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# SKEW$(\beta_1 + u\beta_2 + v\beta_3 + uv\beta_4 + v^2\beta_5 + uv^2\beta_6)$-CONSTACYCLIC CODES OVER $F_q + uF_q + vFq + uvFq + v^2F_q + uv^2F_q$

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ABSTRACT. In this paper, we study $(\theta - \beta)$-constacyclic codes over the ring $R = F_q + uF_q + vF_q + uvF_q + v^2F_q + uv^2F_q$, with $u^2 = 1$, $v^3 = v$, $uv = vu$, $q = p^m$ and p is an odd prime. The structural properties of $(\theta - \beta)$-constacyclic codes over the ring $R$ are studied. Further, *generating polynomials* and *idempotent generators* for $(\theta - \beta)$-constacyclic codes over the ring $R$ are studied.

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## 1. INTRODUCTION

Calderbank et al. [2] studied the structure of cyclic codes over $Z_{p^a}$, Dinh et al. [6] determined the structures of cyclic and negacyclic codes of length $n$ over finite chain ring . Boucher et al. [3, 4] introduced *skew cyclic codes* using *skew polynomial ring* $F_q[x, \theta]$, which is non-commutataive ring and considered the structure of cyclic codes closed under a skew cyclic shift over $F_q[x, \theta]$.

There are a lot of papers that study skew cyclic codes over a ring, Siap et al. [7] examined skew cyclic codes of arbitrary length. Cyclic codes over the ring $R = F_q + uF_q + vF_q + uvF_q$ were studied , and gave a formula for the number of skew cyclic codes of length $n$ over the ring $R = F_q + uF_q + vF_q + uvF_q$, where $u^2 = u$, $v^2 = v$, $uv = vu$, $q = p^m$ and p is an odd prime, see [12].

Skew constacyclic codes over $F_p + vF_p$ with $v^2 = 0$ were studied with two cases when $n$ is even and when $n$ is odd and gave an example that construct constacyclic and skew constacyclic code over $F_3 + vF_3$, see [10].

The properties of skew cyclic codes over the ring $F_{p^m} + uF_{p^m}$ were studied, which generated by *monic right divisor* of $x^n - \lambda$, where $\lambda$ is a unit, see [11].

Skew cyclic and skew $(\alpha_1 + u\alpha_2 + v\alpha_3 + uv\alpha_4)$-constacyclic codes over $F_q + uF_q + vF_q + uvF_q$ with $u^2 = u$, $v^2 = v$, $uv = vu$, $q = p^m$ and p is an odd prime were studied. Also *generated polynomials* and *idempotent generators* for skew cyclic and skew $(\alpha_1 + u\alpha_2 + v\alpha_3 + uv\alpha_4)$-constacyclic codes were determined, see [5] .

Skew cyclic codes over the ring $R = F_q + uF_q + vF_q + uvF_q + v^2F_q + uv^2F_q$ studied, where $v^3 = v$, $u^2 = 1$, $q = p^m$ and p is an odd prime, more over the

*Gray map*, *automorphism* $\theta$ on $F_q$, the skew polynomial ring $F_q[x,\theta]$ was defined, see [9, 13].

In this paper, we study skew constacylic codes over the ring $R = F_q + uF_q + vF_q + uvF_q + v^2F_q + uv^2F_q$, where $v^3 = v$, $u^2 = 1$, $q = p^m$ and p is an odd prime.

The plan of the paper is organized as follows:

In Section 2, we define the *Gray map* $\Psi$: $R \to {F_q}^6$, and use *Chinese Remainder Theorem* to write the ring $R$ as $R = (1-v^2)R \oplus (2^{-1}v^2 + 2^{-1}v)R \oplus (2^{-1}v^2 - 2^{-1}v)R$, and found $\beta_i$ such that ${\beta_i}^2 = \beta_i$, $\beta_i\beta_j = 0$, $\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 = 1$, where $1 \le i, j \le 6$ and $i \ne j$.

In section 3, we define $C$ a linear code of length $n$ over $R$ as $C = \oplus_{i=1}^{6}\beta_i C_i$ and give some familiar structural properties over $R$.

In section 4, we define $(\theta - \beta)$-constacyclic codes over $R$, and generate $(\theta-\beta)$-constacyclic codes by *monic polynomial* $f(x)$ which is a *right divisor* of $(x^n - \beta)$ in $F_q[x,\theta]$. Finally *idempotent generators* of $(\theta-\beta)$-constacyclic codes are determined, and give some examples.

Section 5 concludes the paper.

## 2. Preliminaries

Let $R = F_q + uF_q + vF_q + uvF_q + v^2F_q + uv^2F_q = \{a + ub + vc + uvd + v^2e + uv^2f | a, b, c, d, e, f \in F_q\}$ with $v^3 = v$, $u^2 = 1$, $q = p^m$ and p is an odd prime, this ring is a *Frobenius ring* but not *local*. Recall from [13] a map which defined by

$\Psi$: $R \to {F_q}^6$

$\Psi(r) = \Psi(a + ub + vc + uvd + v^2e + uv^2f) = (a, b, a + c + e, b + d + f, a - c + e, b - d + f)$ called the *Gray map*, which implies that there exist $x$, $y$, $z$, $w$, $l$, $m$ such that

$xa + yb + za + zc + ze + wb + wd + wf + la - lc + le + mb - md + mf = a + ub + vc + uvd + v^2e + uv^2f$ to give the system of 6 equations as follows:

$x + z + l = 1 \qquad y + w + m = u \qquad z - l = v$

$w - m = uv \qquad z + l = v^2 \qquad w + m = uv^2$

one can solve this system by *Maple* as follows:

