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RELATIONS BETWEEN GENERALIZED HERMITE-BASED APOSTOL-BERNOULLI, EULER AND GENOCCHI POLYNOMIALS

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Abstract. In this paper, we intimate connection between generalized Apostol-Hermite-Bernoullli polynomials, Apostol-Hermite-Euler polynomials and Apostol-Hermite-Genocchi polynomials and some implicit summation formulae by applying the generating functions.These results extend some known summations and identities of Apostol-Hermite-Bernoullli, Apostol-Hermite-Euler and Apostol-Hermite-Genocchi polynomials.

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1. Introduction

Luo and Srivastava [5,6,9–12] introduced the generalized Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{C}$ and the generalized Apostol-Genocchi polynomials $G_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{C}$. Luo [4,7,8] also introduced the generalized Apostol-Euler polynomials $E_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{C}$.

Recently [3, 15] and [14] defined a new family of generalized Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials of order α and new classes of generalized Hermite-Based, Apostol-Bernoulli polynomials $B_n^{[m-1,\alpha]}(x;\lambda)$, Apostol-Euler polynomials $E_n^{[m-1,\alpha]}(x;\lambda)$ and Apostol-Genocchi polynomials $G_n^{[m-1,\alpha]}(x;\lambda)$, $m \in N$ of order $\alpha \in \mathbb{C}$.

Definition 1.1. For arbitrary real or complex parameter α and for $a, c \in R^+$, the generalized Apostol-Hermite-Bernoulli polynomials ${}_HB_n^{[m-1,\alpha]}(x,y;a,c,\lambda)$, $m \in N$, $\lambda \in \mathbb{C}$ are defined in a suitable neighborhood of $t = 0$ with $|t\log(a)| < |\log(-\lambda)|$, by means of the following generating function:

$$t^{m\alpha}[A(\lambda,a;t)]^{\alpha}c^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n^{[m-1,\alpha]}(x,y;a,c,\lambda)\frac{t^n}{n!}, \tag{1}$$

where $A\ (\lambda,a;t)$ is given by equation

$$A(\lambda,a;t) = \left(\lambda a^t - \sum_{h=0}^{m-1}\frac{(t\log a)^h}{h!}\right)^{-1}. \tag{2}$$

It is easy to see that if we set $y = 0$ in (1), we get the result given by [15, p. 3, Eq. (1.8)] which involves the generalized Apostol-Bernoulli polynomials

$$t^{m\alpha}[A(\lambda, a; t)]^{\alpha} c^{xt} = \sum_{n=0}^{\infty} B_n^{[m-1,\alpha]}(x; a, c, \lambda)\frac{t^n}{n!}, \tag{3}$$

for $c = e$ in (1) gives

$$t^{m\alpha}[A(\lambda, a; t)]^{\alpha} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H B_n^{[m-1,\alpha]}(x, y; a, e, \lambda)\frac{t^n}{n!}. \tag{4}$$

Moreover if we set $y = 0$, $m = 1$, $a = c = e$ in (1), we arrive at the following result

$$\left[\frac{t}{\lambda e^t - 1}\right]^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{[0,\alpha]}(x; e, e, \lambda)\frac{t^n}{n!}, \quad (|t| < 2\pi, 1^{\alpha} = 1) \tag{5}$$

which is a generating function for the generalized Apostol-Bernoulli polynomials of order α. Thus we have

$$B_n^{[0,\alpha]}(x; e, e, \lambda) = B_n^{[\alpha]}(x; \lambda). \tag{6}$$

Definition 1.2. For arbitrary real or complex parameter α and for $a, c \in R^+$, the generalized Apostol-Hermite-Euler polynomials ${}_H E_n^{[m-1,\alpha]}(x, y; a, c, \lambda)$, $m \in N$, $\lambda \in \mathbb{C}$ are defined in a suitable neighborhood of $t = 0$ with $|t\log(a)| < |\log(-\lambda)|$, by means of the following generating function:

$$2^{m\alpha}[B(\lambda, a; t)]^{\alpha} c^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H E_n^{[m-1,\alpha]}(x, y; a, c, \lambda)\frac{t^n}{n!}, \tag{7}$$

where $B\ (\lambda, a; t)$ is given by equation

$$B(\lambda, a; t) = \left(\lambda a^t + \sum_{h=0}^{m-1} \frac{(t\log a)^h}{h!}\right)^{-1}. \tag{8}$$

