{2.5} > a_n (Q_{n-2}^a)^5 \Rightarrow (Q_n^a)^{0.5} = (Q_n^a)^{\frac{2.5}{5}} > a_n^{\frac{1}{5}} Q_{n-2}^a.$$

Hence,

$$(Q_n^a)^{0.5} > a_n^{\frac{1}{5}} Q_{n-2}^a.$$

Therefore,

$$2Q_{n-1}^a Q_{n-2}^a < (Q_n^a)^{0.5} = \sqrt{Q_n^a} \tag{4}$$

Using the same argument as in Lemma 2.7, we get the result □

Lemma 2.8. *Let* $C = e_1 + \cfrac{d_2}{e_2 + \cfrac{d_3}{e_3 + \ldots + \cfrac{d_n}{e_n + \ldots}}}$ *be the continued fraction expansion of either* $A \pm B, A.B$ *or* $\frac{A}{B}$ *(as in Lemma 2.1), and let* $\frac{P_n^c}{Q_n^c} = e_1 + \cfrac{d_2}{e_2 + \cfrac{d_3}{e_3 + \ldots + \cfrac{d_n}{e_n}}}$ *be the* n^{th} *convergent of* C.

If $\frac{a_n}{2} > b_n > a_{n-1}^{12}, \forall n \geq 2,$ *then* $E_n > E_{n-1}^2, G_n > G_{n-1}^2, K_n > K_{n-1}^2, I_n > I_{n-1}^2$

Proof. We prove for I_n and the rest can be done similarly. Using Lemmas 2.3 and 2.6, we find for all large n, the following:

$$\frac{a_{n+1}}{2} > b_{n+1} > a_n^{12} = (a_n^{\frac{11.75}{0.5} + \frac{0.25}{10.5}})^{10.5} > (Q_n^a)^{10.5} > I_n.$$

Using Theorem 4.9 in [2], we find that

$$I_n - J_n > I_{n-1} J_n, \text{for all large } n.$$

So, $I_n > I_{n-1}J_n$. Using (4) and the property: for any positive real numbers $a, b > 0$. If $a > 4b$, then $a - b > \frac{a}{2}$,(where in our case $a_n > 2b_n \Rightarrow Q_n^a > 2Q_n^b$ and $P_n^a > 2P_n^b \Rightarrow Q_n^a P_{n-1}^a > 4Q_n^b P_{n-1}^b \Rightarrow Q_n^a P_{n-1}^a - Q_n^b P_{n-1}^b > \frac{1}{2}Q_n^a P_{n-1}^a$), we find that $J_n > I_{n-1}$. Therefore,

$$I_n > I_{n-1}^2$$

□

Lemma 2.9. *Let* $C = e_1 + \frac{d_2}{e_2 + \frac{d_3}{e_3 + \ldots + \frac{d_n}{e_n + \ldots}}}$ *be the continued fraction expansion of either* $A \pm B, A.B$ *or* $\frac{A}{B}$ *(as in Lemma 2.1), and let* $\frac{P_n^c}{Q_n^c} = e_1 + \frac{d_2}{e_2 + \frac{d_3}{e_3 + \ldots + \frac{d_n}{e_n}}}$ *be the* n^{th} *convergent of* C.
If $\frac{a_n}{2} > b_n > a_{n-1}^{12}, \forall n \geq 2$, *then* $Q_n^c < (Q_n^a)^{5.2}$

Proof. We prove for $A + B$ and the rest can be done similarly.
Since $a_n > b_n$, then the elements of the continued fraction of $A + B$ are positive integers. This implies that (Q_n^c) forms an increasing sequence. So,

$$Q_n^c = e_n Q_{n-1}^c + d_n Q_{n-2}^c < (e_n + d_n)Q_{n-1}^c < (e_n + d_n)(e_{n-1} + d_{n-1})Q_{n-2}^c < .... < \prod_{i=2}^{n}(e_i + d_i)$$

Hence,

$$Q_n^c < M \prod_{i=4}^{n}(E_{i-1}F_i + E_i - F_i),$$

Where M is some positive constant depending on the first elements $a_1, a_2, a_3, b_1, b_2, b_3$. According to the assumption $\frac{a_n}{2} > b_n > a_{n-1}^{12}, \forall n \geq 2$, (as in the previous lemma) we find that $E_n - F_n > E_{n-1}F_n$, for all large $n \geq N$. So, $E_n > E_{n-1}F_n$ Thus for all large n we have

$$\begin{aligned}
Q_n^c &< M \prod_{i=4}^{n}(E_{i-1}F_i + E_i - F_i) \\
&< M \prod_{i=4}^{n}(E_{i-1}F_i + E_i) \\
&< M \prod_{i=4}^{n} E_i(1 + \frac{E_{i-1}F_i}{E_i}) \\
&< M_1 \prod_{i=N}^{n} E_i(1 + 1) \\
&< M_1 2^n E_N \ldots E_n \\
\text{(using lemma 2.8)} &< M_1 2^n E_n^{1 + \frac{1}{2} + .. + \frac{1}{2^{n-N}}} \\
&< M_1 2^n E_n^{1 + \frac{1}{2} + .. + \frac{1}{2^{n-N}} + \ldots} \\
&< M_1 E_n^{\frac{1}{20}} E_n^2 \\
\text{(using lemma 2.7)} &< M_1 (Q_n^a)^{2.05(2.5)} < (Q_n^a)^{5.2}
\end{aligned}$$