$solve(\{x+z+l = 1, y+w+m = u, z-l = v, w-m = uv, z+l = v^2, w+m = uv^2\}, [x, y, z, w, l, m])$to have the solution

$x = 1 - v^2 \qquad y = -uv^2 + u \qquad z = 2^{-1}v^2 + 2^{-1}v$

$w = 2^{-1}uv^2 + 2^{-1}uv \qquad l = 2^{-1}v^2 - 2^{-1}v \qquad m = 2^{-1}uv^2 - 2^{-1}uv$

$1 - v^2$, $2^{-1}v^2 + 2^{-1}v$, $l = 2^{-1}v^2 - 2^{-1}v$ satisfies ([8], Lemma 2.1) and by *Chinese Remainder Theorem* the ring $R$ can be written as

$$R = (1 - v^2)R \oplus (2^{-1}v^2 + 2^{-1}v)R \oplus (2^{-1}v^2 - 2^{-1}v)R.$$

Let $R_u = F_q + uF_q$ with $u^2 = 1$ be non chain ring. Then $R = R_u + vR_u + v^2R_u$, and let

$\beta_1 = 2^{-1}(1 + u)(1 - v^2) \qquad \beta_2 = 2^{-1}(1 - u)(1 - v^2)$

$\beta_3 = 4^{-1}(1 + u)(v + v^2) \qquad \beta_4 = 4^{-1}(1 - u)(v + v^2)$

$\beta_5 = 4^{-1}(1 + u)(-v + v^2) \qquad \beta_6 = 4^{-1}(1 - u)(-v + v^2)$

Note that ${\beta_i}^2 = \beta_i$, $\beta_i\beta_j = 0$, $\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 = 1$, where $1 \le i, j \le 6$ and $i \ne j$, and every element $r$ in $R$ can be written uniquely as $r = \sum_{i=1}^{6} a_i\beta_i$, where $a_i \in F_q$.

For any element $r = a + ub + vc + uvd + v^2e + uv^2f$ in $R$, we define the

*Lee weight* of $r$ as $w_L(r) = w_H(a, b, a + c + e, b + d + f, a - c + e, b - d + f)$, where $w_H$ is the *Hamming weight* for $q$-ary codes, and the *Lee weight* for the codeword $x = (x_1, x_2, \ldots, x_n) \in R^n$ denoted by $w_L(x) = w_L(x_1) + w_L(x_2) + \cdots + w_L(x_n) = \sum_{i=1}^{n} w_L(x_i)$.

The *Lee distance* between $x$ and $y$ defined as $d_L(x, y) = w_L(x - y) = \sum_{i=1}^{n} w_L(x_i - y_i)$ and the *Lee distance* for the code $C$ is defined by $d_L(C) = min\{d_L(x, y)|x \neq y, \forall x, y \in c\}$.

Let $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n)$, then the *Euclidean inner product* of $x$ and $y$ in $R^n$ is $x \cdot y = \sum_{i=1}^{n} x_i y_i$.

If $C$ is a code, the *dual code of* $C$ denoted by $C^{\perp} = \{x \in R^n | x \cdot y = 0, \forall y \in C\}$, if $C \subseteq C^{\perp}$, then $C$ is called $self$-$orthogonal$, if $C = C^{\perp}$, then $C$ is called $self$-$dual$.

**Theorem 2.1.** *The Gray map* $\Psi$*:* $R \to {F_q}^6$ *is linear and* $d_L(x, y) = d_H(\Psi(x), \Psi(x))$.

*Proof.* Let $x = a + ub + vc + uvd + v^2e + uv^2f$ and $y = \acute{a} + u\acute{b} + v\acute{c} + uv\acute{d} + v^2\acute{e} + uv^2\acute{f}$, then $\Psi(x) + \Psi(y) = (a + \acute{a}, b + \acute{b}, (a + c + e) + (\acute{a} + \acute{c} + \acute{e}), (b + d + f) + (\acute{b} + \acute{d} + \acute{e}), (a - c + e)\ (\acute{a} - \acute{c} + \acute{e}), (b - d + f)(\acute{b} - \acute{d} + \acute{f}) = \Psi(x + y)$.

For any $\alpha \in F_q$, we have $\Psi(\alpha x) = (\alpha a, \alpha b, \alpha(a + c + e), \alpha(b + d + f), \alpha(a - c + e), \alpha(b - d + f)) = \alpha\Psi(x)$, which implies that $\Psi$ is linear .

Now $d_L(x, y) = w_L(x - y) = w_L(\Psi(x - y)) = w_L(\Psi(x) - \Psi(y)) = d_L(\Psi(x), \Psi(y))$. $\square$

**Theorem 2.2** ([9], Proposition 3.2)**.** *Let $C$ be a code of length $n$ over $R$, if $C$ is self-orthogonal, so is $\Psi(C)$.*

**Lemma 2.3** ([13], Lemma 2.3)**.** *Let $C$ be a code of length $n$ over $R$ with rank $K$ and minimum Lee distance $d$, then $\Psi(C)$ is a $[6n, k, d]$ linear code over $F_q$.*

## 3. Linear and Skew Cyclic Codes over $R$

In this section we use the *decomposition method* over the ring $R$ to show some familiar structural properties over $R$.

Let $B_i$, $1 \leq i \leq 6$ are codes over $F_q$, we define their *direct sum* by $\oplus_{i=1}^{6} B_i = \{\sum_{i=1}^{6} b_i | b_i \in B_i\}$.