If we set $y = 0$ in (7), we get the result given by [15, p. 3, Eq. (2.1)] which involves the generalized Apostol-Euler polynomials

$$2^{m\alpha}[B(\lambda, a; t)]^{\alpha} c^{xt} = \sum_{n=0}^{\infty} E_n^{[m-1,\alpha]}(x; a, c, \lambda)\frac{t^n}{n!}, \tag{9}$$

for $c = e$ in (7) gives

$$2^{m\alpha}[B(\lambda, a; t)]^{\alpha} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H E_n^{[m-1,\alpha]}(x, y; a, e, \lambda)\frac{t^n}{n!}. \tag{10}$$

It reduces for $y = 0$, $m = 1$, $a = c = e$ in (7) in the following result:

$$\left[\frac{2}{\lambda e^t + 1}\right]^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{[0,\alpha]}(x; e, e, \lambda)\frac{t^n}{n!}, \quad (|t| < \pi, 1^{\alpha} = 1) \tag{11}$$

which is a generating function for the generalized Apostol-Euler polynomials of order α. Thus we have

$$E_n^{[0,\alpha]}(x; e, e, \lambda) = E_n^{[\alpha]}(x; \lambda). \tag{12}$$

Definition 1.3. For arbitrary real or complex parameter α and for $a, c \in R^+$, the generalized Apostol-Hermite-Genocchi polynomials ${}_HG_n^{[m-1,\alpha]}(x, y; a, c, \lambda)$, $m \in N$, $\lambda \in \mathbb{C}$ are defined in a suitable neighborhood of $t = 0$ with $|t\log(a)| < |\log(-\lambda)|$, by means of the following generating function:

$$2^{m\alpha}t^{m\alpha}[B(\lambda, a; t)]^{\alpha}c^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HG_n^{[m-1,\alpha]}(x, y; a, c, \lambda)\frac{t^n}{n!}, \tag{13}$$

where $B\ (\lambda, a; t)$ is given by equation (8). If we put $y = 0$ in (13), we have obtained the result given by [15, p. 5, Eq. (2.4)] which involves the generalized Apostol-Genocchi polynomials

$$2^{m\alpha}t^{m\alpha}[B(\lambda, a; t)]^{\alpha}c^{xt} = \sum_{n=0}^{\infty} G_n^{[m-1,\alpha]}(x; a, c, \lambda)\frac{t^n}{n!}, \tag{14}$$

for $c = e$ in (7) gives

$$2^{m\alpha}t^{m\alpha}[B(\lambda, a; t)]^{\alpha}e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HG_n^{[m-1,\alpha]}(x, y; a, e, \lambda)\frac{t^n}{n!}. \tag{15}$$

Moreover if we set $y = 0$, $m = 1$, $a = c = e$ in (13), we have obtained the following result:

$$\left[\frac{2t}{\lambda e^t + 1}\right]^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{[0,\alpha]}(x; e, e, \lambda)\frac{t^n}{n!}, \quad (|t| < 2\pi, 1^{\alpha} = 1) \tag{16}$$

which is a generating function for the generalized Apostol-Genocchi polynomials of order α. Thus we have

$$G_n^{[0,\alpha]}(x; e, e, \lambda) = G_n^{[\alpha]}(x; \lambda). \tag{17}$$

2. Generalization of Hermite-based Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials

In one of our paper [2] we consider a natural generalization of the Apostol-Hermite-Bernoulli polynomials ${}_HB_n^{[m-1,\alpha]}(x, y; a, c, \lambda)$, $m \in N$, $\lambda \in \mathbb{C}$, Apostol-Hermite-Euler polynomials ${}_HE_n^{[m-1,\alpha]}(x, y; a, c, \lambda)$, $m \in N$, $\lambda \in \mathbb{C}$ and Apostol-Hermite-Genocchi polynomials ${}_HG_n^{[m-1,\alpha]}(x, y; a, c, \lambda)$, $m \in N$, $\lambda \in \mathbb{C}$.

$$\begin{aligned} f_n^{[m-1,\alpha]}(a_1, a_2 \ldots a_p; t; k, l) &= 2^{mk\alpha}t^{ml\alpha}(-1)^{k\alpha}[A(\lambda, a; t)]^{\alpha}c^{a_1t+a_2t^2 \ldots a_pt^p} \\ &= \sum_{n=0}^{\infty} {}_HP_{n,\lambda}^{[m-1,\alpha]}(a_1, a_2 \ldots a_p; k, l)\frac{t^n}{n!}, \\ &\qquad (k, l, m, p \in N_0; \alpha \in C) \end{aligned} \tag{18}$$

which are defined in a suitable neighborhood of $t = 0$ with $|t\log(a)| < |\log(-\lambda)|$.

