Proceedings of the Jangjeon Mathematical Society
23 (2020), No. 1. pp. 65 - 69

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2020.23.1.65

Cartesian products over intuitionistic fuzzy index matrices

Krassimir Atanassov and Tania Pencheva

Institute of Biophysics and Biomedical Engineering
Bulgarian Academy of Sciences, Sofia-1113, Bulgaria
e-mails: *krat@bas.bg*, *tania.pencheva@biomed.bas.bg*

Abstract: In this paper, two forms of a new operation, called a Cartesian product, is introduced over intuitionistic fuzzy index matrices. Some of its properties are discussed.

Keywords: Index matrix, Intuitionistic fuzzy index matrix, Intuitionistic fuzziness.

AMS Classification: 03E72, 11C20.

1 Introduction

Here, as a continuation of the development of the Index Matrix (IM) theory [1, 5, 7], we discuss Cartesian type of products over Intuitionistic Fuzzy IMs (IFIMs), that extend the standard IMs, described in details in [5].

In [7], a new operation called "concatenation" and denoted by $\textcircled{\wedge}$, was introduced over standard IM.

Here, we modify it to the form of two types of Cartesian products.

2 Basic definition

Firstly, we give some remarks on Intuitionistic Fuzzy Sets (IFSs, see, e.g., [3]) and especially, of their particular case, Intuitionistic Fuzzy Pairs (IFPs; see [6]). The IFP is an object in the form $\langle a, b\rangle$, where $a, b \in [0, 1]$ and $a + b \leq 1$ which used as an evaluation of some object or process and which components (a and b) are interpreted as degrees of membership and non-membership, or degrees of validity and non-validity, or degrees of correctness and non-correctness, etc.

Let us have two IFPs $x = \langle a, b\rangle$ and $y = \langle c, d\rangle$. We define some examples for definitions of operations "conjunction" and "disjunction":

$$x \vee_1 y = \langle \max(a, c)), \min(b, d)\rangle,$$
$$x \wedge_1 y = \langle \min(a, c), \max(b, d)\rangle,$$
$$x \vee_2 y = \langle a + c - a.c, b.d\rangle,$$
$$x \wedge_2 y = \langle a.c, b + d - b.d\rangle,$$
$$x@y = \langle \tfrac{a+c}{2}, \tfrac{b+d}{2}\rangle.$$

It is seen easy that the fifth operation is simultaneously "conjunction" and "disjunction".

Secondly, following [5], the definition of an IFIM is proposed.

Let $\mathcal{I}$ be a fixed set of indices,

$$\mathcal{I}^n = \{\langle i_1, i_2, ..., i_n\rangle | (\forall j : 1 \leq j \leq n)(i_j \in \mathcal{I})\}$$

and

$$\mathcal{I}^* = \bigcup_{1 \leq n < \infty} \mathcal{I}^n.$$

In the present research, $n = 1$ or 2.

Let everywhere below $\mathcal{X}$ be a fixed set of some objects. In particular cases, they can be either real numbers, or just numbers 0 or 1; logical variables, propositions or predicates; IFPs, etc.

Let operations $\circ, * : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be given.

We call the object $[K, L, \{a_{k_i,l_j}\}]$ with index sets K and L $(K, L \subset \mathcal{I}^*)$ and elements – IFPs $\langle \mu, \nu\rangle$, where $\mu, \nu, \mu + \nu \in [0, 1]$ an IFIM. It has the form

$$[K, L, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle\}]$$

$$= \begin{array}{c|ccccc} & l_1 & \dots & l_j & \dots & l_n \\ \hline k_1 & \langle \mu_{k_1,l_1}, \nu_{k_1,l_1}\rangle & \dots & \langle \mu_{k_1,l_j}, \nu_{k_1,l_j}\rangle & \dots & \langle \mu_{k_1,l_n}, \nu_{k_1,l_n}\rangle \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_i & \langle \mu_{k_i,l_1}, \nu_{k_i,l_1}\rangle & \dots & \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle & \dots & \langle \mu_{k_i,l_n}, \nu_{k_i,l_n}\rangle \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m & \langle \mu_{k_m,l_1}, \nu_{k_m,l_1}\rangle & \dots & \langle \mu_{k_m,l_j}, \nu_{k_m,l_j}\rangle & \dots & \langle \mu_{k_m,l_n}, \nu_{k_m,l_n}\rangle \end{array},$$

where for every $1 \leq i \leq m, 1 \leq j \leq n$: $a_{k_i,l_j} = \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle$ and $\mu_{k_i,l_j}, \nu_{k_i,l_j}, \mu_{k_i,l_j} + \nu_{k_i,l_j} \in [0, 1]$.