For any element $r = a + ub + vc + uvd + v^2e + uv^2f \in R$, we can written $r$ as $r = \beta_1 a + \beta_2 b + \beta_3(a + c + e) + \beta_4(b + d + f) + \beta_5(a - c + e) + \beta_6(b - d + f)$, where $a, b, c, d, e, f \in F_q$ and $r = a + ub + vc + uvd + v^2e + uv^2f \in R$ is unit if and only if $a$, $b$, $(a + c + e)$, $(b + d + f)$, $(a - c + e)$, and $(b - d + f)$ are units in $F_q$.

Let $C$ be a linear code of length $n$ in $R$ and let

$C_1 = \{a \in {F_q}^n | a + ub + vc + uvd + v^2e + uv^2f \in C,\ for\ some\ b, c, d, e, f\}$
$C_2 = \{b \in {F_q}^n | a + ub + vc + uvd + v^2e + uv^2f \in C,\ for\ some\ a, c, d, e, f\}$
$C_3 = \{a + c + e \in {F_q}^n | a + ub + vc + uvd + v^2e + uv^2f \in C,\ for\ some\ b, d, f\}$
$C_4 = \{b + d + f \in {F_q}^n | a + ub + vc + uvd + v^2e + uv^2f \in C,\ for\ some\ a, c, e\}$
$C_5 = \{a - c + e \in {F_q}^n | a + ub + vc + uvd + v^2e + uv^2f \in C,\ for\ some\ b, d, f\}$
$C_6 = \{b - d + f \in {F_q}^n | a + ub + vc + uvd + v^2e + uv^2f \in C,\ for\ some\ a, c, e\}$.

Then $C_1$, $C_2$, $C_3$, $C_4$, $C_5$ and $C_6$ are linear codes of length $n$ over $F_q$, with $C = \oplus_{i=1}^{6} \beta_i C_i$ and $|\ C\ |=|\ C_1\ | \cdot |\ C_2\ | \cdot |\ C_3\ | \cdot |\ C_4\ | \cdot |\ C_5\ | \cdot |\ C_6\ |$.

Let $C = \oplus_{i=1}^{6}\beta_i C_i$ be $F_q$ module, and let $G_i$ be the *generator matrix* of $q$-ary linear codes $C_i$ respectively, where $1 \leq i \leq 6$, then the *generator matrix* of $C$ is $G = \begin{pmatrix} \beta_1 C_1 \\ \beta_2 C_2 \\ \vdots \\ \beta_6 C_6 \end{pmatrix}$, and the *generator matrix* of $\Psi(C)$ is $\Psi(G) = \begin{pmatrix} \Psi(\beta_1 C_1) \\ \Psi(\beta_2 C_2) \\ \vdots \\ \Psi(\beta_6 C_6) \end{pmatrix}$.

Let $\theta_i$ be the *automorphisms* of $R$ defined as
$\theta_i(a+ub+vc+uvd+v^2e+uv^2f) = a^{p^i}+ub^{p^i}+vc^{p^i}+uvd^{p^i}+v^2e^{p^i}+uv^2f^{p^i}$, the *skew polynomial ring* is $R[x,\theta_i] = \{f(x) = a_0 + a_1x + \cdots + a_nx^n | a_i \in F_q, 0 \leq i \leq n\} = \{\sum_{i=0}^{n} a_i x^i | a_i \in F_q\}$, where the *multiplication* is defined by the basic rule $xa = \theta_i(a)x$, where $a \in F_q$, while the *addition* is the usual polynomial addition.

**Definition 3.1.** *A subset $C$ of $R^n$ is called skew cyclic code of length $n$ if $C$ is a submodule of $R^n$, and if $c = (c_0, c_1, \ldots, c_{n-1}) \in C$, then $\sigma_{\theta_i}(c) = (\theta_i(c_{n-1}), \theta_i(c_0), \ldots, \theta_i(c_{n-2})) \in C$.*

**Theorem 3.2.** *Let $C = \oplus_{i=1}^{6}\beta_i C_i$ be a linear code of length $n$ over $R$. Then $C^{\perp} = \oplus_{i=1}^{6}\beta_i {C_i}^{\perp}$.*

*Proof.* The proof is similar to proof of ([5], Theorem 7). □

**Theorem 3.3.** *Let $C = \oplus_{i=1}^{6}\beta_i C_i$ be a linear code of length $n$ over $R$, where $C_i$, $1 \leq i \leq 6$ are linear codes of length $n$ over $F_q$. Then $C$ is skew cyclic code with respect to the automorphism $\theta_i$ over $R$ if and only if $C_i$, $1 \leq i \leq 6$ are all skew cyclic codes over $F_q$.*

*Proof.* The proof is similar to the proof of ([1], Theorem 3.4).

Let $({c_1}^i, {c_2}^i, \ldots, {c_n}^i) \in C$, $1 \leq i \leq 6$. Assume that $c_j = \sum_{i=1}^{6} \beta_i c_j$, then $c = (c_1, c_2, \ldots, c_n) \in C$. Let $C$ be a skew cyclic code with respect to the *automorphism* $\theta_i$ over $R$, then $\sigma_{\theta_i}(c) = (\theta_i(c_n), \theta_i(c_1), \ldots, \theta_i(c_{n-1})) \in C$. We have that $\sigma_{\theta_i}(c) = \sum_{i=1}^{6} \beta_i(\theta_i(c_n^i), \theta_i(c_1^i), \ldots, \theta_i(c_{n-1}^i))$. Hence
$(\theta_i(c_n^i), \theta_i(c_1^i), \ldots, \theta_i(c_{n-1}^i)) \in C_i$, for $1 \leq i \leq 6$. We have $C_1$, $C_2$, $C_3$, $C_4$, $C_5$, and $C_6$ are skew cyclic codes with respect to the *automorphism* $\theta_i$ over $F_q$.