For $l = 1$, $k = 0$ in (18), we define the following:

Definition 2.1. For arbitrary real or complex parameter α and for $a, c \in R^+$, the generalized Apostol-Hermite-Bernoulli polynomials ${}_HB_n^{[m-1,\alpha]}$ $(a_1, a_2 \ldots a_p;\ a, c, \lambda)$, $m \in N$, $\lambda \in \mathbb{C}$ are defined in a suitable neighborhood

of $t = 0$ with $|t\log(a)| < |\log(-\lambda)|$, by means of the following generating function:
(19)

$$t^{m\alpha}[A(\lambda,a;t)]^{\alpha}c^{a_1t+a_2t^2\ldots+a_pt^p} = \sum_{n=0}^{\infty} {}_HB_n^{[m-1,\alpha]}(a_1,a_2\ldots a_p;a,c,\lambda)\frac{t^n}{n!},$$

where A $(\lambda,a;t)$ is given by equation (2). It is clear from (18) that

$$f_n^{[m-1,\alpha]}(a_1,a_2\ldots a_p;t;0,1) = t^{m\alpha}[A(\lambda,a;t)]^{\alpha}c^{a_1t+a_2t^2\ldots a_pt^p},$$
$${}_HP_{n,\lambda}^{[m-1,\alpha]}(a_1,a_2\ldots a_p;0,1) = {}_HB_n^{[m-1,\alpha]}(a_1,a_2\ldots a_p;a,c,\lambda).$$

We get result given by [15, p. 3, Eq. (1.8)] by putting $a_2 = a_3 \cdots = a_p = 0$ in (19).

For $a_3 = a_4 \cdots = a_p = 0$ in (19), we obtained the result given by [14, p. 158, Eq. (24)].

If we put $m = 1$, $a = c = e$ and $a_4 = a_5 \cdots = a_p = 0$ in (19), result reduces to known result [13, p. 3, Definition 2.1].

For $l = 0$, $k = 1$ and replacing λ by $-\lambda$ in (18), we define the following:

Definition 2.2. For arbitrary real or complex parameter α and for $a, c \in R^+$, the generalized Apostol-Hermite-Euler polynomials ${}_HE_n^{[m-1,\alpha]}(a_1,a_2\ldots a_p;$ $a,c,\lambda)$, $m \in N$, $\lambda \in \mathbb{C}$ are defined in a suitable neighborhood of $t = 0$ with $|t\log(a)| < |\log(-\lambda)|$, by means of the following generating function:
(20)

$$2^{m\alpha}[B(\lambda,a;t)]^{\alpha}c^{a_1t+a_2t^2\ldots+a_pt^p} = \sum_{n=0}^{\infty} {}_HE_n^{[m-1,\alpha]}(a_1,a_2\ldots a_p;a,c,\lambda)\frac{t^n}{n!},$$

where B $(\lambda,a;t)$ is given by equation (8). It is clear from (18) that

$$f_n^{[m-1,\alpha]}(a_1,a_2\ldots a_p;t;1,0) = 2^{m\alpha}[B(\lambda,a;t)]^{\alpha}c^{a_1t+a_2t^2\ldots a_pt^p},$$
$${}_HP_{n,\lambda}^{[m-1,\alpha]}(a_1,a_2\ldots a_p;1,0) = {}_HB_n^{[m-1,\alpha]}(a_1,a_2\ldots a_p;a,c,\lambda).$$

We get result given by [15, p. 3, Eq. (2.1)] by putting $a_2 = a_3 \cdots = a_p = 0$ in (20).

If we set $a_3 = a_4 \cdots = a_p = 0$ in (20), we obtained the result given by [14, p. 158, Eq. (29)].

If we put $m = 1$, $a = c = e$ and $a_4 = a_5 \cdots = a_p = 0$ in (20), result reduces to known result of [13, p. 3, Definition 2.2].