In fact, we have $E_n > a_n.b_n > (2^{12^n})(2^{12^n}) = 2^{2(12^n)} > 2^{20n}$. So, $2^n < E_n^{\frac{1}{20}}$ □

3. PROOFS OF THE MAIN RESULTS

Now, we are able to prove the main theorems as follows:

Proof. Proof of Theorem 1.4. First of all, Irrationality of A and B follows from the fact that any simple continued fraction is irrational.
Now, we prove for $\frac{A}{B}$. According to [2], and the condition $a_n > b_n, \forall n \geq 1$, we find that the elements of the continued fraction of $\frac{A}{B}$ (as explained in Lemma 2.1) are *positive* integers. Using Lemmas 2.2 and 2.6, and the assumption of theorem, we find that

$$\frac{a_{n+1}}{2} > b_{n+1} > a_n^5 = (a_n^{\frac{4.75}{3}+\frac{0.25}{3}})^3 > (Q_n^a)^3 > I_n.$$

Using Theorem 4.9 in [2], we find that

$$I_n - J_n > I_{n-1}J_n, \text{for all large } n.$$

Using Theorem 2.4, we find that $\frac{A}{B}$ is an irrational number.
For $A+B$: Since $a_n > b_n, \forall n \geq 1$, it follows that the elements of the continued fraction of $A+B$ (as explained in Lemma 2.1) are *positive* integers. Using the same argument, we find that

$$\frac{a_{n+1}}{2} > b_{n+1} > a_n^5 = (a_n^{\frac{4.75}{3}+\frac{0.25}{3}})^3 > (Q_n^a)^3 > E_n.$$

Using Theorem 4.3 in [2], we find that

$$E_n - F_n > E_{n-1}F_n, \text{for all large } n.$$

So, $A+B$ is an irrational number (using Theorem 2.4). The same argument can be applied to prove the irrationality of $A-B$ and $A.B$. □

Example 3.1. Let

$$A = 2^{7^1} + \cfrac{1}{2^{7^2} + \cfrac{1}{2^{7^3} + \dots_{+2^{7^n}_{+..}}}}, B = 2^{6(7^0)} + \cfrac{1}{2^{6(7^1)} + \cfrac{1}{2^{6(7^2)} + \dots_{+2^{6(7^{n-1})}_{+..}}}}.$$

It is clear that $\frac{2^{7^n}}{2} > 2^{6(7^{n-1})} > 2^{5(7^{n-1})}, \forall n \geq 2$. Using Theorem 1.4, we find that the six numbers $A, B, A \pm B, A.B^{\pm 1}$ are all irrationals.

Proof. Proof of Theorem 1.5. First, transcendence of A and B follows from Theorem 1.3 and Lemma 2.3 (where we put $k = 1, \epsilon = 1$ and $c = 1$).
Now, For $A+B$: Let $C := A + B = e_1 + \cfrac{d_2}{e_2 + \cfrac{d_3}{e_3 + \dots + \cfrac{d_n}{e_n + \dots}}}$ and $\frac{P_n^c}{Q_n^c} = \frac{P_n^a}{Q_n^a} + \frac{P_n^a}{Q_n^a}, \forall n \geq 1$. Then, we have

$$\left|C - \frac{P_n^c}{Q_n^c}\right| \leq \left|A - \frac{P_n^a}{Q_n^a}\right| + \left|B - \frac{P_n^b}{Q_n^b}\right| < \frac{1}{Q_n^a Q_{n+1}^a} + \frac{1}{Q_n^b Q_{n+1}^b} < \frac{2}{Q_n^b Q_{n+1}^b} < \frac{1}{b_{n+1}} < \frac{1}{a_n^{12}}.$$

Using lemmas 2.3 and 2.9, we find for all large n

$$\left|C - \frac{P_n^c}{Q_n^c}\right| < \frac{1}{a_n^{12}} = \frac{1}{\left(a_n^{\frac{11.75}{10.5}+\frac{0.25}{10.5}}\right)^{10.5}} < \frac{1}{(Q_n^a)^{10.5}} < \frac{1}{(Q_n^c)^{\frac{1}{5.2}(10.5)}} < \frac{1}{(Q_n^c)^{2.019}}.$$

Using Roth Theorem, we find that C is transcendental.