Conversely, assume that $C_1$, $C_2$, $C_3$, $C_4$, $C_5$, and $C_6$ are skew cyclic codes with respect to the *automorphism* $\theta_i$ over $F_q$, and let $c = (c_1, c_2, \ldots, c_n) \in C$, with $c_j = \sum_{i=1}^{6} \beta_i c_j$, then $({c_1}^i, {c_2}^i, \ldots, {c_n}^i) \in C$, where $1 \leq i \leq 6$.

Now $\sigma_{\theta_i}(c) = (\theta_i(c_n), \theta_i(c_1), \ldots, \theta_i(c_{n-1})) \in C$. □

As we know that it is not easy to find the exact number of skew cyclic codes over $[R, \theta_i]$, and there idempotent generator over $R$ for the important reason that the skew polynomial is non-commutative. For this the conditions $gcd(n,k) = 1$, and $gcd(n,q) = 1$ allow us to know the existence of an idempotent generator $e(x) \in [R, \theta_i]$. Also if $f(x)$ is a monic right divisor of $x^n - 1$ with $C =< f(x) >$, $gcd(n,k) = 1$, and $gcd(n,q) = 1$, then there exist an idempotent polynomial, such that $C =< e(x) >$, see [5, 9, 13].

## 4. Skew $(\beta_1 + u\beta_2 + v\beta_3 + uv\beta_4 + v^2\beta_5 + uv^2\beta_6)$-Constacyclic Codes over $R$

In this section we recall the definition of $\beta$ - constacyclic code and $(\theta-\beta)$-constacyclic codes over $R = F_q + uF_q + vF_q + uvF_q + v^2F_q + uv^2F_q$ and give some results on skew $\beta$-constacyclic code over $R$.

**Definition 4.1.** *Let $\beta$ be a unit in $R$. A linear code $C$ of length $n$ over $R$ is called $\beta$-constacyclic code if for every $c = (c_0, c_1, \ldots, c_{n-1}) \in C$, we have $(\beta c_{n-1}, c_0, \ldots, c_{n-2}) \in C$.*

Note that if $\beta = 1$, then a $\beta$-constacyclic codes is cyclic codes, while if $\beta = -1$, then a $\beta$-constacyclic codes is called negacyclic codes.

**Definition 4.2.** *Let $\beta = \beta_1+u\beta_2+v\beta_3+uv\beta_4+v^2\beta_5+uv^2\beta_6$ be a unit in $R$, where $\beta_i \in {F_q}^*$ and $\theta$ be the automorphism on $R$. A linear code $C$ of length $n$ is said to be skew constacyclic code or specifically $(\theta-\beta)$-constacyclic Codes over $R$ if and only if $C$ is invariant under the $(\theta-\beta)$-constacyclic shift vector $\tau_{\theta,\beta} : R^n \to R^n$ defined as $\tau_{\theta,\beta}(c) = \tau_{\theta,\beta}(c_0, c_1, \ldots, c_{n-1}) = (\beta\theta(c_{n-1}), \theta(c_0), \ldots, \theta c_{n-2})$.*

For any codewords as a polynomial, a skew $\beta$-constacyclic code $C$ of length $n$ over $F_q$ with respect to automorphism $\theta$ is left $F_q[x,\theta]$-submodule of $F_q[x,\theta]/ < x^n - \theta >$ generated by a monic polynomial $f(x)$ which is a right divisor of $(x^n - \beta)$ in $F_q[x,\theta]$, see [7, 13] .

Note that there is a one-to-one correspondence between the skew cyclic codes and skew $\beta$-constacyclic codes over $R$ of odd length, see [5].

**Theorem 4.3.** *Let $\beta = \beta_1 + u\beta_2 + v\beta_3 + uv\beta_4 + v^2\beta_5 + uv^2\beta_6$ be a unit in $R$, $C = \oplus_{i=1}^{6}\beta_i C_i$ be a linear code of length $n$ over $R$. Then $C$ is $\beta$-constacyclic codes over $R$ if and only if $C_1$, $C_2$, $C_3$, $C_4$, $C_5$ and $C_6$ are skew $\beta_1$-constacyclic code, skew $\beta_2$-constacyclic code, skew $(\beta_1+\beta_3+\beta_5)$-constacyclic code, skew $(\beta_2+\beta_4+\beta_6)$-constacyclic code, skew $(\beta_1-\beta_3+\beta_5)$-constacyclic code and skew $(\beta_2-\beta_4+\beta_6)$-constacyclic code of length $n$ over $F_q$ respectively.*

*Proof.* Let $r = (r_0, r_1, \ldots, r_{n-1}) \in C$, where $r_i = \beta_1 a_i + \beta_2 b_i + \beta_3 c_i + \beta_4 d_i + \beta_5 e_i + \beta_6 f_i$, $0 \le i \le n-1$.

Let $a = (a_0, a_1, \ldots, a_{n-1})$, $b = (b_0, b_1, \ldots, b_{n-1})$, $c = (c_0, c_1, \ldots, c_{n-1})$, $d = (d_0, d_1, \ldots, d_{n-1})$, $e = (e_0, e_1, \ldots, e_{n-1})$ and $f = (f_0, f_1, \ldots, f_{n-1})$, so $a \in C_1$, $b \in C_2$, $c \in C_3$, $d \in C_4$, $e \in C_5$ and $f \in C_6$.