For $l = 1$, $k = 1$ and replacing λ by $-\lambda$ in (18), we define the following:

Definition 2.3. For arbitrary real or complex parameter α and for $a, c \in R^+$, the generalized Apostol-Hermite-Genocchi polynomials ${}_HG_n^{[m-1,\alpha]}$ $(a_1,a_2\ldots a_p;\ a,c,\lambda)$, $m \in N$, $\lambda \in \mathbb{C}$ are defined in a suitable neighborhood of $t = 0$ with $|t\log(a)| < |\log(-\lambda)|$, by means of the following generating function:
(21)

$$2^{m\alpha}t^{m\alpha}[B(\lambda,a;t)]^{\alpha}c^{a_1t+a_2t^2\ldots+a_pt^p} = \sum_{n=0}^{\infty} {}_HG_n^{[m-1,\alpha]}(a_1,a_2\ldots a_p;a,c,\lambda)\frac{t^n}{n!},$$

where $B\,(\lambda, a; t)$ is given by equation (8). It is clear from (18) that

$$f_n^{[m-1,\alpha]}(a_1, a_2 \ldots a_p; t; 1, 1) = 2^{m\alpha} t^{m\alpha} [B(\lambda, a; t)]^\alpha c^{a_1 t + a_2 t^2 \ldots a_p t^p},$$
$$_H P_{n,\lambda}^{[m-1,\alpha]}(a_1, a_2 \ldots a_p; 1, 1) = {}_H G_n^{[m-1,\alpha]}(a_1, a_2 \ldots a_p; a, c, \lambda).$$

It is easy to see that if we set $a_2 = a_3 \cdots = a_p = 0$ in (21), we obtained the result of [15, p. 5, Eq. (2.4)].

It we set $a_3 = a_4 \cdots = a_p = 0$ in (21), we obtained the result given by [14, p. 158, Eq. (34)].

$$t^{m\alpha}[A(\lambda, a; t)]^\alpha c^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H B_n^{[m-1,\alpha]}(x, y; a, c, \lambda) \frac{t^n}{n!}.$$

For $m = 1$, $a = c = e$ and $a_4 = a_5 \cdots = a_p = 0$ in (21), the result reduces to known result of [13, p. 3, Definition 2.3].

Theorem 2.1. *For each $n \in N$, the following relation*

$$\sum_{r=0}^{n} \binom{n}{r} {}_H B_{n-r}^{[m-1,\alpha]}(a_1, a_2 \ldots a_p; a, c, \lambda) . {}_H E_r^{[m-1,\alpha]}(b_1, b_2 \ldots b_p; a, c, \lambda)$$
$$= 2^n {}_H B_n^{[m-1,\alpha]} \left(\frac{a_1 + b_1}{2}, \frac{a_2 + b_2}{4} \ldots \frac{a_p + b_p}{2^p}; a, c, \lambda^2 \right) \tag{22}$$

holds true between the Hermite-based generalized Apostol-Bernoulli and Euler polynomials.

Proof. We have

$$\sum_{n=0}^{\infty} {}_H B_n^{[m-1,\alpha]} \left(\frac{a_1 + b_1}{2}, \frac{a_2 + b_2}{4} \ldots \frac{a_p + b_p}{2^p}; a, c, \lambda^2 \right) \frac{(2t)^n}{n!}$$
$$= (2t)^{m\alpha} \left[A\left(\lambda^2, a; 2t\right) \right]^\alpha c^{\left[\left(\frac{a_1+b_1}{2} \right) 2t + \left(\frac{a_2+b_2}{4} \right)(2t)^2 \ldots + \left(\frac{a_p+b_p}{2^p} \right)(2t)^p \right]},$$
$$= t^{m\alpha} \left[A\left(\lambda, a; t\right) \right]^\alpha c^{\left(a_1 t + a_2 t^2 \ldots + a_p t^p \right)} . 2^{m\alpha} \left[B\left(\lambda, a; t\right) \right]^\alpha c^{\left(b_1 t + b_2 t^2 \ldots + b_p t^p \right)},$$
$$= \sum_{r=0}^{\infty} {}_H B_r^{[m-1,\alpha]} \left(a_1, a_2 \ldots a_p; a, c, \lambda \right) \frac{t^r}{r!} \cdot \sum_{n=0}^{\infty} {}_H E_n^{[m-1,\alpha]} \left(b_1, b_2 \ldots b_p; a, c, \lambda \right) \frac{t^n}{n!},$$
$$= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} {}_H B_n^{[m-1,\alpha]} \left(a_1, a_2 \ldots a_p; a, c, \lambda \right) \cdot {}_H E_r^{[m-1,\alpha]} \left(b_1, b_2 \ldots b_p; a, c, \lambda \right) \frac{t^{n+r}}{n!r!}.$$