3 Main results

Let us have two IFIMs $A = [K, L, \{a_{k_i,l_j}\}]$ and $B = [P, Q, \{b_{p_r,q_s}\}]$, where a_{k_i,l_j} and b_{p_r,q_s} are IFPs or real numbers.

The first type of Cartesian product is the following

$$A \times_C B = [K \times P, L \times Q, \{c_{\langle k_i,p_r\rangle,\langle l_j,q_s\rangle}\}],$$

where

$$c_{\langle\langle k_i,p_r\rangle,\langle l_j,q_s\rangle\rangle} = \langle a_{k_i,l_j}, b_{p_r,q_s}\rangle$$

and operation $\times$ between K and P, and between L and Q is the standard set-theoretical Cartesian product.

Let for the three elements $a, b, c \in \mathcal{X}$ the equalities

$$\langle\langle a, b\rangle, c\rangle = \langle a, b, c\rangle = \langle a, \langle b, c\rangle\rangle$$

are valid.

Theorem 1. The operation $\times_C$ is associative, but not commutative.

Proof. Let us have three IMs A_1, A_2, A_3 so that

$$A_\rho = [K_\rho, L_\rho, \{a^\rho_{k^\rho_i, l^\rho_j}\}]$$

for $\rho = 1, 2, 3$. Then

$$(A_1 \times_C A_2) \times_C A_3$$

$$= ([K_1 \times K_2, L_1 \times L_2, \{\langle a^1_{k^1_i, l^1_j}, a^2_{k^2_i, l^2_j}\rangle\}]) \times_C A_3$$

$$= [(K_1 \times K_2) \times K_3, (L_1 \times L_2) \times L_3, \{\langle\langle a^1_{k^1_i, l^1_j}, a^2_{k^2_i, l^2_j}\rangle, a^3_{k^3_i, l^3_j}\rangle\}]$$

$$= [K_1 \times (K_2 \times K_3), L_1 \times (L_2 \times L_3), \{\langle a^1_{k^1_i, l^1_j}, \langle a^2_{k^2_i, l^2_j}, a^3_{k^3_i, l^3_j}\rangle\rangle\}]$$

$$= A_1 \times_C [K_2 \times K_3, L_2 \times L_3, \langle a^2_{k^2_i, l^2_j}, a^3_{k^3_i, l^3_j}\rangle\}]$$

$$= A_1 \times_C (A_2 \times_C A_3).$$

From the definition of operation $\times_C$ is clear that it is not commutative, because from set-theoretical point of view, for example, $K_1 \times K_2 \neq K_2 \times K_1$ and the same is valid for L-index sets.

Let

$$I_\emptyset = [\emptyset, \emptyset, \langle \perp \rangle],$$

where symbol $\perp$ represents the lack of a symbol from set $\mathcal{X}$.

Theorem 2. For each IM A:

$$A \times_C I_\emptyset = A = I_\emptyset \times_C A.$$

The proof follows directly from the definition of $I_\emptyset$.

Therefore, the set of the IM is a monoid about operation $\times_C$ and unit element $I_\emptyset$.

Having in mind that for each two nonempty IM A and B, $A \times_C B$ is a nonempty IM, then there is no IM B so that $A \times_C B = I_\emptyset$. Therefore, the set of the IM cannot be a group.

When the matrices A and B are IFIMs, then the elements of the IFIM $A \times_C B$ have the form

$$c_{\langle\langle k_i, p_r\rangle, \langle l_j, q_s\rangle\rangle} = \langle a_{k_i, l_j}, b_{p_r, q_s}\rangle = \langle\langle \mu_{k_i, l_j}, \nu_{k_i, l_j}\rangle, \langle \varphi_{p_r, q_s}, \psi_{p_r, q_s}\rangle\rangle,$$

i.e., pair of pairs, where

$$a_{k_i, l_j} = \langle \mu_{k_i, l_j}, \nu_{k_i, l_j}\rangle,$$

$$b_{p_r, q_s} = \langle \varphi_{p_r, q_s}, \psi_{p_r, q_s}\rangle.$$

Let $(\circ, *) \in \{(\max, \min), (\min, \max), (+, .), (., +), (@, @)\}$ and let for two IFPs $\langle a, b\rangle, \langle c, d\rangle$:

$$\langle a, b\rangle(\circ, *)\langle c, d\rangle = \langle a \circ c, b * d\rangle.$$