Suppose that $C_1$, $C_2$, $C_3$, $C_4$, $C_5$ and $C_6$ are skew $\beta_1$-constacyclic code, skew $\beta_2$-constacyclic code, skew $(\beta_1+\beta_3+\beta_5)$-constacyclic code, skew $(\beta_2+\beta_4+\beta_6)$-constacyclic code, skew $(\beta_1-\beta_3+\beta_5)$-constacyclic code and skew $(\beta_2-\beta_4+\beta_6)$-constacyclic code of length $n$ over $F_q$ respectively. So $\tau_{\beta_1}(a) \in C_1$, $\tau_{\beta_2}(b) \in C_2$, $\tau_{\beta_1+\beta_3+\beta_5}(c) \in C_3$, $\tau_{\beta_2+\beta_4+\beta_6}(d) \in C_4$, $\tau_{\beta_1-\beta_3+\beta_5}(e) \in C_5$ and $\tau_{\beta_2-\beta_4+\beta_6}(f) \in C_6$.

Now $\tau_\beta(r) = \tau_{\beta_1+u\beta_2+v\beta_3+uv\beta_4+v^2\beta_5+uv^2\beta_6}(r) = (\beta\theta_i(r_{n-1}), \theta_i(r_0), \ldots, \theta_i(c_{r-2})) = ((\beta_1+u\beta_2+v\beta_3+uv\beta_4+v^2\beta_5+uv^2\beta_6)\theta_i(r_{n-1}), \theta_i(r_0), \ldots, \theta_i(c_{r-2})) = \beta_1\tau_{\beta_1}(a) + \beta_2\tau_{\beta_2}(b) + \beta_3\tau_{\beta_1+\beta_3+\beta_5}(c) + \beta_4\tau_{\beta_2+\beta_4+\beta_6}(d) + \beta_5\tau_{\beta_1-\beta_3+\beta_5}(e) + \beta_6\tau_{\beta_2-\beta_4+\beta_6}(f) \in \oplus_{i=1}^{6}\delta_i C_i = C$, which implies that $C$ is $\beta$-constacyclic code over $R$.

Conversely, Let $a = (a_0, a_1, \ldots, a_{n-1}) \in C_1$, $b = (b_0, b_1, \ldots, b_{n-1}) \in C_2$, $c = (c_0, c_1, \ldots, c_{n-1})$
$\in C_3$, $d = (d_0, d_1, \ldots, d_{n-1}) \in C_4$, $e = (e_0, e_1, \ldots, e_{n-1}) \in C_5$ and $f = (f_0, f_1, \ldots, f_{n-1}) \in C_6$, let $r_i = \beta_1 a_i + \beta_2 b_i + \beta_3 c_i + \beta_4 d_i + \beta_5 e_i + \beta_6 f_i$, where $0 \leq i \leq n-1$, then $r = (r_0, r_1, \ldots, r_{n-1}) \in C$.

Suppose that $C$ is $\beta$-constacyclic code over $R$, so
$\tau_\beta(r) = \tau_{\beta_1+u\beta_2+v\beta_3+uv\beta_4+v^2\beta_5+uv^2\beta_6}(r) \in C$.
$\tau_\beta(r) = \beta_1\tau_{\beta_1}(a) + \beta_2\tau_{\beta_2}(b) + \beta_3\tau_{\beta_1+\beta_3+\beta_5}(c) + \beta_4\tau_{\beta_2+\beta_4+\beta_6}(d) + \beta_5\tau_{\beta_1-\beta_3+\beta_5}(e)$ $+\beta_6\tau_{\beta_2-\beta_4+\beta_6}(f)$. Which implies directly that $\tau_{\beta_1}(a) \in C_1$, $\tau_{\beta_2}(b) \in C_2$, $\tau_{\beta_1+\beta_3+\beta_5}(c) \in C_3$, $\tau_{\beta_2+\beta_4+\beta_6}(d) \in C_4$, $\tau_{\beta_1-\beta_3+\beta_5}(e) \in C_5$ and $\tau_{\beta_2-\beta_4+\beta_6}(f) \in C_6$. Hence $C_1$, $C_2$, $C_3$, $C_4$, $C_5$ and $C_6$ are skew $\beta_1$-constacyclic code, skew $\beta_2$-constacyclic code, skew $(\beta_1 + \beta_3 + \beta_5)$-constacyclic code, skew $(\beta_2 + \beta_4 + \beta_6)$-constacyclic code, skew $(\beta_1 - \beta_3 + \beta_5)$-constacyclic code and skew $(\beta_2 - \beta_4 + \beta_6)$-constacyclic code of length $n$ over $F_q$ respectively. □

Now, we want to study skew constacyclic codes generated by $Monic\ Right\ Divisor$ of $x_n - \beta$ as $\beta = \beta_1 + u\beta_2 + v\beta_3 + uv\beta_4 + v^2\beta_5 + uv^2\beta_6$ is a unit in $R$, with $n = km$, where $n$ is the length of codes and $k$ is the order of the automorphism $\theta$. A generator matrix of the $(\theta - \beta)$-constacyclic code generated by $g(x)$ is given. See [11], where $g(x)$ is a $Monic\ Right\ Divisor$ of $x_n - (\beta_1 + u\beta_2 + v\beta_3 + uv\beta_4 + v^2\beta_5 + uv^2\beta_6)$.