Replacing n by $(n - r)$ in RHS of above equation, we get

$$2^n \sum_{n=0}^{\infty} {}_H B_n^{[m-1,\alpha]} \left(\frac{a_1 + b_1}{2}, \frac{a_2 + b_2}{4} \ldots \frac{a_p + b_p}{2^p}; a, c, \lambda^2 \right) \frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} \binom{n}{r} {}_H B_{n-r}^{[m-1,\alpha]} \left(a_1, a_2 \ldots a_p; a, c, \lambda \right) \cdot {}_H E_r^{[m-1,\alpha]} \left(b_1, b_2 \ldots b_p; a, c, \lambda \right) \frac{t^n}{n!}.$$

Comparing the coefficient of $\frac{t^n}{n!}$ on both sides, we get the result (22). □

Remark 2.1. *If we set $m = 1$, $a = c = e$ and $a_4 = a_5 \cdots = a_p = 0$ and $b_4 = b_5 \cdots = b_p = 0$ in (22), the result reduces to the known result of [13].*

Theorem 2.2. *The generalized Apostol-Hermite-Bernoulli polynomials ${}_HB_n^{[m-1,\alpha]}(a_1, a_2 \ldots a_p; a, c, \lambda)$ and the generalized Apostol-Hermite-Euler polynomials ${}_HE_n^{[m-1,\alpha]}(a_1, a_2 \ldots a_p; a, c, \lambda)$, $m, \alpha \in N_0$, $\lambda \in \mathbb{C}$ are related by*

$$\begin{aligned} &{}_HB_n^{[m-1,\alpha]}(a_1, a_2 \ldots a_p; a, c, -\lambda) \\ &= \frac{(-1)^\alpha n!}{2^{m\alpha}(n-m\alpha)} {}_HE_{n-m\alpha}^{[m-1,\alpha]}(a_1, a_2 \ldots a_p; a, c, \lambda) \end{aligned} \tag{23}$$

or equivalently by

$$\begin{aligned} &{}_HE_n^{[m-1,\alpha]}(a_1, a_2 \ldots a_p; a, c, -\lambda) \\ &= \frac{(-2)^\alpha n!}{(n-m\alpha)} {}_HB_{n+m\alpha}^{[m-1,\alpha]}(a_1, a_2 \ldots a_p; a, c, \lambda) \end{aligned} \tag{24}$$

Proof. Considering the generating function (19)

$$\begin{aligned} &\sum_{n=0}^{\infty} {}_HB_n^{[m-1,\alpha]}(a_1, a_2 \ldots a_p; a, c, -\lambda) \frac{t^n}{n!} \\ &= t^{m\alpha}[A(-\lambda, a; t)]^\alpha c^{a_1t+a_2t^2 \ldots + a_pt^p} \\ &= \frac{(-1)^\alpha 2^{m\alpha}}{2^{m\alpha}} t^{m\alpha}[B(\lambda, a; t)]^\alpha c^{a_1t+a_2t^2 \ldots + a_pt^p} \\ &= \frac{(-1)^\alpha}{2^{m\alpha}} \sum_{n=0}^{\infty} {}_HE_n^{[m-1,\alpha]}(a_1, a_2 \ldots a_p; a, c, \lambda) \frac{t^{n+m\alpha}}{n!}. \end{aligned}$$

Replacing n by $n - m\alpha$ in RHS of the above equation

$$\begin{aligned} &\sum_{n=0}^{\infty} {}_HB_n^{[m-1,\alpha]}(a_1, a_2 \ldots a_p; a, c, -\lambda) \frac{t^n}{n!} \\ &= \frac{(-1)^\alpha}{2^{m\alpha}} \sum_{n=0}^{\infty} {}_HE_{n-m\alpha}^{[m-1,\alpha]}(a_1, a_2 \ldots a_p; a, c, \lambda) \frac{t^n}{n-m\alpha!}. \end{aligned}$$

Comparing the coefficient of t^n on both sides of the above equation, we obtain the result (23). $\square$

Remark 2.2. *If we set $a_3 = a_4 \cdots = a_p = 0$ in (23), the result reduces to the known result given by [14, p. 161, Eq. (39)].*