The second type of Cartesian product is the following

$$A \times_{\circ,*} B = [K \times P, L \times Q, \{c_{\langle k_i,p_r\rangle,\langle l_j,q_s\rangle}\}],$$

where

$$c_{\langle\langle k_i,p_r\rangle,\langle l_j,q_s\rangle\rangle} = (\circ,*)\langle a_{k_i,l_j}, b_{p_r,q_s}\rangle,$$

and for the suitable variables t, u, v, w, in some cases (e.g., conjunction or disjunction)

$$(\circ,*)\langle\langle t,u\rangle,\langle v,w\rangle\rangle = \langle\circ(t,v),*(u,w)\rangle$$

and in others (e.g., implication)

$$(\circ,*)\langle\langle t,u\rangle,\langle v,w\rangle\rangle = \langle\circ(u,v),*(t,w)\rangle$$

in respect of the type of the operation that the pair $(\circ,*)$ represents.

Therefore, when

$$a_{k_i,l_j} = \langle\mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle,$$
$$b_{p_r,q_s} = \langle\varphi_{p_r,q_s}, \psi_{p_r,q_s}\rangle,$$

then, as above, in some cases (e.g., conjunction or disjunction) we have

$$c_{\langle\langle k_i,p_r\rangle,\langle l_j,q_s\rangle\rangle} = \langle\circ(\mu_{k_i,l_j}, \varphi_{p_r,q_s}), *(\nu_{k_i,l_j}, \psi_{p_r,q_s})\rangle$$

and in others (e.g., implication)

$$c_{\langle\langle k_i,p_r\rangle,\langle l_j,q_s\rangle\rangle} = \langle\circ(\nu_{k_i,l_j}, \varphi_{p_r,q_s}), *(\mu_{k_i,l_j}, \psi_{p_r,q_s})\rangle.$$

Theorem 3. The operation $\times_{\circ,*}$ over three IFIMs is associative, but not commutative.

The proof is similar to the proof of Theorem 1.

4 Conclusion

The IFIM constructed in this manner can also be used in graph theory, e.g., for representation of intuitionistic fuzzy graphs, in multi-criteria multi-person decision making procedures, in which the experts (persons) have provided their own scores as estimates, in intercriteria analysis, in which the different criteria have intuitionistic fuzzy weights, and in other applications.

The two forms of the new operation can be defined over the extensions of the IMs and IFIMs, e.g. Extended IFIMs, Interval-Valued IFIMs, n-dimensional IM, and others. These operators can be combined with some ideas, described in [8, 9].

Acknowledgments

The authors are grateful for the support under Grant Ref. No. DN 17/6 "A New Approach, Based on an Intercriteria Data Analysis, to Support Decision Making in *in silico* Studies of Complex Biomolecular Systems" of the Bulgarian National Science Fund.

References

[1] Atanassov, K. (1987) Generalized index matrices, *Comptes rendus de l'Academie Bulgare des Sciences,* 40(11), 15–18.

[2] Atanassov, K. (2010) On index matrices, Part 2: Intuitionistic fuzzy case. *Proceedings of the Jangjeon Mathematical Society,* 13(2), 121–126.

[3] Atanassov, K. (2012) *On Intuitionistic Fuzzy Sets Theory,* Springer, Berlin.

[4] Atanassov, K. (2013) On extended intuitionistic fuzzy index matrices, *Notes on Intuitionistic Fuzzy Sets,* 19(4), 27–41.

[5] Atanassov, K. (2014) *Index Matrices: Towards an Augmented Matrix Calculus,* Springer, Cham.

[6] Atanassov, K., Szmidt, E., & Kacprzyk, J. (2013) On intuitionistic fuzzy pairs, *Notes on Intuitionistic Fuzzy Sets,* 19(3), 1–13.

[7] Shannon, A., V. Atanassova, K. Atanassov. Operation "concatenation" over intuitionistic fuzzy index matrices. Notes on Intuitionistic Fuzzy Sets, Vol. 22, 2016, No. 3, 106-111.

[8] Traneva, V. Internal operations over 3-dimensional extended index matrices, Proceedings of the Jangjeon Mathematical Society, Vol. 18, 2015, No. 4, 547-569.

[9] Traneva, V., E. Sotirova, V. Bureva, K. Atanassov, Aggregation operations over 3-dimensional extended index matrices, Advanced Studies in Contemporary Mathematics, Vol. 25, 2015, No. 3, 407-416.