**Lemma 4.4** ([11], Lemma 3.1). *Let $C$ be a code of length $n$ over $R$. Then $C$ is $(\theta - \beta)$-constacyclic if and only if $C^\perp$ is $(\theta - \beta^{-1})$-constacyclic. In particular, if $\beta^2 = 1$, then $C$ is $(\theta - \beta)$-constacyclic if and only if $C^\perp$ is $(\theta - \beta)$-constacyclic.*

**Corollary 4.5.** *Let $C = \oplus_{i=1}^{6}\beta_i C_i$ be a skew $\beta$-constacyclic code of length $n$ over $R$. Then the dual code $C^\perp = \oplus_{i=1}^{6}\beta_i {C_i}^\perp$ is skew $\beta^{-1}$-constacyclic code over $R$, where ${C_1}^\perp$, ${C_2}^\perp$, ${C_3}^\perp$, ${C_4}^\perp$, ${C_5}^\perp$ and ${C_6}^\perp$ are skew ${\beta_1}^{-1}$-constacyclic code, skew ${\beta_2}^{-1}$-constacyclic code, skew $(\beta_1 + \beta_3 + \beta_5)^{-1}$-consta cyclic code, skew $(\beta_2 + \beta_4 + \beta_6)^{-1}$-constacyclic code, skew $(\beta_1 - \beta_3 + \beta_5)^{-1}$-constacyclic code and skew $(\beta_2 - \beta_4 + \beta_6)^{-1}$-constacyclic code of length $n$ over $F_q$ respectively, where $\beta = \beta_1 + u\beta_2 + v\beta_3 + uv\beta_4 + v^2\beta_5 + uv^2\beta_6$ and $n = mk$, where $k =|< \theta >|$ the order of the ring automorphism.*

*Proof.* Let $\beta = \beta_1 + u\beta_2 + v\beta_3 + uv\beta_4 + v^2\beta_5 + uv^2\beta_6$ be fixed by $\theta$ and $n = mk$, where $n$ is the length of a code $C$ and $k =|< \theta >|$, then be Lemma 4.4, $C^\perp$ is$(\theta - \beta^{-1})$-constacyclic over $R$.

Since $\beta = \beta_1 + u\beta_2 + v\beta_3 + uv\beta_4 + v^2\beta_5 + uv^2\beta_6 = \beta_1\beta_1 + \beta_2\beta_2 + \beta_3(\beta_1 + \beta_3 + \beta_5) + \beta_4(\beta_2 + \beta_4 + \beta_6) + \beta_5(\beta_1 - \beta_3 + \beta_5) + \beta_6(\beta_2 - \beta_4 + \beta_6)$. It follows$\beta^{-1} = \beta_1{\beta_1}^{-1} + \beta_2{\beta_2}^{-1} + \beta_3(\beta_1 + \beta_3 + \beta_5)^{-1} + \beta_4(\beta_2 + \beta_4 + \beta_6)^{-1} + \beta_5(\beta_1 - \beta_3 + \beta_5)^{-1} + \beta_6(\beta_2 - \beta_4 + \beta_6)^{-1}$, so we have ${C_1}^\perp, {C_2}^\perp, {C_3}^\perp, {C_4}^\perp, {C_5}^\perp$ $and {C_6}^\perp$ are skew ${\beta_1}^{-1}$-constacyclic code, skew ${\beta_2}^{-1}$-constacyclic code, skew $(\beta_1 + \beta_3 + \beta_5)^{-1}$-constacyclic code, skew $(\beta_2 + \beta_4 + \beta_6)^{-1}$-constacyclic code, skew $(\beta_1 - \beta_3 + \beta_5)^{-1}$-constacyclic code and skew $(\beta_2 - \beta_4 + \beta_6)^{-1}$-constacyclic code of length $n$ over $F_q$ respectively. □

**Theorem 4.6.** *Let $\beta = \beta_1 + u\beta_2 + v\beta_3 + uv\beta_4 + v^2\beta_5 + uv^2\beta_6$ and $C = \oplus_{i=1}^{6}\beta_i C_i$ be $(\theta - \beta)$-constacyclic code over $R$, where $\theta$ the automorphism of*

*$R$, then there exist a polynomial $f(x)$ in $R[x, \theta]$, which is a right divisor of $x^n - \beta = x^n - (\beta_1 + u\beta_2 + v\beta_3 + uv\beta_4 + v^2\beta_5 + uv^2\beta_6)$ and $C =< f(x) >$.*

*Proof.* Firstly want to show that $C =< f(x) >$

Let $f_i(x)$ be generator of $C_i$ for $i = 1, 2, 3, 4, 5, 6$. Then $\beta_i f_i(x)$ are generators of $C$ for $1 \leq i \leq 6$. Take $f(x) = \sum_{i=1}^{6} \beta_i f_i(x)$ and $\pounds =< f(x) >$, then $\pounds \subseteq C$.

On the other hand $\beta_i f_i(x) = \beta_i f(x) \in \pounds$, for $1 \leq i \leq 6$, which implies that $C \subseteq \pounds$ . Hence $C = \pounds =< f(x) >$.

Since $f_i(x)$ are right divisors of $x^n - \beta_1$, $x^n - \beta_2$, $x^n - (\beta_1 + \beta_3 + \beta_5)$, $x^n - (\beta_2 + \beta_4 + \beta_6)$, $x^n - (\beta_1 - \beta_3 + \beta_5)$ and $x^n - (\beta_2 - \beta_4 + \beta_6)$ respectively, so there exist $h_i(x)$, where $1 \leq i \leq 6$ such that $x^n - \beta_1 = h_1(x) * f_1(x)$, $x^n - \beta_2 = h_2(x) * f_2(x)$, $x^n - (\beta_1 + \beta_3 + \beta_5) = h_3(x) * f_3(x)$, $x^n - (\beta_2 + \beta_4 + \beta_6) = h_4(x) * f_4(x)$, $x^n - (\beta_1 - \beta_3 + \beta_5) = h_5(x) * f_5(x)$ and $x^n - (\beta_2 - \beta_4 + \beta_6) = h_6(x) * f_6(x)$. Also $[\sum_{i=1}^{6} \beta_i h_i(x)] * f(x) = \beta_i h_i(x) * f_i = x^n - (\beta_1 + u\beta_2 + v\beta_3 + uv\beta_4 + v^2\beta_5 + uv^2\beta_6)] = x^n - \beta$, which implies that $f(x)$ is a right divisor of $x^n - \beta$. □