Remark 2.3. *For $a_2 = a_3 = a_4 \cdots = a_p = 0$ in (23), we obtained the result of [15].*

For proving the second part of the theorem, we consider the generating function (20)

$$\begin{aligned} &\sum_{n=0}^{\infty} {}_HE_n^{[m-1,\alpha]}(a_1, a_2 \ldots a_p; a, c, \lambda) \frac{t^n}{n!} \\ &= 2^{m\alpha}[B(\lambda, a; t)]^\alpha c^{a_1t+a_2t^2 \ldots + a_pt^p} \\ &= \frac{(-1)^\alpha t^{m\alpha}}{t^{m\alpha}} 2^{m\alpha}[A(\lambda, a; t)]^\alpha c^{a_1t+a_2t^2 \ldots + a_pt^p} \\ &= (-2)^\alpha \sum_{n=0}^{\infty} {}_HB_n^{[m-1,\alpha]}(a_1, a_2 \ldots a_p; a, c, \lambda) \frac{t^{n-m\alpha}}{n!}. \end{aligned}$$

Replacing n by $n+m\alpha$ in RHS of the above equation, we get

$$\sum_{n=0}^{\infty} {}_HE_n^{[m-1,\alpha]}(a_1,a_2\dots a_p;a,c,-\lambda)\frac{t^n}{n!}$$
$$=(-2)^\alpha\sum_{n=0}^{\infty} {}_HB_n^{[m-1,\alpha]}(a_1,a_2\dots a_p;a,c,\lambda)\frac{t^n}{n+m\alpha!}.$$

Comparing the coefficient of t^n on both sides of the above equation, we obtain the result (24).

Remark 2.4. *If we set $a_3=a_4\cdots=a_p=0$ in* (24)*, we obtained the known result of [14, p. 161, Eq. (40)].*

Remark 2.5. *If we set $a_2=a_3=a_4\cdots=a_p=0$ in* (24)*, the result reduces to the result of [15].*

Theorem 2.3. *The generalized Apostol-Hermite-Genocchi polynomials ${}_HG_n^{[m-1,\alpha]}(a_1,a_2\dots a_p;a,c,\lambda)$, the generalized Apostol-Hermite-Bernoulli polynomials ${}_HB_n^{[m-1,\alpha]}(a_1,a_2\dots a_p;a,c,\lambda)$ and the generalized Apostol-Hermite-Euler polynomials ${}_HE_n^{[m-1,\alpha]}(a_1,a_2\dots a_p;a,c,\lambda)$, are related by*

$$\begin{aligned}&{}_HG_n^{[m-1,\alpha]}(a_1,a_2\dots a_p;a,c,-\lambda)\\ &\quad=(-2^m)^\alpha{}_HB_n^{[m-1,\alpha]}(a_1,a_2\dots a_p;a,c,\lambda)\end{aligned}\tag{25}$$

or equivalently by

$$\begin{aligned}&{}_HG_n^{[m-1,\alpha]}(a_1,a_2\dots a_p;a,c,\lambda)\\ &\quad=\frac{n!}{(n-m\alpha)!}{}_HE_{n-m\alpha}^{[m-1,\alpha]}(a_1,a_2\dots a_p;a,c,\lambda),\end{aligned}\tag{26}$$

($m,n,\alpha\in N_0$, $\lambda\in\mathbb{C}$ and $n>m\alpha$).

Proof. Considering the generating function (19)

$$t^{m\alpha}[A(\lambda,a;t)]^\alpha c^{a_1t+a_2t^2\dots+a_pt^p}$$
$$=\sum_{n=0}^{\infty}{}_HB_n^{[m-1,\alpha]}(a_1,a_2\dots a_p;a,c,\lambda)\frac{t^n}{n!},$$
$$t^{m\alpha}[B(-\lambda,a;t)]^\alpha c^{a_1t+a_2t^2\dots+a_pt^p}$$
$$=(-2^m)^\alpha\sum_{n=0}^{\infty}{}_HB_n^{[m-1,\alpha]}\,(a_1,a_2\dots a_p;a,c,\lambda)\,\frac{t^n}{n!},$$
$$\sum_{n=0}^{\infty}{}_HG_n^{[m-1,\alpha]}(a_1,a_2\dots a_p;a,c,-\lambda)\frac{t^n}{n!}$$
$$=(-2^m)^\alpha\sum_{n=0}^{\infty}{}_HB_n^{[m-1,\alpha]}(a_1,a_2\dots a_p;a,c,\lambda).$$