For $\frac{A}{B}$. Let $C := \frac{A}{B} = e_1 + \frac{d_2}{e_2 + \frac{d_3}{e_3 + \ldots + \frac{d_n}{e_n + \ldots}}}$ and $\frac{P_n^c}{Q_n^c} = \frac{\frac{P_n^a}{Q_n^a}}{\frac{P_n^b}{Q_n^b}}, \forall n \geq 1$. We have $\frac{P_n^a}{Q_n^a} \leq a_1 + \frac{1}{a_2} \leq 2a_1$ and $\frac{P_n^b}{Q_n^b} \geq \frac{1}{2b_1}, \forall n \geq 3$. So, we get

$$
\begin{aligned}
\Big|C - \frac{P_n^c}{Q_n^c}\Big| &= \Big|\frac{A}{B} - \frac{P_n^a/Q_n^a}{P_n^b/Q_n^b}\Big| \\
&= \Big|\frac{A\frac{P_n^b}{Q_n^b} - B\frac{P_n^a}{Q_n^a}}{B\frac{P_n^b}{Q_n^b}}\Big| \\
&= \frac{\Big|A\frac{P_n^b}{Q_n^b} - \frac{P_n^a}{Q_n^a}\frac{P_n^b}{Q_n^b} + \frac{P_n^a}{Q_n^a}\frac{P_n^b}{Q_n^b} - B\frac{P_n^a}{Q_n^a}\Big|}{B\frac{P_n^b}{Q_n^b}} \\
&= \frac{\Big|\frac{P_n^b}{Q_n^b}(A - \frac{P_n^a}{Q_n^a}) + \frac{P_n^a}{Q_n^a}(B - \frac{P_n^b}{Q_n^b})\Big|}{B\frac{P_n^b}{Q_n^b}} \\
&\leq \frac{\Big|\frac{P_n^b}{Q_n^b}(A - \frac{P_n^a}{Q_n^a})\Big|}{B\frac{P_n^b}{Q_n^b}} + \frac{\Big|\frac{P_n^a}{Q_n^a}(B - \frac{P_n^b}{Q_n^b})\Big|}{B\frac{P_n^b}{Q_n^b}} \\
&\leq \frac{1}{B}\Big|A - \frac{P_n^a}{Q_n^a}\Big| + \frac{2a_1}{B\frac{1}{2b_1}}\Big|A - \frac{P_n^a}{Q_n^a}\Big| \\
&\leq \frac{4a_1b_1}{B}\left[\Big|A - \frac{P_n^a}{Q_n^a}\Big| + \Big|B - \frac{P_n^b}{Q_n^b}\Big|\right] \\
&\leq \frac{4a_1b_1}{B}\frac{2}{Q_n^bQ_{n+1}^b}.
\end{aligned}
$$

We can choose n sufficiently large such that $Q_n^b > 2\frac{4a_1b_1}{B}$. So, $\frac{2}{Q_n^b}\frac{4a_1b_1}{B} < 1$. Hence, we obtain the following:

$$
\Big|C - \frac{P_n^c}{Q_n^c}\Big| < \frac{1}{Q_{n+1}^b} < \frac{1}{b_{n+1}} < \frac{1}{a_n^{12}} < \frac{1}{(Q_n^c)^{2.019}}.
$$

Using Roth Theorem, we find that C is transcendental. The cases of $A.B$ and $A - B$ can be done similarly. □

Example 3.2. Let

$$
A = 2^{14^1} + \frac{1}{2^{14^2} + \frac{1}{2^{14^3} + \ldots_{+2^{14^n}_{+\ldots}}}}, B = 2^{13(14^0)} + \frac{1}{2^{13(14^1)} + \frac{1}{2^{13(14^2)} + \ldots_{+2^{13(14^{n-1})}_{+\ldots}}}}.
$$

It is clear that $\frac{2^{14^n}}{2} > 2^{13(14^{n-1})} > 2^{12(14^{n-1})}, \forall n \geq 2$. Using Theorem 1.5, we find that the six numbers $A, B, A \pm B, A.B^{\pm 1}$ are all transcendentals.

Acknowledgments. I would like to thank the anonymous referees for their valuable suggestions and comments. Also, Also, I would like to thank Professor Hasan Abdo Sankari, Tishreen University, for proposing this subject.

References

[1] Hanĉl, Jaroslav, Continued fractional algebraic independence of sequences. Publ. Math. Debrecen 46 (1995), no. 1-2, 27-31.

[2] [N] G. Nettler, On transcendental numbers whose sum, difference, quotient and product are transcendental numbers, Math Student 41 (1973). 339-348.

[3] [N1] G. Nettler, Transcendental continued fractions, J. Number Theory, 13 (1981), 456-462.

[4] T.Okano, A Note on the Transcendental Continued Fractions, TOKYO J. MATH. VOL. 10, No. 1,(1987), 151-156.

[5] T.Töpfer, Transcendence and Algebraic Independence of Certain Continued Fractions, Mh. Math. 117, (1994) 255-262.

PhD at Beirut Arab University, Beirut, Lebanon
Email address: amrandalloul@hotmail.com