**Corollary 4.7.** *Let $\beta = \beta_1 + u\beta_2 + v\beta_3 + uv\beta_4 + v^2\beta_5 + uv^2\beta_6$ be a unit in $R$, each left submodule of $R[x, \theta]/ < x^n - \beta >$ is generated by single element.*

**Theorem 4.8.** *Let $\beta = \beta_1 + u\beta_2 + v\beta_3 + uv\beta_4 + v^2\beta_5 + uv^2\beta_6$, and $C = \oplus_{i=1}^{6} \beta_i C_i$ be $(\theta{-}\beta)$-constacyclic code over $R$, let $gcd(n, k) = 1$ and $gcd(n, q) = 1$. Then there exists an idempotent generator $e(x) = \sum_{i=1}^{6} \beta_i e_i(x) \in R[x, \theta]/ < x^n - \beta >$ such that $C =< e(x) >$, where $e_1(x) \in F_q[x, \theta]/ < x^n - \beta_1 >$, $e_2(x) \in F_q[x, \theta]/ < x^n - \beta_2 >$, $e_3(x) \in F_q[x, \theta]/ < x^n - (\beta_1 + \beta_3 + \beta_5) >$, $e_4(x) \in F_q[x, \theta]/ < x^n - (\beta_2 + \beta_4 + \beta_6) >$, $e_5(x) \in F_q[x, \theta]/ < x^n - (\beta_1 - \beta_3 + \beta_5) >$ and $e_6(x) \in F_q[x, \theta]/ < x^n - (\beta_2 - \beta_4 + \beta_6) >$ are idempotent generator of $C_1$, $C_2$, $C_3$, $C_4$, $C_5$ and $C_6$ respectively.*

*Proof.* By the same argument of ([7], Theorem 16), we have that $C_1$, $C_2$, $C_3$, $C_4$, $C_5$ and $C_6$ are cyclic codes, which implies that $C_i =< e_i(x) >$, where $1 \leq i \leq 6$ and $e_i(x)$ are idempotent generators of $C_i$ respectively $\in F_q[x, \theta]$ . Then we have $e(x) = \sum_{i=1}^{6} \beta_i e_i(x)$ is an idempotent generator of $C$. □

**Theorem 4.9.** *Let $C$ be a skew $\beta$-constacyclic code of length $n$ over $R$, let $k$ be the order of the automorphism and $n$ be the length of the code with $gcd(n, k) = 1$. Then $C$ is a $\beta$-constacyclic code of length $n$ over $R$.*

*Proof.* The proof is similar to the proof of([5], Theorem 22).

Let $gcd(n, k) = 1$, then there exist an integers $a, b$, such that $ak = 1 + bn$. Let $c(x) = c_0 + c_1x^1 + \cdots + c_{n-1}x^{n-1} \in C$. Then $x^i c(x) \in C$, $1 \leq i \leq ak$.

Now $x^{ak}c(x) = x^{ak} \sum_{i=0}^{n-1} c_i x^i = \theta_i^{ak} c_0 x^{ak} + \theta_i^{ak} c_1 x^{ak+1} + \cdots + \theta_i^{ak} c_{n-1}$ $x^{ak+n-1} = c_0 x^{1+bn} + c_1 x^{2+bn} + \cdots + c_{n-1} x^{n+bn} = \beta^b (c_0 x + c_1 x^2 + \cdots + c_{n-2} x^{n-1} + \beta x^{n-1})$.

Hence $\beta^b x^{ak} c(x) = c_0 x + c_1 x^2 + \cdots + c_{n-2} x^{n-1} + c_{n-1}\beta \in C$. So $C$ is a $\beta$-constacyclic code of length $n$ over $R$. □

**Corollary 4.10.** *Let $gcd(n, k) = 1$. If $f(x)$ is a right divisor of $x^n - \beta$ in the skew polynomial ring $R[x, \theta_i]$, then $f(x)$ is a factor of $x^n - \beta$ in the polynomial ring $R[x]$.*