Comparing the coefficient of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain the result (25). □

Remark 2.6. *If we set $a_3=a_4\cdots=a_p=0$ in* (25)*, the result reduces to the known result of [14, p. 164, Eq. (48)].*

Remark 2.7. *For the setting $a_2 = a_3 = a_4 \cdots = a_p = 0$ in (25), we obtained the known result of [15].*

For proving the second part of the theorem, we consider the generating function (21)

$$2^{m\alpha} t^{m\alpha} \left[B(\lambda, a; t)\right]^{\alpha} c^{a_1 t + a_2 t^2 \ldots + a_p t^p} = \sum_{n=0}^{\infty} {}_H G_n^{[m-1,\alpha]} (a_1, a_2 \ldots a_p; a, c, \lambda) \frac{t^n}{n!},$$

$$\sum_{n=0}^{\infty} {}_H E_n^{[m-1,\alpha]} (a_1, a_2 \ldots a_p; a, c, \lambda) \frac{t^{n+m\alpha}}{n!} = \sum_{n=0}^{\infty} {}_H G_n^{[m-1,\alpha]} (a_1, a_2 \ldots a_p; a, c, \lambda) \frac{t^n}{n!}.$$

Replacing n by $n - m\alpha$ in LHS of the above equation, we get

$$\sum_{n=m\alpha}^{\infty} {}_H E_{n-m\alpha}^{[m-1,\alpha]} (a_1, a_2 \ldots a_p; a, c, \lambda) \frac{t^n}{n - m\alpha!} = \sum_{n=0}^{\infty} {}_H G_n^{[m-1,\alpha]} (a_1, a_2 \ldots a_p; a, c, \lambda) \frac{t^n}{n!}.$$

Comparing the coefficient of t^n on both sides of the above equation, we obtain the result (26).

Remark 2.8. *If we put $a_3 = a_4 \cdots = a_p = 0$ in (26), we obtained the known result given [14, p. 164, Eq. (49)].*

Remark 2.9. *If we put $a_2 = a_3 = a_4 \cdots = a_p = 0$ in (26), the result reduces to the result of [15].*

Theorem 2.4. *The generalized Apostol-Hermite-Euler polynomials ${}_H E_n^{[m-1,\alpha]}$ $(a_1, a_2 \ldots a_p; a, c, \lambda)$ satisfy the following recurrence relation:*

$$\begin{aligned} &\lambda {}_H E_n^{[m-1,\alpha]}(a_1 + 1, a_2 \ldots a_p; a, c, \lambda) + {}_H E_n^{[m-1,\alpha]}(a_1, a_2 \ldots a_p; a, c, \lambda) \\ &= 2 \sum_{r=0}^{n} \binom{n}{r} {}_H E_n^{[m-1,\alpha]}(a_1, a_2 \ldots a_p; a, c, \lambda) E_{n-r}^{[-1,\alpha]}(0, a, \lambda) \end{aligned} \tag{27}$$

Proof. Let

$$\begin{aligned} &\lambda {}_H E_n^{[m-1,\alpha]}(a_1 + 1, a_2 \ldots a_p; a, c, \lambda) + {}_H E_n^{[m-1,\alpha]}(a_1, a_2 \ldots a_p; a, c, \lambda) \\ &= (2^m)^{\alpha} [B(\lambda, a; t)]^{\alpha} c^{a_1 t + a_2 t^2 \ldots + a_p t^p} (\lambda a^t + 1) \\ &= 2.2^{m\alpha} [B(\lambda, a; t)]^{\alpha} c^{a_1 t + a_2 t^2 \ldots + a_p t^p} \left(\frac{2}{\lambda a^t + 1}\right)^{(-1)} \\ &= 2. \sum_{r=0}^{\infty} {}_H E_n^{[m-1,\alpha]}(a_1, a_2 \ldots a_p; a, c, \lambda) \frac{t^r}{r!}. \sum_{n=0}^{\infty} E_n^{[-1]}(0, a, \lambda) \frac{t^n}{n!} \\ &= 2. \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} {}_H E_n^{[m-1,\alpha]}(a_1, a_2 \ldots a_p; a, c, \lambda). E_n^{[-1]}(0, a, \lambda) \frac{t^{n+r}}{n! r!}. \end{aligned}$$

Replacing n by $n-r$ in the above equation, we get

$$\sum_{n=0}^{\infty}(\lambda_H E_n^{[m-1,\alpha]}(a_1+1,a_2\dots a_p;a,c,\lambda) +_H E_n^{[m-1,\alpha]}(a_1,a_2\dots a_p;a,c,\lambda))\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\left(2.\sum_{r=0}^{n}{}_H E_r^{[m-1,\alpha]}(a_1,a_2\dots a_p;a,c,\lambda).E_{n-r}^{[-1]}(0,a,\lambda)\frac{t^n}{n-r!r!}\right).$$

Comparing the coefficient of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain the result (27). □

Remark 2.10. *If we set* $a_3=a_4\cdots=a_p=0$ *in* (27)*, we obtained the known result given by [14, p. 162, Eq. (43)].*

Remark 2.11. *For the setting* $a_2=a_3=a_4\cdots=a_p=0$ *in* (27)*, the result reduces to the result of [15].*

Theorem 2.5. *The generalized Apostol-Hermite-Genocchi polynomials* ${}_H G_n^{[m-1,\alpha]}(a_1,a_2\dots a_p;a,c,\lambda)$ *satisfy the following recurrence relation:*

$$\lambda_H G_n^{[m-1,\alpha]}(a_1+1,a_2\dots a_p;a,c,\lambda) +_H G_n^{[m-1,\alpha]}(a_1,a_2\dots a_p;a,c,\lambda)$$
$$=2n\sum_{r=0}^{n}\binom{n-1}{r}{}_H G_n^{[m-1,\alpha]}(a_1,a_2\dots a_p;a,c,\lambda)G_{n-1-r}^{[-1,\alpha]}(0,a,\lambda)\,. \tag{28}$$

Proof. Let

$$\lambda_H G_n^{[m-1,\alpha]}(a_1+1,a_2\dots a_p;a,c,\lambda) +_H G_n^{[m-1,\alpha]}(a_1,a_2\dots a_p;a,c,\lambda)$$
$$=(2^m t^m)^{\alpha}[B(\lambda,a;t)]^{\alpha}c^{a_1t+a_2t^2\dots+a_pt^p}(\lambda a^t+1)$$
$$=2t.2^{m\alpha}t^{m\alpha}[B(\lambda,a;t)]^{\alpha}c^{a_1t+a_2t^2\dots+a_pt^p}\left(\frac{2t}{\lambda a^t+1}\right)^{(-1)}$$
$$=2t.\sum_{r=0}^{\infty}{}_H G_n^{[m-1,\alpha]}(a_1,a_2\dots a_p;a,c,\lambda)\frac{t^r}{r!}.\sum_{n=0}^{\infty}G_n^{[-1]}(0,a,\lambda)\frac{t^n}{n!}$$
$$=2.\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}{}_H G_n^{[m-1,\alpha]}(a_1,a_2\dots a_p;a,c,\lambda).G_n^{[-1]}(0,a,\lambda)\frac{t^{n+r+1}}{n!r!}\,.$$

Replacing n by $(n-r-1)$ in the above equation, we get

$$\sum_{n=0}^{\infty}(\lambda_H G_n^{[m-1,\alpha]}(a_1+1,a_2\dots a_p;a,c,\lambda) +_H G_n^{[m-1,\alpha]}(a_1,a_2\dots a_p;a,c,\lambda))\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\left(2n.\sum_{r=0}^{n}{}_H G_r^{[m-1,\alpha]}(a_1,a_2\dots a_p;a,c,\lambda).G_{n-r-1}^{[-1]}(0,a,\lambda)\frac{t^n}{n-1-r!r!}\right).$$

Comparing the coefficient of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain the result (28). □

Remark 2.12. *If we set* $a_3=a_4\cdots=a_p=0$ *in* (28)*, we obtained the known result given by [14, p. 162, Eq. (51)].*

Remark 2.13. *For putting* $a_2=a_3=a_4\cdots=a_p=0$ *in* (28)*, the result reduces to the known result of [15].*

3. Conclusion

In this paper we found number of interesting applications of Generalised Apostol Hermite-Bernoulli, Generalised Apostol-Hermite-Euler polynomials and Generalised Apostol Gennochi Polynomials. The polynomials found in this paper can be further taken into the consideration for more properties and applications with the use of linear positive operators in real and complex domain given by Aral et al [1] and Gupta and Agarwal [3]. The concepts discussed by them can be extended to few more interesting polynomials.

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