**Example 4.11.** *We construct the field $F_{25} = F_{5^2} = F_5[\alpha]$ as a ring of polynomials over $F_5$ modulo the irreducible polynomial $x^2+x+1$, with $x = \alpha = 0.\alpha^2+1.\alpha+0.1$. Now, $q = 5$ and take $n = 4$ with $Frobenius$ automorphism $\theta : F_{25} \to F_{25}$ defined by $\theta(\alpha) = \alpha^5$ , the factorization of $x^4 - 1$ modulo 5 is $(x^4 - 1) = (x - 1)(x + 1)(x + 2)(x + 3)$, and the factorization of $x^4 + 1$ modulo 5 is $(x^4 + 1) = (x^2 + 2)(x^2 + 3)$. Take $f_1(x) = f_2(x) = f_3(x) = f_4(x) = f_5(x) = x + 1$, $f_6(x) = x^2 + 3$, let $\beta_1 = 1$, $\beta_1 + \beta_2 = 1$, $\beta_1 + \beta_2 + \beta_3 = 1$, $\beta_1 + \beta_2 + \beta_3 + \beta_4 = -1$, $\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 = -1$ and $\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 = -1$.*
*Now this system can be solved by $Maple$ as follows:*
*solve$(\beta_1 = 1, \beta_1 + \beta_2 = 1, \beta_1 + \beta_2 + \beta_3 = 1, \beta_1 + \beta_2 + \beta_3 + \beta_4 = -1, \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 = -1, \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 = -1), [\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6]$. mod 5 to have $[[\beta_1 = 1, \beta_2 = 0, \beta_3 = 0, \beta_4 = 3, \beta_5 = 0, \beta_6 = 0]]$, so we compute $\beta$ by Maple as follows:*
*$\beta_1 := 1, \beta_2 := 0, \beta_3 := 0, \beta_4 := -2, \beta_5 := 0, \beta_6 := 0, evala(\beta_1 + u\beta_2 + v\beta_3 + uv\beta_4+v^2\beta_5+uv^2\beta_6)$ mod 5 to have $\beta = \beta_1+u\beta_2+v\beta_3+uv\beta_4+v^2\beta_5+uv^2\beta_6 = 1 + 3uv$.By the same way one can have $C =< \beta_1 f_1(x) + \beta_2 f_2(x) + \beta_3 f_3(x) + \beta_4 f_4(x)+\beta_5 f_5(x)+\beta_6 f_6(x) >=< (\beta_1+\beta_2+\beta_3+\beta_4+\beta_5)f_1(x)+\beta_6 f_6(x) >=< (1 + 4v + uv + v^2 + 4uv^2)(x + 1) + (1 - u)(v - v^2)(x^2 + 3) >$. Then $C =< (1 + 4v + uv + v^2 + 4uv^2)(x + 1) + (1 - u)(v - v^2)(x^2 + 3) >$ is a self-dual skew $(1 + 3uv)$-constacyclic code of length 4 over $R = F_{25} + uF_{25} + vF_{25} + uvF_{25} + v^2F_{25} + uv^2F_{25}$, where $u^2 = 1$, $v^3 = v$, $uv = vu$.*

**Example 4.12.** *We construct the field $F_9 = F_{3^2} = F_3[2\alpha + 1]$ as a ring of polynomials over $F_3$ modulo the irreducible polynomial $x^2+1$, with $\alpha^2+1 = 0$. Now, $q = 3$ and take $n = 5$ with $Frobenius$ automorphism $\theta : F_9 \to F_9$ defined by $\theta(\alpha) = \alpha^3$ , the factorization of $x^5 - 1$ modulo 3 is $(x^5 - 1) = (x - 1)(x^4 + x^3 + x^2 + x + 1)$, and the factorization of $x^5 + 1$ modulo 3 is $(x^5 + 1) = (x + 1)(x^4 - x^3 + x^2 - x + 1)$.*

*Take $f_1(x) = f_2(x) = f_3(x) = (x^4 + x^3 + x^2 + x + 1)$, $f_4(x) = f_5(x) = f_6(x) = (x^4 - x^3 + x^2 - x + 1)$, let $\beta = 1 - 2u + v - uv$, we have $\beta_1 = 1$, $\beta_1+\beta_2 = -1$, $\beta_1+\beta_2+\beta_3 = 1$, $\beta_1+\beta_2+\beta_3+\beta_4 = -1$, $\beta_1+\beta_2+\beta_3+\beta_4+\beta_5 = 1$ and $\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 = -1$.*

*Compute $\beta_1 + \beta_2 + \beta_3$ and $\beta_4 + \beta_5 + \beta_6$ by Maple as follows:*
*Expand$(2^{-1}(1 + u)(1 - v^2) + 2^{-1}(1 - u)(1 - v^2) + 4^{-1}(1 + u)(v + v^2))$ mod 3 to have $1 + v + uv + uv^2$*
*Expand$(4^{-1}(1 - u)(v + v^2) + 4^{-1}(1 + u)(-v + v^2) + 4^{-1}(1 - u)(-v + v^2))$ mod 3 to have $2v + 2uv + 2uv^2$.*

*Then we have $f(x) = \beta_1 f_1(x) + \beta_2 f_2(x) + \beta_3 f_3(x) + \beta_4 f_4(x) + \beta_5 f_5(x) + \beta_6 f_1(6) = (\beta_1 + \beta_2 + \beta_3)f_1(x) + (\beta_4 + \beta_5 + \beta_6)f_3(x) = (1+v+uv+uv^2)(x^4 + x^3 + x^2 + x + 1) + (2v + 2uv + 2uv^2)(x^4 - x^3 + x^2 - x + 1) = x^4 + (1 - v - uv - uv^2)x^3 + x^2 + (1 - v - uv - uv^2)x + 1$ is aright divisor of $x^5 - (1 - 2u + v - uv)$ in $R[\theta, x]$ and by Theorem 4.9 since $gcd(n, k) = gcd(5, 3) = 1$, then $C =< f(x) >$ is a $(1 - 2u + v - uv)$- constacyclic code of length 5 over $R = F_9 + uF_9 + vF_9 + uvF_9 + v^2F_9 + uv^2F_9$, where $u^2 = 1$, $v^3 = v$, $uv = vu$.*

## 5. Conclusion

In this paper, we considered $(\theta - \beta)$ - Constacyclic codes over the ring $R = F_q + uF_q + vF_q + uvF_q + v^2F_q + uv^2F_q$, with $u^2 = 1$, $v^3 = v$, $uv = vu$,

$q = p^m$ and p is an odd prime. For future resaearch one can study skew constacyclic codes over the rings $F_p[u,v,w]/ < u^2, v^2, w^2, uv - vu, vw - wv, uw - wu >= F_p + uF_p + vF_p + wF_p + uvF_p + uwF_p + vwF_p + uvwF_p$ or $F_q[u,v,w]/ < u^2, v^2, w^2, uv - vu, vw - wv, uw - wu >= F_q + uF_q + vF_q + wF_q + uvF_q + uwF_q + vwF_q + uvwF_q$, where $u^2 = v^2 = w^2 = 0$, $uv = vu, vw = wvanduw = wu$.

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